PARAMETRIC AMPLIFICATION OF LIGHT IN THE FIELD OF A MODULATED LASER WAVE

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The theory of parametric amplification of two waves in an amplitude and phase modulated laser field is developed. Particular attention is paid to the exponential signal growth under various conditions. A new class of solutions valid for all possible cases of group velocity mismatch is obtained for a bell-shaped inhomogeneous pumping amplitude. Stationary mode formation of parametric signals is analyzed in detail. The possibility of applying the technique to the analysis of stimulated Raman scattering in a picosecond pulse field is considered.

1. INTRODUCTION

PARAMETRIC processes are important in nonlinear optics because they appear in a broad class of the most diverse phenomena. In addition to purely parametric processes due to electron nonlinearity this includes the majority of scatterings (Mandel'shtam-Brillouin, Raman, etc.). In the case of spontaneous scattering and generation of the sum and difference of optical frequencies the increase of signal wave amplitude along the nonlinear medium fails to exceed the linear case even under optimum conditions.

The present work deals with theoretical analysis of exponential amplification of waves excited by narrow beams and short pulses of laser emission.[1)] We consider different variants of ray vector geometry and group velocity relations of the interacting waves.

We obtained a new class of solutions for the three-frequency parametric amplifier with amplitude-modulated (AM) bell-shaped pumping covering all possible cases of phase and group velocity mismatch. These solutions allow us to trace the dynamics of amplification of outgoing and incoming waves with initial amplitude and phase modulation of any type (AM and PM signals).

The developed theory of parametric amplification in a quadratic medium can be applied in a modified form to SRS. The analysis of non-stationary SRS is of interest in itself in view of picosecond pumping experiments.

Among other work on this subject we note papers of Freĭdman et al.[7-11] as well as[12-16].

2. QUASI-OPTICS EQUATIONS

Thus let there by three waves propagating in a medium with quadratic nonlinearity:

$$\mathbf{E}=\sum_{n=1}^{3}\mathbf{e}_nA_n(\mathbf{r},t)\exp[i(\omega_nt-k_nz)]+\text{c.c.} \tag{1}$$

The wave frequencies are $\omega_3 = \omega_1 + \omega_2$ and wave numbers are $k_3 = k_1 + k_2 + \Delta$. The interaction of weakly modulated waves (1) is described in quasi-optical approximation by parabolic equations for complex amplitudes (field E_3 is considered given):

$$\partial A_1/\partial z+[\mathscr{L}_1(\eta)+\mathscr{L}_1(x,y)]A_1=\sigma_1A_3A_2^*e^{-i\Delta z}-\delta_1A_1, \tag{2}$$

$$\partial A_2/\partial z+[\mathscr{L}_2(\eta)+\mathscr{L}_2(x,y)]A_2=\sigma_2A_3A_1^*e^{-i\Delta z}-\delta_2A_2, \tag{3}$$

$$\partial A_3/\partial z+[\mathscr{L}_3(\eta)+\mathscr{L}_3(x,y)]A_3=0. \tag{4}$$

Here σ is the nonlinear coupling coefficient, δ are attenuation coefficients, and $\mathscr{L}_n(\eta)$ are differential operators describing the frequency dispersion of the medium:

$$\mathscr{L}_n(\eta)=\nu_{n3}\frac{\partial}{\partial\eta}-\frac{i}{2}\frac{\partial^2k_n}{\partial\omega_n^2}\frac{\partial^2}{\partial\eta^2}, \tag{5}$$

where $\eta = t - z/u_3$, $u_n = \partial k_n/\partial\omega_n$ is group velocity, $\nu_{n3} = u_n^{-1} - u_3^{-1}$ is the mismatch of group velocity reciprocals, and $\mathscr{L}_n(x, y)$ are operators describing birefringence and diffraction,

$$\mathscr{L}_n(x,y)=\beta_n\frac{\partial}{\partial x}+\frac{i}{2k_n}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \tag{6}$$

β_n are angles between the ray vectors and the z axis.

3. UNMODULATED PUMPING WAVE

In the case of parametric amplification in the field of a plane monochromatic pumping wave ($A_3 = A_{30}$) we can observe the modification of the space-time signal spectra[1,5,11]:

$$S_n=\iiint_{-\infty}^{\infty}A_n(x,y,\eta,z)\exp[i\Omega\eta-ik_xx-ik_yy]\,dx\,dy\,d\eta. \tag{7}$$

The solution of (2) and (3) for arbitrary conditions at the boundary z = 0 has the form:

$$S_1(\Omega,k_x,k_y,z)=\exp\left\{-\frac{(\delta_1+\delta_2)}{2}z+i\frac{(\Delta_1-\Delta_2)}{2}\eta\right\}$$
$$\times\left\{S_{10}(\Omega,k_x,k_y)\left[\operatorname{ch}\Gamma z-i\frac{\Delta_2}{\Gamma}\operatorname{sh}\Gamma z\right]+S_{20}^*(-\Omega,-k_x,-k_y)\times\frac{\sigma_1A_{30}\operatorname{sh}\Gamma z}{\Gamma}\right\}. \tag{8}$$

