ASYMPTOTIC BEHAVIOR OF THE FLUCTUATIONS OF INTENSITY OF A PLANE LIGHT WAVE PROPAGATING IN A TURBULENT MEDIUM

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The problem of propagation of a plane light wave in a turbulent medium is considered in the parabolic-equation and Markov-model approximation. It is shown that as $\sigma_0^2 \to \infty$ all the intensity moments are saturated, with the quantity σ_0^2 (amplitude-level dispersion in the gradual perturbation method) is a parameter describing the fluctuation intensity. The asymptotic values for the intensity moments indicate the absence of a limiting probability distribution for the intensity.

We shall describe the propagation of a light wave in the approximation of the scalar parabolic equation

$$\frac{\partial u(x,\rho)}{\partial x}=\frac{i}{2k}\Delta_\perp u+\frac{ik}{2}\varepsilon u, \tag{1}$$

where the x axis is chosen in the direction of the initial propagation of the wave, ρ and $\Delta_\perp$ denote the transverse coordinates and the Laplace operator with respect to them, and ϵ the deviation of the dielctric constant from its mean value of unity. Equation (1) for a plane wave must be solved with the boundary condition

$$u(0,\rho)=1. \tag{2}$$

The solution of Eq. (1) with condition (2) can be written, following Fradkin's method[1,2], in operator form[3]

$$u(x,\rho)=\exp\left\{\frac{i}{2k}\int_0^x d\xi\frac{\delta^2}{\delta\tau^2(\xi)}\right\}\exp\left\{\frac{ik}{2}\int_0^x d\xi\,\varepsilon\left(\xi,\rho+\int_\xi^x d\eta\,\tau(\eta)\right)\right\}\Bigg|_{\tau=0} \tag{3}$$

or in the form of a Feynman continual integral

$$u(x,\rho)=\int D\nu\exp\left\{\frac{ik}{2}\int_0^x d\xi\left[\nu^2(\xi)+\varepsilon\left(\xi,\rho+\int_\xi^x d\eta\,\nu(\eta)\right)\right]\right\},$$

$$D\nu=\prod_{\xi=0}^x d\nu(\xi)\Big/\int\ldots\int\prod_{\xi=0}^x d\nu(\xi)\exp\left\{\frac{ik}{2}\int_0^x d\xi\,\nu^2(\xi)\right\}. \tag{4}$$

In[3-6] we investigated the propagation of light in the approximation of a Markov random process (AMRP) and the condition for the applicability of the AMRP as well as of the parabolic-equation approximation.

In the case of a Gaussian field of the dielectric constant ϵ, the AMRP corresponds to a correlation function of the field ϵ in the form

$$B_\varepsilon(x-x',\ \rho-\rho')=\langle\varepsilon(x,\rho)\varepsilon(x',\rho')\rangle=\delta(x-x')A(\rho-\rho'),$$

$$A(\rho)=\int_{-\infty}^{\infty}dx\,B_\varepsilon(x,\rho),\quad A(\rho)=2\pi\int d\varkappa\,\Phi_\varepsilon(\varkappa)\exp\{i\varkappa\rho\}, \tag{5}$$

where $\Phi_\epsilon(\kappa)$ is the three-dimensional spectrum of the field ϵ as a function of the two-dimensional vector $\boldsymbol{\kappa}$. In this case all the moments of the field $u(x,\rho)$ constructed with the aid of formula (3) represent the operator form of the solution of the corresponding equation obtained in[4]. Thus, for example, for the second-order coherence function $\Gamma(x,\rho_1,\rho_2) = \langle u(x,\rho_1)u^*(x,\rho_2)\rangle$, we have, as shown in[3]

$$\Gamma(x,\rho)=\exp\left\{\frac{i}{2k}\int_0^x d\xi\left[\frac{\delta^2}{\delta\tau_1^2(\xi)}-\frac{\delta^2}{\delta\tau_2^2(\xi)}\right]\right\}\exp\left\{-\frac{k^2}{4}\int_0^x d\xi\cdot\right. \tag{6}$$

