ANOMALOUS SKIN EFFECT IN METALS IN AN INCLINED MAGNETIC FIELD

I. E. ARONOV, É. A. KANER, and R. I. SHEKHTER

Institute of Radiophysics and Electronics, Ukrainian Academy of Sciences

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An asymptotically exact solution of the problem of the anomalous skin effect in metals in slightly inclined magnetic fields is obtained for diffuse reflection of the electrons from the boundary.

1. INTRODUCTION

THE problem of the penetration of an electromagnetic field into a metal in an inclined magnetic field turns out to be much more difficult than in the case of normal or parallel orientation of the magnetic field relative to the surface of the sample. These difficulties are due to violation of the symmetry of the problem when the magnetic field is inclined, leading to considerable complications when account is taken of the reflection of the electrons from the separation boundary. Therefore in the analysis of the penetration of electromagnetic fields into a metal in inclined geometry the scattering of electrons by the boundary has so far been neglected.[1,2] Although the results obtained thereby describe qualitatively correctly the real picture of the penetration of the field, there is undisputed interest attached to a rigorous solution of the boundary-value problem. In the present paper we obtain an asymptotically exact solution for the distribution of the high-frequency field and of the surface impedance for diffuse reflection of the electrons under conditions of the anomalous skin effect.

2. FORMULATION OF PROBLEM

We consider a metal with a spherical Fermi surface. Let the magnetic field **H** be oriented at an angle Φ to the separation boundary. We choose the coordinate system $\xi\eta\zeta$ such that the η axis is directed along the inward normal to the surface and the ζ axis along the projection of the vector **H** on the surface $\eta = 0$; consequently, the magnetic field **H** lies in the plane $\eta\zeta$.

The complete system of equations consists of Maxwell's equations for the electromagnetic field and the kinetic equation for the electron distribution function. We write out Maxwell's equations for the spatial Fourier components of the electric field $\mathcal{E}_\mu(k)$ in the form

$$k^2\mathcal{E}_\mu(k) + 2E'(0) = 4\pi i\omega c^{-2} j_\mu(k) \qquad (\mu = \xi, \zeta); \qquad (2.1)$$

$$j_\eta(k) = 0. \qquad (2.2)$$

Here we have neglected the displacement current and introduced the notation

$$\mathcal{E}_\mu(k) = 2\int_0^\infty d\eta E_\mu(\eta)\cos k\eta, \qquad E_\mu(\eta) = \frac{1}{\pi}\int_0^\infty dk\mathcal{E}_\mu(k)\cos k\eta; \quad (2.3)$$

$\mathcal{E}(\eta)$ is the electric field intensity inside the metal, which we continue in an even fashion to the region outside of the metal $\eta < 0$, $\mathbf{j}(k)$ is the Fourier component of the current density, ω is the frequency, and k is the wave number of the electromagnetic field. Equation (2.2) expresses the condition of electric quasineutrality of the metal and serves for the determination of the component $\mathcal{E}_\eta(k)$.

The current density is expressed in terms of the electron distribution function which should be obtained from the kinetic equation in the corresponding boundary conditions. We shall assume that the scattering of the electrons from the boundary is diffuse. The case of specular reflection, which is apparently more realistic for electrons that skip along the surface,[3] is much more difficult. The distribution functions of the electrons in an inclined magnetic field was found for diffuse scattering by Chambers.[4] Using this distribution function, we write down immediately a formula for the Fourier component of the current density:

$$j_\mu(k) = K_{\mu\nu}(k)\mathcal{E}_\nu(k) - \frac{1}{\pi}\int_0^\infty dk' Q_{\mu\nu}(k,k')\mathcal{E}_\nu(k'), \qquad (2.4)$$

where

$$K_{\mu\nu}(k) = \sigma_H \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\tau n_\mu(\tau) e^{-\gamma\tau} \int_{-\infty}^{\tau} d\tau' n_\nu(\tau') e^{\gamma\tau'} \cos[kR\alpha(\tau',\tau)]; \quad (2.5)$$

$$Q_{\mu\nu}(k\,k') = \sigma_H R \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\tau n_\mu(\tau) \int_{-\infty}^{\tau} d\tau' |n_\eta(\tau')| \cos[kR\alpha(\tau',\tau)]$$
$$\times \int_{\lambda(\tau')}^{\tau'} d\tau'' e^{\gamma(\tau''-\tau)} n_\nu(\tau'') \cos[k'R\alpha(\tau'',\tau')]. \qquad (2.6)$$