Here $S^*(-\Omega, -k_x, -k_y)$ is the spectrum of complex conjugate amplitude and Δ_n are wave mismatch values due to the action of operators (5) and (6):

$$\Delta_n=\frac{\Omega}{u_n}+\frac{1}{2}\frac{\partial^2k_n}{\partial\omega_n^2}\Omega^2-\frac{(k_x^2+k_y^2)}{2k_n}+k_x\beta_n. \tag{9}$$

The quantity Γ determines the amplifier increment

$$\Gamma=[\sigma_1\sigma_2A_{30}^2+{}^1/_4(\delta_1-\delta_2)^2-{}^1/_4(\Delta_1-\Delta_2)^2]^{1/2}. \tag{10}$$

Solution (8) shows that the presence of the component with parameters $\omega_2 + \Omega$, k_x, k_y in the initial spec-

[1)]A part of our results was reported earlier to some extent [1-6].

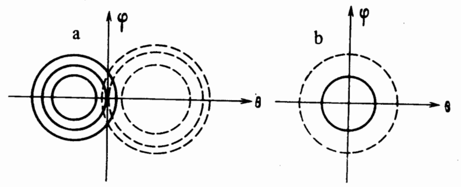

FIG. 1. Frequency-angle pattern of parametric scattering of light waves for the interactions of (a) $\gamma_e(\omega_3) \rightarrow \gamma_e(\omega_1) + \gamma_0(\omega_2)$, and (b)–$\gamma_e(\omega_3) \rightarrow \gamma_0(\omega_2) + \gamma_0(\omega_2)$. Circles represent cross sections of cones of two-dimensional velocity match. Solid lines correspond to a wave with a frequency $\omega_1 + \Omega$, and dashed lines to $\omega_2 - \Omega$.

trum of the second signal excites the component $\omega_1 - \Omega$, $-k_x$, $-k_y$ in the spectrum of the first signal and vice versa. The excitation effectiveness of such a wave pair depends on the wave mismatch Δ_n (9). The maximum increment is reached when mismatch values are equal $\Delta_1 = \Delta_2$, thus determining the frequency-angle characteristics of the amplifier. For example for an input of the type $\gamma_e(\omega_1) + \gamma_0(\omega_2) = \gamma_e(\omega_3)$ maximum gain of the signal component with the frequency $\omega_1 + \Omega$ occurs on a cone (see Fig. 1a):

$$\theta_1^2 + \varphi_1^2 + \frac{2k_2}{k_3}\beta_1\theta_1 = \frac{2k_2}{k_3k_1}\nu_{12}\Omega + \frac{k_2}{k_3k_1}\left(\frac{\partial^2 k_1}{\partial\omega_1^2} + \frac{\partial^2 k_2}{\partial\omega_2^2}\right)\Omega^2. \quad (11)$$

Here $\theta_1 = k_x/k_1$ and $\varphi_1 = k_y/k_1$ are angles of observation. It is interesting to note that the $\Omega = 0$ components can be amplified not only in a single dimension along the z axis ($\theta = \varphi = 0$) but also in two dimensions if the angular components are located on cones tangent to each other in the point of one-dimensional velocity match ($\theta = \varphi = 0$), and the cone axis for the extraordinary wave of the first signal is inclined at an angle $\beta_1 k_2/k_3$ to the z axis in the direction of the optical axis of the negative crystal.

In the case of ordinary polarization of both signal waves the velocity match cones leave traces in the form of concentric circles with the center at $\theta = \varphi = 0$ (Fig. 1b). The above spectral radiation structure is observed in parametric scattering of light[14]. It is fairly well described quantitatively by (11)[4].

The signals grow exponentially in the region of high gain: $|S|^2 \sim e^{Gz}$, $G = 2\Gamma - \delta_1 - \delta_2$. The value of mismatch for which intensity decreases e times is reduced with rising z:

$$\Delta_1 - \Delta_2 = \pm\sqrt{2}z^{1/2}[4\sigma_1\sigma_2 A_{30}^2 + (\delta_1 - \delta_2)^2]^{1/4}. \quad (12)$$

This corresponds to a narrowing of both the frequency and angular spectra of the signal waves (at the same time the scale of the space-time modulation of wave amplitudes increases).

The narrowing of spectra according to (12) pertains to parametric amplification in the field of unmodulated pumping. Actually the pumping wave is always spatially (bounded beam rather than a plane wave front) and often time modulated (pulse emission, phase and frequency modulation). When z is large enough the pumping wave modulation affects the nature of emission invalidating the above analysis for this area.