$$\left.\times D\left(\rho+\int_\xi^x d\eta\,[\tau_1(\eta)-\tau_2(\eta)]\right)\right\}\Bigg|_{\tau_i=0}\quad(\rho=\rho_1-\rho_2),$$

where $D(\rho) = A(0) - A(\rho)$, which is equivalent to the solution of the equation

$$\frac{\partial}{\partial x}\Gamma(x,\rho_1,\rho_2)=\frac{i}{2k}[\Delta_1-\Delta_2]\Gamma-\frac{k^2}{4}D(\rho_1-\rho_2)\Gamma \tag{7}$$

with the condition $\Gamma(0,\rho_1,\rho_2) = 1$. The solution of Eq. (7) or the direct calculation of (6) yield the following expression for the coherence function:

$$\Gamma(x,\rho_1,\rho_2)=\exp\left\{-\frac{k^2}{4}D(\rho_1-\rho_2)x\right\}. \tag{8}$$

In the general case the methods developed in[3,4] are equivalent. However, the expression for $u(x,\rho)$ in the form (3) or (4) makes it possible to write an expression for the wave intensity in explicit form:

$$I(x,\rho)=u(x,\rho)u^*(x,\rho)=L_{1,2}\exp\{iz_{1,2}\}|_{\tau_i=0}, \tag{9}$$

where the operator

$$L_{1,2}=\exp\left\{\frac{i}{2k}\int_0^x d\xi\left[\frac{\delta^2}{\delta\tau_1^2(\xi)}-\frac{\delta^2}{\delta\tau_2^2(\xi)}\right]\right\},$$

$$z_{1,2}=\frac{k}{2}\int_0^x d\xi\left[\varepsilon\left(\xi,\rho+\int_\xi^x d\eta\,\tau_1(\eta)\right)-\varepsilon\left(\xi,\rho+\int_\xi^x d\eta\,\tau_2(\eta)\right)\right],$$

which is more convenient for study.

We shall assume below that the field of the dielectric constant is Gaussian and δ-correlated along the x axis. The random inhomogeneous medium will be regarded as a turbulent medium described by a spectral function in the form (see[7])

$$\Phi_\varepsilon(\varkappa)=AC_\varepsilon^2\varkappa^{-11/3}\exp\{-\varkappa^2/\varkappa_m^2\}, \tag{10}$$

where C_ϵ^2 characterizes the intensity of the fluctuations of the field ϵ, and κ_m determines the internal scale of the turbulence. In the case $\kappa_m^2 x/k \gg 1$, which is of interest and will be considered below, we have

$$\Phi_\varepsilon(\varkappa)=AC_\varepsilon^2\varkappa^{-11/3},\quad D(\rho)\sim AC_\varepsilon^2\rho^{5/3}. \tag{11}$$

We note first that there is an equality $\langle I\rangle = 1$, which expresses the law of energy conservation. For the square of the intensity we have

$$I^2(x,\rho)=L_{1,2}L_{3,4}\exp\{i(z_{1,2}+z_{3,4})\}|_{\tau_i=0}. \tag{12}$$

Let us average (12) over the ensemble of realizations of the field ϵ. By virtue of the Gaussian and δ-correlated character of the field ϵ, we obtain

$$\langle I^2\rangle = L_{1,2}L_{3,4}\exp\{-{}^1/_2[\langle z_{1,2}^2\rangle + 2\langle z_{1,2}z_{3,4}\rangle + \langle z_{3,4}^2\rangle]\}|_{\tau_i=0}, \quad (13)$$

where

$$\langle z_{1,2}^2\rangle = -\frac{k^2}{4}\int_0^x d\xi\, D\left(\int_\xi^x d\eta[\tau_1-\tau_2]\right),$$