Here $\mathbf{n} = \mathbf{v}/v$ is the unit vector of the electron velocity on the Fermi surface, θ is the polar angle and τ is the azimuthal angle in the velocity space with the polar axis parallel to the magnetic field **H**,

$$n_\eta(\theta,\tau) = \cos\theta\sin\Phi + \sin\theta\cos\Phi\sin\tau,$$
$$n_\xi(\theta,\tau) = \sin\theta\cos\tau, \quad n_\zeta(\theta,\tau) = \cos\theta\cos\Phi - \sin\theta\sin\Phi\sin\tau; \quad (2.7)$$

$$\sigma_H = \frac{3}{4\pi}\frac{Ne^2}{m\Omega}, \qquad \gamma = \frac{\nu - i\omega}{\Omega}, \qquad R = \frac{v}{\Omega}, \qquad (2.8)$$

where N is the concentration, e the absolute value of the electron charge, m the effective mass, $\Omega = eH/mc$ the cyclotron frequency, R the maximum radius of the orbit of the conduction electron, and ν the frequency of the collisions between the electrons and the scatterers. The function $\alpha(\tau', \tau)$ is defined by the formula

$$\alpha(\tau',\tau)=\int_{\tau}^{\tau'} d\tau'' n_\eta(\tau'')=q(\tau')-q(\tau), \qquad (2.9)$$
$$q(\tau)=\tau\cos\theta\sin\Phi-\sin\theta\cos\Phi\cos\tau.$$

The quantity $\lambda(\tau)$ is the root of the equation

$$\alpha(\lambda,\tau)\equiv q(\lambda)-q(\tau)=0. \qquad (2.10)$$

If Eq. (2.10) has many solutions then it is necessary to choose that solution which is smaller than τ and is closest to τ. If such a root does not exist, then $\lambda(\tau)$ should be set equal to $-\infty$.

The elements of the tensor $K_{\mu\nu}(k)$ constitute the Fourier components of the conductivity operators of an unbounded metal with allowance for spatial and temporal dispersion, and also for the dependence on the constant magnetic field. The kernel of the nonlocal operator $Q_{\mu\nu}(k, k')$ is due to the presence of the interface and takes into account the contribution made to the current by the electrons colliding with the surface of the metal and those not colliding with it.

It is impossible to solve Eqs. (2.1), (2.4), (2.5), and (2.6) in general form. We shall therefore investigate henceforth the region of the anomalous skin effect when

$$kR\gg 1. \qquad (2.11)$$

We assume the magnetic field sufficiently strong so that

$$|\gamma|=|\nu-i\omega|/\Omega\ll 1. \qquad (2.12)$$

In addition, we confine ourselves to the region of relatively small angles Φ of the inclination of the vector **H** relative to the surface, namely

$$\Phi\ll 1/kR. \qquad (2.13)$$

These inequalities enable us to simplify the expressions for $K_{\mu\nu}$ and $Q_{\mu\nu}$, if we replace the exact formulas (2.5) and (2.6) by their asymptotic forms. Owing to the condition (2.11), the main contribution of the current is made by those sections of the electron trajectories in which the electron moves almost parallel to the surface of the metal, and

$$v_\eta(\tau,\theta)=0. \qquad (2.14)$$

However, for electrons from the vicinity of the limiting points, i.e., for angles θ in the intervals $(0, \Phi)$ and $(\pi-\Phi, \pi)$, Eq. (2.14) has no solutions. Therefore we confine ourselves henceforth to integration over the interval $\Phi\le\theta\le\pi-\Phi$.