4. MODULATED PUMPING WAVE

An analysis of effects occurring in this case can be performed in good detail in the geometric optics approximation neglecting second derivatives in the system (2)–(4). Thus we investigate the following equations:

$$\frac{\partial A_1}{\partial z} - \beta_1\frac{\partial A_1}{\partial x} - \nu_1\frac{\partial A_1}{\partial\eta} = \sigma_1 A_{30}(x,\eta)A_2^* e^{-i\Delta z} - \delta_1 A_1, \quad (13)$$

$$\frac{\partial A_2^*}{\partial z} + \beta_2\frac{\partial A_2^*}{\partial x} + \nu_2\frac{\partial A_2^*}{\partial\eta} = \sigma_2^* A_{30}^*(x\ \eta)A_1 e^{i\Delta z} - \delta_2 A_2^*. \quad (14)$$

The effect of spatial and time pumping modulation on parametric amplification is considered separately for two cases: (1) absence of group lag effects (when $z \ll \tau/\nu$, τ is the characteristic scale of time pumping modulation), and (2) absence of effects due to the difference in wave propagation directions ($z \ll a/\beta$, a is the transverse scale of pump modulation). According to (13) and (14) the two variants of the system of equations have the same formal appearance, a fact that can be utilized in the development of a space-time analogy[14].

Thus we assume that $\nu_1 = \nu_2 = 0$. Introducing new amplitudes B_1 and B_2 by means of the relations

$$A_1 = B_1(x,z)\exp\left[-i\frac{\Delta(x+\beta_1 z)}{\beta_1+\beta_2} - \frac{(\delta_1\beta_2+\delta_2\beta_1)z}{\beta_1+\beta_2} + \frac{(\delta_1-\delta_2)x}{\beta_1+\beta_2}\right], \quad (15)$$

$$A_2^* = B_2(x,z)\left(\frac{\sigma_2}{\sigma_1}\right)^{1/2}\exp\left[-i\frac{\Delta(x-\beta_2 z)}{\beta_1+\beta_2} - \frac{(\delta_1\beta_2+\delta_2\beta_1)z}{\beta_1+\beta_2} + \frac{(\delta_1-\delta_2)x}{\beta_1+\beta_2}\right], \quad (16)$$

we arrive finally at the following hyperbolic system:

$$\frac{\partial B_1}{\partial z} - \beta_1\frac{\partial B_1}{\partial x} = \sigma A_{30}(x)B_2, \quad (17)$$

$$\frac{\partial B_2}{\partial z} + \beta_2\frac{\partial B_2}{\partial x} = \sigma A_{30}^*(x)B_1, \quad (18)$$

where $\sigma = (\sigma_1\sigma_2^*)^{1/2}$.

Further the wave propagating along the z axis together with the pumping wave ($\beta = 0$) is called the accompanying wave, and the wave propagating at an angle $\beta \neq 0$ to the z axis is called the outgoing wave. According to the following analysis the nature of their amplification depends not only on the type of pump modulation but also on the direction of propagation of signal waves (on the sign of β). We separate AM from PM in the complex pump amplitude using the following notation:

$$A_{30} = E_{30}(x)e^{i\varphi_3(x)}. \quad (19)$$

A. Amplification of Accompanying Waves

Waves propagating at the same velocity as the pumping ($\beta_n = 0$) experience the greatest gain

$$B_1 = B_{10}\,\mathrm{ch}\,\Gamma_{00}(x)z + B_{20}\,\mathrm{sh}\,\Gamma_{00}(x)z. \quad (20)$$

Here the increment $\Gamma_{00} = \sigma E_{30}(x)$ depends on the pumping intensity modulation and pump PM has no effect on gain. In the AM pump field amplification is not uniform: increasing z narrows the signals and therefore expands their spectra. For example for a Gaussian pump beam $E_{30}^2(x) = E_3^2\exp(-x^2/a_3^2)$ the width of signal beams narrows down according to the law $a_{1,2} \approx a_3(2\Gamma_{00}(0)z)^{1/2}$.

B. Amplification of Accompanying and Outgoing Waves

Now let one wave propagate as before along with the pump beam ($\beta_1 = 0$) while the other departs from it at an angle β_2. Such a situation is typical of the interaction of light waves of the type $\gamma_e(\omega_3) \to \gamma_e(\omega_1) + \gamma_0(\omega_2)$ where all wave vectors are directed along the z axis. In its time-base variant ($x \to \eta$, $\beta \to \nu$) this problem plays an important role in the theory of stimulated Mandel'shtam-Brillouin scattering in which the outgoing wave is an acoustic one[12,13]. The solution of this problem was discussed many times in relation to various viewpoints. For the accompanying wave we have

$$B_1(x,z)=B_{10}(x)+\frac{\sigma A_{30}(x)}{\beta_2}\int_\xi B_{20}(x_1)I_0(G_{0\beta}(x,z,x_1))\,dx_1$$
$$+\frac{2\sigma^2 A_{30}(x)}{\beta_2^2}\int_\xi^x B_{10}(x_1)A_{30}^*(x_1)(x_1-\xi)\frac{I_1(G_{0\beta})}{G_{0\beta}}\,dx_1, \quad (21)$$

and for the outgoing wave

$$B_2(x,z)=B_{20}(\xi)+\frac{\sigma}{\beta_2}\int_\xi^x A_{30}^*(x_1)B_{10}(x_1)I_0(G_{0\beta})\,dx_1$$
$$+\frac{\sigma}{2}\int_\xi^x B_{20}(x_1)\frac{G_{0\beta}I_1(G_{0\beta})}{x_1-\xi}\,dx_1, \quad (22)$$

where $\xi = x - \beta_2 z$, and I_0 and I_1 are modified Bessel functions of the first kind and zero and first orders whose argument is

$$G_{0\beta}(x,z,x_1)=\frac{2\sigma}{\beta_2}\left[(x_1-\xi)\int_{x_1}^x E_{30}^2(\tau)\,d\tau\right]^{1/2}. \quad (23)$$