$$\langle z_{3,4}^2\rangle = -\frac{k^2}{4}\int_0^x d\xi\, D\left(\int_\xi^x d\eta[\tau_3-\tau_4]\right), \quad (14)$$

$$\langle z_{1,2}z_{3,4}\rangle = -\frac{k^2}{4}\int_0^x d\xi\left\{D\left(\int_\xi^x d\eta[\tau_1-\tau_4]\right)\right.$$

$$\left.+D\left(\int_\xi^x d\eta[\tau_2-\tau_3]\right)-D\left(\int_\xi^x d\eta[\tau_1-\tau_3]\right)-D\left(\int_\xi^x d\eta[\tau_2-\tau_4]\right)\right\}.$$

Expression (13) cannot be calculated, and represents the solution of the equation for the fourth-order coherence function:

$$\Gamma_4(x,\rho_i) = \langle u(x,\rho_1)u^*(x,\rho_2)u(x,\rho_3)u^*(x,\rho_4)\rangle,$$

$$\frac{\partial}{\partial x}\Gamma_4 = \frac{i}{2k}[\Delta_1-\Delta_2+\Delta_3-\Delta_4]\Gamma_4 - \frac{k^2}{4}Q(\rho_1,\rho_2,\rho_3,\rho_4)\Gamma_4,$$

$$\Gamma_4|_{x=0} = 1, \quad (15)$$

obtained in[4] for $\rho_1 = \rho_2 = \rho_3 = \rho_4$. We note that (14) and (15) contain the quantity $D(\rho)$, which is determined by the small-scale fluctuations of the field ϵ.

Formula (13) for $\langle I^2\rangle$ is convenient for investigations. We rewrite it in the form

$$\langle I^2(x,\rho)\rangle = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_n(x), \quad (16)$$

where

$$a_n = L_{1,2}L_{3,4}\exp\{-{}^1/_2[\langle z_{1,2}^2\rangle + \langle z_{3,4}^2\rangle]\}\langle z_{1,2}z_{3,4}\rangle^n|_{\tau_i=0}. \quad (17)$$

In principle all the quantities $a_n(x)$ can be directly calculated. Let us illustrate this with a_0 and a_1 as an example:

$$a_0 = \langle I\rangle\langle I\rangle = 1,$$
$$a_1 = L_{1,2}L_{3,4}\exp\{-{}^1/_2[\langle z_{1,2}^2\rangle + \langle z_{3,4}^2\rangle]\}\langle z_{1,2}z_{3,4}\rangle|_{\tau_i=0}. \quad (18)$$

The quantities $\langle z_{1,2}z_{3,4}\rangle$ can be rewritten in the form

$$\langle z_{1,2}z_{3,4}\rangle = \frac{\pi k^2}{2}\int_0^x d\xi\int d\varkappa\,\Phi_\epsilon(\varkappa)\left[\exp\left\{i\varkappa\int_\xi^x d\eta[\tau_1-\tau_3]\right\}\right.$$

$$\left.+\exp\left\{i\varkappa\int_\xi^x d\eta[\tau_2-\tau_4]\right\}-\exp\left\{i\varkappa\int_\xi^x d\eta[\tau_1-\tau_4]\right\}-\exp\left\{i\varkappa\int_\xi^x d\eta[\tau_2-\tau_3]\right\}\right]. \quad (19)$$

Further calculations are analogous to the calculation of the second-order coherence function in[3].