We consider the effect of pump AM and PM on the amplification of sufficiently broad beams (long pulses) of excited waves.

In the case of pump PM ($E_{30}(x)$ = const), in addition to the above formulas, we can find exact expressions for the spectral signal components analogous to (8); now however $\Delta_1 = 0$, $\Delta_2 = k_x\beta_2$ and (8) are valid not for the spectrum of S_1 but for its convolution with the spectrum of the complex conjugate pump amplitude $A_{30}^* = E_{30}e^{-i\varphi(x)}$.

The manifestation of pump phase modulation depends on the boundary conditions at the input to the parametric amplifier. In the case of a single outgoing wave ($B_{10} = 0$) amplification is in no way connected with pump PM; pump AM is effective instead. In the amplification process the pump PM is completely superimposed on the accompanying wave: $B_1 \sim e^{i\varphi_3(x)}$ (see (21)). In particular this results in the situation where in the field of the PM fundamental frequency wave (without AM) the spectrum of the outgoing wave narrows and that of the accompanying wave tends toward the pump spectrum as z increases. In an amplifier with an accompanying input signal, amplification at the initial stage may depend on pump PM.

We now consider amplification in the field of a bounded pump beam (AM, $E_{30}(x)$ has a bell shape).

At high gain we can find asymptotic expressions for integrals (21) and (22):

$$B_1=E_{30}(x)\left[B_{10}(x_{10})+B_{20}(x_{10})\right]\left[4E_{30}^2(x_{10})\right.$$
$$\left.+2(\beta_2 z-x-x_{10})\partial E_3^2(x_{10})/\partial x_{10}\right]^{-1/2}\exp\left[C_{0\beta}(x,z,x_{10})\right], \quad (24)$$
$$B_2=B_1(x,z)E_{30}(x_{10})/E_{30}(x), \quad (25)$$

where parameter x_{10} is determined by the condition that argument $C_{0\beta}$ be maximum

$$\int_{x_{10}}^x E_{30}^2(\tau)\,d\tau=(\beta_2 z-x+x_{10})E_{30}^2(x_{10}). \quad (26)$$

The accompanying wave is localized here about the main beam (near its boundary where the second wave B_2 leaves). The maximum gain over the beam cross section is

$$G_{0\beta}^{(max)}\approx 2\sigma(zP_3/\beta_2)^{1/2} \quad (27)$$

and depends on beam power $P_3 = \int_{-\infty}^{\infty} E_{30}^2(x)\,dx$ and the exit angle β_2 of the second signal (in the time-based variant of the problem the maximum gain depends on the pump pulse and the mismatch of group velocites ν_2). The outgoing wave has the same increment (27) but strongly broadens with increasing z

$$(\Delta x)_\beta\approx\frac{\beta_2}{\sigma}\left(\frac{\beta_2 z}{P_3}\right)^{1/2}=\frac{2\beta_2 z}{G_{0\beta}}.$$

The amplitude of the second signal is lower than that of the accompanying wave because of the escape of second signal energy from the pump beam region.

Below are some specific data on signal amplification in the pumping field with various forms of amplitude modulation.

a. Parabolic profile $A_{30}^2 = E_3^2(1 - x^2/a^2)$. The accompanying signal narrows down according to the law $\Delta x_1 = 0.67aG_{0\beta}^{-1/2}$; its maximum lies at a distance of the same order from the pump beam boundary ($x = a$).

b. Gaussian profile $A_{30}^2 = E_3^2 e^{-x^2/a^2}$. Here the accompanying signal narrows down more slowly, $\Delta x_1 \approx \sqrt{2}a/\sqrt{\ln G_{0\beta}}$, and its peak gradually departs from the main beam axis towards the outgoing beam by a magnitude of $x_{m,1} = \frac{1}{2}a\sqrt{\ln G_{0\beta}}$.

c. Profile $A_{30}^2 = E_3^2\cosh^{-2}(x/a)$. A still gentler front of the pumping beam results in the establishment of a constant width of the accompanying wave $\Delta x_1 \approx \sqrt{2}a$; its peak is shifted as in the case of the Gaussian pump beam: $x_{m,1} = \frac{1}{2}a\ln G_{0\beta}$.

Thus pump AM is much more significant in the amplification of the accompanying and outgoing waves. The AM shape plays a role in the formation of signal beams (especially of the incoming beam) and to a lesser extent affects the gain (power is significant here).

C. Amplification of Outgoing Waves. Stationary Modes

In the majority of cases the signal waves and the pumping wave propagate with different velocities. The most interesting is the case of amplification of waves that leave the pumping beam in different directions. It turns out that under such conditions both signals are localized near the main beam leading to an exponential amplification ($B \sim \exp G_{\beta\beta}$) in which gain grows in proportion to z($G_{\beta\beta} = \Gamma_{\beta\beta}z$)[2,3,7,8].

In fact we now assume the existence of such an amplification regime that results in the formation of stationary amplitude profiles of outgoing waves

$$B_n=B_{n,M}(x)\exp(\Gamma_{\beta\beta}z)\quad(n=1,2). \quad (28)$$

We consider the shape of AM signals $B_{n,M})x)$ for two types of AM pump beam:

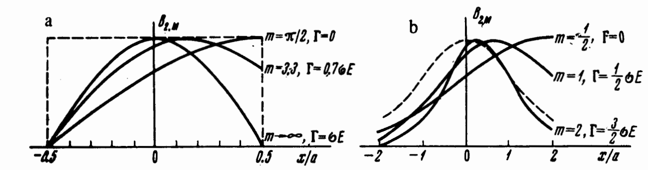

FIG. 2. Stationary modes of parametric signals leaving the pumping beam in different directions. a—uniformly bound pumping field (dashed line); b—bell-shaped pump amplitude distribution of the type $E_{30} = E_3\cosh^{-1}(x/a)$. Diagram illustrates the case of symmetric propagation of waves with $\beta_1 = \beta_2 = \beta$. Parameter of curves is quantity $m = \sigma E_3 a/\beta$.

a. Rectangular profile $E_{20} = E_3$ for $|x| \le a/2$ and $E_{30} = 0$ for $|x| > a/2$. The signal modes of a parametric amplifier with such a pumping have the following form: in the pumping beam region $|x| < a/2$,

$$B_{n,M} = \frac{B_M}{\sqrt{\beta_n}} \exp\left\{\frac{\Gamma_{\beta\beta}(\beta_2-\beta_1)}{2\beta_1\beta_2}x\right\}\sin\left(\frac{x}{a} \pm \frac{1}{2}\right) b. \tag{29}$$

Here parameter b determining the spatial modulation of signal beams is given by the transcendental equation

$$b^{-1}\sin b = \pm 1/m, \tag{30}$$

where $m = (m_1 m_2)^{1/2}$ and $m_n = \sigma E_3 a/\beta_n$[2]. The gain increment depends on the same parameter b:

$$\Gamma_{\beta\beta} = -\sigma E_3 \cos b. \tag{31}$$

Outside the pumping beam region signal amplitudes decrease: for $x < -a/2$

$$B_{1,M} = B_{1,M}(x = -{}^1\!/_2 a)\exp\left(\Gamma_{\beta\beta}(x + a/2)/\beta_1\right), \tag{32}$$

and for $x > a/2$

$$B_{1,M} = 0,\quad B_{2,M} = B_{2,M}(x = a/2)\exp(-\Gamma_{\beta\beta}(x - a/2)/\beta_2). \tag{33}$$

The modes of the first signal $B_{1,M}(x)$ are shown in Fig. 2a. It is clear that the degree of localization of the signal increases with increasing parameter m.

If the gain is very large over the aperture length $a/(\beta_1\beta_2)^{1/2}$, i.e., $m \gg 1$, we have $b = \pi k$, $k = \pm 1, \pm 2$, and $\Gamma_{\beta\beta} = \sigma E_3$ for all modes. In this case (30) corresponds to an expansion of signal amplitudes into Fourier series

$$\sim \sum_{k=1}^{\infty} \cos\left(\frac{x}{a} k\pi\right).$$

Mode amplification decreases with increasing angles βn (parameter m decreases) as a result of intensive escape of signal energy from the pumping region. In this case the highest gain falls on the principal mode ($0 < b < \pi$); mode gain vanishes altogether for $m = 1$ (without taking wave attenuation into acconnt).

b. Pumping profile $E_{30} = E_3\cosh^{-1}(x/a)$. The principal mode of the first signal has the form

$$B_{1,M} = B_M \exp\left[-\frac{x}{2a} - \frac{\Gamma_{\beta\beta}(\beta_2-\beta_1)x}{2\beta_1\beta_2 a}\right]\mathrm{ch}^{-m}\frac{x}{a}. \tag{34}$$

Here $\Gamma_{\beta\beta} = (2m-1)\sigma E_3/(m_1+m_2)$ and gain vanishes for $m = 1/2$. As m increases the degree of signal localization also increases (see Fig. 2b) leading to a rise in gain. In the limit $m_1 = m_2 \to \infty$ we arrive at the case of accompanying waves and $\Gamma_{\beta\beta} = \sigma E_3$ although signal beams narrow down without limit due to inhomogeneity of amplification (compare with A above). Thus inhomogeneous distribution of pumping amplitude strongly affects the mode structure of parametric signals.