We make the functional substitution

$$\tau_1-\tau_2=\tau, \qquad \tau_1+\tau_2 = 2\mathrm{T},$$
$$\tau_3-\tau_4=\tilde\tau, \qquad \tau_3+\tau_4=2\tilde{\mathrm{T}}. \quad (20)$$

Then the operator $L_{1,2}L_{3,4}$ takes the form

$$L_{1,2}L_{3,4} = \exp\left\{\frac{i}{k}\int_0^x d\xi\left[\frac{\delta^2}{\delta\tau(\xi)\delta\mathrm{T}(\xi)} - \frac{\delta^2}{\delta\tilde\tau(\xi)\delta\tilde{\mathrm{T}}(\xi)}\right]\right\}. \quad (21)$$

Since T and $\tilde{\mathrm{T}}$ enter linearly under the exponential sign, we can perform the operation of variational differentiation with respect to T and $\tilde{\mathrm{T}}$. Then the action of the operator (21) reduces to a simple functional shift with respect to τ and $\tilde\tau$. Setting then all τ_i equal to zero, we obtain the final expression for a_1 in the form

$$a_1(x) = 2\pi k^2\int_0^x d\xi\int d\varkappa\,\Phi_\epsilon(\varkappa)\sin^2\left[\frac{\varkappa^2}{k}(x-\xi)\right]\cdot$$
$$\times\exp\left\{-\frac{k^2}{4}\int_0^x d\xi_1 D\left[\frac{\varkappa}{k}(x-\max\{\xi,\xi_1\})\right]\right\}. \quad (22)$$

It is possible to calculate analogously the other quantities $a_n(x)$, but the expressions obtained for them are exceedingly cumbersome. We note that the determination of the values of $a_n(x)$ by the method described above is equivalent in essence to the solution of (15) by the method of successive approximations developed in[8]. The quantity $a_1(x)$ is equivalent in this sense to a determination of the solution of Eq. (15) in the "single-scattering" approximation for equations of the transport type (see[8]).

Taking into account the expressions for $\Phi_\epsilon(\kappa)$ and $D(\rho)$, determined by formulas (11), we can write (22) in the form

$$a_1(x) = 2\pi k^2\beta^2\int_0^1 d\xi\int d\varkappa\,\tilde\Phi_\epsilon(\varkappa)\sin^2[\varkappa^2(1-\xi)]$$
$$\times\exp\left\{-\frac{\beta^2}{2}\int_0^1 d\xi_1\tilde D[\varkappa(1-\max\{\xi,\xi_1\})]\right\}, \quad (23)$$

where $\beta^2 = AC_\epsilon^2k^{7/6}x^{11/6} \sim \sigma_0^2$ (σ_0^2—is the variance of the level of the amplitude in the smooth perturbation method) and

$$\tilde\Phi_\epsilon(\varkappa) = \varkappa^{-11/3}, \quad D(\rho)\sim\rho^{5/3}. \quad (24)$$

It is obvious that an analogous form can be used also for all the other quantities $a_n(x)$. In view of this, transformation of the series (16) enables us to write

$$\langle I^2\rangle = \tilde L_{1,2}\tilde L_{3,4}\exp\left\{-\frac{\beta^2}{2}[\langle\tilde z_{1,2}^2\rangle + 2\langle\tilde z_{1,2}\tilde z_{3,4}\rangle + \langle\tilde z_{3,4}^2\rangle]\right\}\Bigg|_{\tau_i=0}, \quad (25)$$

where

$$\tilde L_{1,2} = \exp\left\{\frac{i}{2}\int_0^1 d\xi\left[\frac{\delta^2}{\delta\tau_1{}^2(\xi)} - \frac{\delta^2}{\delta\tau_2{}^2(\xi)}\right]\right\}.$$

It is also obvious that an expression in the form (25) is valid also for all other moments of the intensity. We can therefore state that

$$\langle I^n(x,\rho)\rangle = \langle\psi^n(\beta)\rangle, \quad (26)$$

where the random function $\psi(\beta)$ is described by the formula

$$\psi(\beta) = \tilde L_{1,2}\exp\left\{\frac{i\beta}{2}\tilde z_{1,2}\right\}\Bigg|_{\tau_i=0}, \quad (27)$$

where $\epsilon(\xi,\rho)$ is a Gaussian random field, δ-correlated with respect to ξ, determined by its spectral function in the form $\tilde\Phi_\epsilon(\kappa) = \kappa^{-11/3}$.