[2]It is of interest to note that such a parameter appears in the theory of giant parametric pulse formation [15].

D. Dynamics of Amplification of Outgoing Waves

If amplification is high enough both outgoing waves can be localized in the pumping beam region, as we noted above. Generally speaking, however, mode existence does not imply practical realizability of modes. Therefore it is necessary to consider the formation of stationary beams for any boundary conditions at the amplifier input.

In the case of a bell-shaped pumping AM of the form $E_{30} = E_3\cosh^{-1}(x/a)$ the cascade Laplace method can be used (see[17] for example) to obtain exact solution of (17) and (18) for any values of parameter m[6]:

$$B_1 = B_{10}(\xi) + \frac{m^2\beta_2\,\mathrm{ch}^{-1}(x/a)}{a(\beta_1+\beta_2)}\int_\eta^\xi B_{10}(x_1).$$
$$\mathrm{sh}\frac{\beta_1(x_1-\eta)}{a(\beta_1+\beta_2)}\mathrm{ch}^{-1}\frac{x_1}{a}F(-m+1,-m+1;2;u)(1-u)^{-m+1}dx_1$$
$$+\frac{\sigma E_3}{(\beta_1+\beta_2)}\mathrm{ch}^{-1}\frac{x}{a}\int_\eta^\xi B_{20}(x_1)\,\mathrm{ch}\frac{\beta_2 x_1+\beta_1\eta}{a(\beta_1+\beta_2)}\mathrm{ch}^{-1}\frac{x_1}{a}$$
$$\times F(-m,-m+1;1;u)(1-u)^{-m+1}dx. \tag{35}$$

Here $\xi = x + \beta_1 z$; $\eta = x - \beta_2 z$; and F is a hypergeometric Gauss function whose argument is

$$u = \mathrm{sh}\frac{\beta_1(x_1-\eta)}{a(\beta_1+\beta_2)}\mathrm{sh}\frac{\beta_2(\xi-x_1)}{a(\beta_1+\beta_2)}\mathrm{ch}^{-1}\frac{\beta_2 x_1+\beta_1\eta}{a(\beta_1+\beta_2)}\mathrm{ch}^{-1}\frac{\beta_2\xi+\beta_1 x_1}{a(\beta_1+\beta_2)}. \tag{36}$$

An analogous expression is obtained from (35) for the amplitude B_2 by the substitutions $\beta_1 \longleftrightarrow -\beta_2$ and $B_{10}(x) \longleftrightarrow B_{20}(x)$.

We note first of all that as one of the angles tends to zero, $\beta_1 \to 0$ for example, the solution (35) is transformed into (21) for $E_{30}(x) = E_3\cosh^{-1}(x/a)$. In fact for $\beta_1 \to 0$ parameter $m \to \infty$ and for the hypergeometric function one can use a transformation into a Bessel function (see[18] for example):

$$I_\nu(z) = \lim_{\alpha,\gamma\to\infty}\left[\frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu F\left(\alpha,\gamma;\nu+1;-\frac{z^2}{4\alpha\gamma}\right)\right]. \tag{37}$$

We illustrate the formation of modes of parametric signals by the important example where the amplification length $\sim(\sigma E_3)^{-1}$ is comparable to the aperture length $\sim a/(\beta_1\beta_2)^{1/2}$, i.e., parameter $m = 1$ and $F \equiv 1$. In the case of a symmetric escape of signals $\beta_1 = \beta_2 = \beta$ and homogeneous distribution of their amplitudes at the amplifier input, B_{10} = const and B_{20} = const, we find the following expression for the amplitude of the first signal after integrating (35):

$$B_1 = \frac{\mathrm{ch}^{-1}(x/a)}{2\sqrt{2}}\left[B_{20}\,\mathrm{ch}\frac{\eta}{a} - B_{10}\,\mathrm{sh}\frac{\eta}{a}\right]\left[\pi - \mathrm{arctg}\frac{\mathrm{sh}^{-1}(\xi/2a)}{\sqrt{2}}\right.$$
$$\left.+\,\mathrm{arctg}\frac{\mathrm{sh}^{-1}(\eta/2\alpha)}{\sqrt{2}}\right] - \frac{1}{2}\left(B_{10}\,\mathrm{ch}\frac{\eta}{2a} + B_{20}\,\mathrm{sh}\frac{\eta}{2a}\right)\ln\left\{\left(\sqrt{2}\,\mathrm{ch}\frac{\xi}{2a}+1\right)\right.$$
$$\left.\left(\sqrt{2}\,\mathrm{ch}\frac{\eta}{2a}-1\right)\left(\sqrt{2}\,\mathrm{ch}\frac{\xi}{2a}+1\right)^{-1}\left(\sqrt{2}\,\mathrm{ch}\frac{\eta}{2a}-1\right)^{-1}\right\}. \tag{38}$$

The change in the width of the beam and the motion of its peak in the course of amplification are shown in Fig. 3.

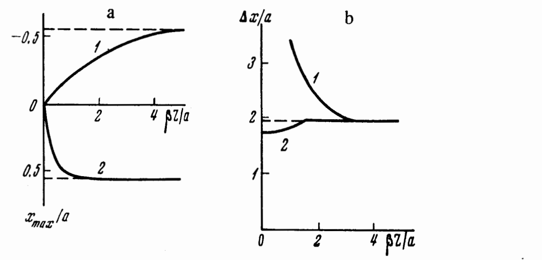

FIG. 3. Mode formation of parametric signal $B_{1,M}$ (see Fig. 2) for waves with the amplitudes $B_{20} = 0$ and $B_{10}(x) = \text{const}$ at amplifier input. Diagram shows position of peak x_m (a) and width (b) of signal as functions of distance traveled in the amplifier. 1—first signal, 2—second signal.

A stationary distribution of the signal is formed at a large distance z from the boundary (comp. (34)):

$$B_1 = (B_{10}+B_{20})\frac{\pi}{2\sqrt{2}}\,\text{ch}^{-1}\frac{x}{a}\exp\left\{\frac{\sigma E_3}{2}z - \frac{x}{2a}\right\}. \qquad (39)$$

The mode peak is located at the point $x_{1,max} = -0.55\,a$, and its width $\Delta x_M \approx 1.95\,a$.

Solution (35) allows us also to determine the amplification of AM signals of the type

$$B_{10} = B_0\,\text{ch}^{-1}\frac{x - x_{10}}{a}, \qquad B_{20} = B_0\,\text{ch}^{-1}\frac{x + x_{20}}{a}.$$

The expression for $B_1(x, z)$ is still more cumbersome than (38) and is not given here. We merely note one interesting detail: there is an optimal distribution of initial beams, i.e., $x_{10} = x_{20} = 0.7\,a$, for which maximum signal amplitudes are reached at the amplifier output. The growth of signals is lower for other initial beam distributions.

An amplifier with parameters $m_1 = m_2 = 2$ forms a stationary signal

$$B_1 \doteq (B_{10}+B_{20})\frac{3\pi}{8\sqrt{2}}\,\text{ch}^{-2}\frac{x}{a}\exp\left\{\frac{3}{4}\sigma E_3 z - \frac{x}{2a}\right\}. \qquad (40)$$

The maximum of its amplitude is located closer to the pump axis ($x_m = -0.25\,a$) than in the preceding case ($m = 1$). The width of the signal beam $\Delta x = a$ is smaller. It follows from (35) that in the general case of an arbitrary magnitude of $m > 1/2$ the amplifier forms a stationary mode of the (32) type if the distance z is large enough (here the argument of the hypergeometric function (36) tends to unity). The amplitude $B_{1,m}$ of the mode is related to the amplitudes of initial signals by (we cite the simplest formula for $\beta_1 = \beta_2$)

$$B_{1,M} = (B_{10}+B_{20})\,\Gamma(m-1/2)\,\Gamma(m+1/2)\,/\,\sqrt{2}\,\Gamma^2(m), \qquad (41)$$

where $\Gamma(m)$ is the Gamma function.

In conclusion of this section we compare the amplification of outgoing and accompanying waves. For accompanying waves $G_{00} = \sigma E_3 z$ and for outgoing waves we have $G_{\beta\beta} = (1 - 1/2m)\sigma E_3 z$. Gains differ by $1/2$: $G_{\beta\beta} = G_{00} - 1/2$ (this corresponds to a reduction of intensity e times) for waves with parameter $m = \sigma E_3 z$, or, in other words, for waves with an exit angle of $\beta = z/a$. The above means that for a given width of the pumping beam and length of amplifier $z = l$ the highest gain occurs in modes propagating within the limits of the geometric angle $\alpha = a/l$. For waves with $\beta > \alpha$ the intensity of the amplified signal falls more than e times in comparison with the intensity of the accompanying wave (assuming of course the same levels of input signals).

E. Saturation in Amplification of Outgoing Waves[3)]

The amplification of waves escaping from the pumping beam to one side proceeds quite differently than in the preceding case. The most significant difference is the fact that, as we know, gain saturation takes place beyond the aperture length ($z > a/\beta$). To analyze the operation of such amplifier we can use the previous solution (35) replacing the exit angle of the first wave β_1 by $-\beta_1$ for example (both signal waves propagate towards positive values of x). The m parameter of the amplifier becomes imaginary in this process $m \to in = i\sigma E_3 a(\beta_1\beta_2)^{-1/2}$.