We proceed now to find the asymptotic form of $\langle I^2(x,\rho)\rangle$ as $\sigma_0^2\to\infty$ or, what is the same, as $\beta^2\to\infty$. It is difficult to find the asymptotic form of $\langle I^2\rangle$ directly from (25). Recognizing that $\langle\psi(\beta)\rangle = 1$, we consider the fluctuations of the function $\psi(\beta)$ relative to its mean value:

$$\psi'(\beta) = \psi(\beta)-1, \quad \langle\psi'(\beta)\rangle = 0. \quad (28)$$

Formula (28) with allowance for (27) can be rewritten in the form

$$\psi'(\beta) = \tilde L_{1,2}\int_0^\beta d\lambda\frac{\partial}{\partial\lambda}\exp\{i\lambda\tilde z_{1,2}\}|_{\tau=0}. \quad (29)$$

Then the quantity

$$\langle[\psi']^2\rangle = \widetilde{L}_{1,2}\widetilde{L}_{3,4}\iint_0^\beta d\lambda_1\, d\lambda_2 \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\exp\left\{-\frac{1}{2}\langle[\lambda_1\widetilde{z}_{1,2}+\lambda_2\widetilde{z}_{3,4}]^2\rangle\right\}\Bigg|_{\tau_i=0}, \tag{30}$$

and as $\beta\to\infty$, by virtue of the fact that the form $\langle[\lambda_1\widetilde{z}_{1,2}+\lambda_2\widetilde{z}_{3,4}]^2\rangle$ is positive definite in λ_i it is possible to carry out the integration and we obtain

$$\langle[\psi'(\beta)]^2\rangle \underset{\beta\to\infty}{\to} \widetilde{L}_{1,2}\widetilde{L}_{3,4}1 = 1. \tag{31}$$

Consequently as $\beta^2\to\infty$

$$\langle I^2(x,\rho)\rangle = \langle\psi^2(\beta)\rangle \to 2. \tag{32}$$

Similarly we obtain for the other moments

$$\langle[\psi']^n\rangle = (-1)^n \quad (n=2,3,\ldots) \text{ for } \beta^2\to\infty \tag{33}$$

and consequently

$$\lim_{\beta^2\to\infty}\langle I^n(x,\rho)\rangle = n. \tag{34}$$

We have thus found that all the moments of the intensity of a plane light wave saturate at $\sigma_0^2\to\infty$ and the level of saturation for each moment is determined by the expression (34). However, the method developed above does not make it possible to determine either the manner in which the moments tend to their asymptotic value nor the region of values of σ_0 for which the asymptotic formulas (34) are valid. We note that if the probability distribution for the light amplitude at $\sigma_0^2\to\infty$ were of the Rayleigh type, then this would denote that $\langle I^n\rangle = n!$, which does not agree with formula (34). Moreover, it follows from (34) that there is no limiting expression at all for the distribution of the intensity probabilities. In fact, the Carleman condition (see, e.g.,[9]), which makes it possible to determine the intensity probability density uniquely for the moments, is satisfied:

$$\sum_{n=1}^{\infty}\langle I^{2n}\rangle^{-1/2n} = \infty.$$

We can formally construct the function

$$p(I) = \delta(I-0) - \frac{\partial}{\partial I}\delta(I-1)$$

such that

$$\int_0^\infty p(I)\,dI = 1, \qquad \int_0^\infty I^n p(I)\,dI = n,$$

but this function cannot be the probability density, since the condition $p(I)\geq 0$ is violated.

The absence of a limiting law of probability distribution for the intensity means that there exists no region of values of σ_0 at which all the moments I assume an asymptotic value, although each moment separately does tend to its asymptotic value.

We note that this fact agrees with the qualitative picture of the behavior of the distribution of the probabilities for the quantity ln I, obtained in[10].

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