In the special case where signal waves leave at the same angle ($\beta_1 = \beta_2 = \beta$, $\xi = \eta = x - \beta z$, $n = \sigma E_3 a/\beta$) the gain

$$G_{\beta\beta} = \frac{\sigma}{\beta}\int_x^\xi E_{30}(x_1)\,dx_1 = 2n\,\text{arctg}\left[\frac{\text{sh}(\beta z/2a)}{\text{ch}(x/a - \beta z/2a)}\right]. \qquad (42)$$

It follows from (42) that for $z \gg a/\beta$ the gain saturates at the level

$$G_{\beta\beta}^{(max)} = \pi n/2. \qquad (43)$$

The saturation effect is also observed in the general case of unequal exit angles ($\beta_1 \neq \beta_2$). We first consider the amplification of signal beams $B(x) = B_0\exp(-x^2/b^2)$ that are so narrow that the integrands in (35) can be brought outside the integral sign for $x_1 = 0$. The amplitude of the first wave then is

$$B_1(x,z) = B_{10}(\xi) - B_{10}\frac{\sqrt{\pi}\beta_2 b_1 n^2}{a(\beta_2-\beta_1)}\,\text{sh}\frac{\beta_1 h}{a(\beta_2-\beta_1)}\,\text{ch}^{-1}\frac{x}{a}$$
$$\times F\left(1 - in, 1 + in; 2; \frac{1-u_0}{2}\right) + B_{20}\frac{\sqrt{\pi}\sqrt{\beta_1\beta_2}\,b^2 n}{a(\beta_2-\beta_1)}\,\text{ch}\frac{\beta_1\eta}{a(\beta_2-\beta_1)}$$
$$\times\,\text{ch}^{-1}\frac{x}{a}F\left(1 - in, 1 + in; 1; \frac{1-u_0}{2}\right), \qquad (44)$$

where the argument of the hypergeometric functions is

$$u_0 = \text{ch}\frac{\beta_1\eta + \beta_2\xi}{a(\beta_2-\beta_1)}\,\text{ch}^{-1}\frac{x}{a}. \qquad (45)$$

In the first stage of amplification when the signal waves have not yet left the pumping beam the maximum gain occurs along a straight line inclined to the pumping beam at an angle of

$$\beta_0 = (\beta_1 + \beta_2)/2. \qquad (46)$$

Thus the peaks of signal beams propagate at first at an average angle that lies between the directions of ray vectors: $\beta_1 < \beta_0 < \beta_2$. This effect was considered earlier in[10] in a single-wave approximation. To investigate the further behavior of the signal beam we use the asymptote of the hypergeometric function valid for high gain $n \gg 1$ (for the sake of simplicity we assume that $B_{20} = 0$),

$$B_1 \approx B_{10}\frac{\sqrt{2}\beta_2 b_1 n^2}{a(\beta_2-\beta_1)}\,\text{sh}\frac{\beta_1\eta}{a(\beta_2-\beta_1)}$$
$$\times\,\text{ch}^{-1}\frac{x}{a}(1-u_0)^{-1}(1+u_0)^{-1/2}\exp(n\arccos u_0). \qquad (47)$$

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3)The entire Section E is a supplement dated February 22, 1971.

At a long distance ($z \gg a/\beta_1$) the amplitude profile (47) of the beam assumes the form

$$B_1 \sim \exp\left[-\frac{\beta_2\xi}{a(\beta_2-\beta_1)} + n \arccos \exp\left(\frac{-2\beta_2\xi}{a(\beta_2-\beta_1)}\right)\right]. \quad (48)$$

It follows from (48) that gain saturation occurs after exit from the pumping beam and the wave propagates along its ray vector. The beam peak is located on a straight line

$$\xi = \frac{a(\beta_2-\beta_1)}{2\beta_2}\ln(2n). \quad (49)$$

The gain $G_{\beta\beta} = n \operatorname{arc\,cos}(1/2n) \approx \pi n/2$ (compare with saturation (43)). The width of the formed beam is $\Delta x_1 \approx a(\beta_2 - \beta_1)/\beta_2$. When broad signal beams (B_{10}, $B_{20} \approx$ const) enter the amplifier gain saturation occurs at the previous level $G_{\beta\beta} = \pi n/2$. Beam widths grow with increasing traveled distance $\Delta x_{1,2} \approx \beta_{1,2} z/2$.

5. CONCLUSION

In the present work we considered several typical problems from the theory of parametric amplification. The methodology developed here can be fairly useful in the analysis of stimulated Raman scattering in the field of picosecond pulses. In particular the results of Section 4D allow us to analyze SRS in the presence of nonstationary phenomena of both types: wave type due to group delay of Stokes and pumping pulses and local type due to the relaxation of molecular vibrations. We note that when the group velocity mismatch is not accounted for ($\nu = 0$) molecular relaxation results in a square-root dependence of gain on distance: $G \sim \sqrt{x}$ (see[19]). At the same time the group delay effect can cause a formation of stationary modes of the Stokes pulse and oscillations of the medium so that $G \sim z$. Such a situation occurs in SRS forward ($u_s > u_p$); thus we can expect here a significant improvement of gain in comparison to the case of $\nu = 0$.

The solution of the type (35) is also valid for the case of amplification of waves leaving the pumping beam in one direction. Here the m parameter in the hypergeometric function becomes imaginary ($\beta_1\beta_2 < 0$). In a single-wave approximation this amplifier variant was considered in[10].

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