3. ASYMPTOTIC CURRENT DENSITY

We proceed to the calculation of the asymptotic current density. The Fourier component of the volume conductivity $K_{\mu\nu}(k)$ is conveniently represented in the form

$$K_{\mu\nu}(k)=\frac{\sigma_H}{4\pi}\int_{\Phi}^{\pi-\Phi} d\theta\sin\theta\int_0^{2\pi} d\tau n_\mu(\tau)e^{-\gamma\tau}\left[\frac{e^{-ikRq(\tau)}}{\gamma+ikR\Phi\cos\theta}+\frac{e^{ikRq(\tau)}}{\gamma-ikR\Phi\cos\theta}\right]\int_{\tau-2\pi}^{\tau} d\tau' n_\nu(\tau')\exp[\gamma\tau'+ikRq(\tau')]. \qquad (3.1)$$

This formula is obtained from (2.5) if the cosine is represented in the form of a half-sum of two exponentials, the integral with respect to τ' is transformed with the aid of the identity

$$\int_{-\infty}^{\tau} d\tau' G(\tau')e^{\alpha\tau'}=[1-e^{-2\pi\alpha}]^{-1}\int_{\tau-2\pi}^{\tau} d\tau' G(\tau')e^{\alpha\tau'},$$
$$G(\tau)=G(\tau+2\pi), \qquad (3.2)$$

and the inequalities (2.12) and (2.13) are used. The terms with $[\gamma\pm ikR\Phi\cos\theta]^{-1}$ have a singularity at $\gamma, \Phi\to 0$. All the remaining quantities in (3.1) are smooth functions of γ and Φ, and owing to (2.12) and (2.13) we can put in them $\gamma=\Phi=0$. The integrals with respect to τ and τ' can then be calculated exactly. It is convenient in what follows to represent them with the aid of the following symbols:

$$K_{\mu\nu}(k)=\frac{\sigma_H}{4\pi}\int_0^{\pi} d\theta\sin\theta\left[\frac{F_\mu(k,k)}{\Gamma(k)}+\frac{F_\mu(k,-k)}{\Gamma(-k)}\right]F_\nu(-k,-k), \qquad (3.3)$$

where

$$\Gamma(k)=\gamma+ikR\Phi\cos\theta,$$

$$F_j(x,y)=\exp(ixR\sin\theta)\int_0^{2\pi} d\tau m_j(\tau,\theta)\exp(-iyR\sin\theta\cos\tau); \qquad (3.4)$$

$m_j(\tau, \theta)$ are the components of the unit vector of the electron velocity (2.7) at $\Phi = 0$;

$$F_\xi(x,y)=-2\pi i\sin\theta\exp(ixR\sin\theta)J_1(yR\sin\theta),$$
$$F_\eta(x,y)=0,\quad F_\zeta(x,y)=2\pi\cos\theta\exp(ixR\sin\theta)J_0(yR\sin\theta). \qquad (3.5)$$

To simplify the kernel $Q_{\mu\nu}$, we use the identity (3.2) and the inequalities (2.12) and (2.13). As a result we obtain

$$Q_{\mu\nu}(k,k')={}^1/_4[P_{\mu\nu}(k,k')+P_{\mu\nu}(k,-k')+P_{\mu\nu}(-k,k')+P_{\mu\nu}(-k,-k')], \qquad (3.6)$$

where

$$P_{\mu\nu}(k,k')=\frac{\sigma_H R}{2\pi}\int_{\Phi}^{\pi-\Phi} d\theta\frac{\sin\theta}{\Gamma(k)}\int_0^{2\pi} d\tau n_\mu(\tau)\exp[-\gamma\tau-ikRq(\tau)]\cdot$$
$$\times\int_{\tau-2\pi}^{\tau} d\tau'|n_\eta(\tau')|\exp[i(k-k')Rq(\tau')]\int_{\lambda(\tau')}^{\tau'} d\tau'' n_\nu(\tau'')\exp[\gamma\tau''+ik'Rq(\tau'')].$$

Just as in the case of the kernel $K_{\mu\nu}$, the function $\Gamma^{-1}(k)$ has a singularity at $\gamma, \Phi\to 0$. However, unlike (3.1), the remaining quantities in (3.7) are no longer smooth functions at small values of γ and Φ. In other words, in (3.7) there are still factors of the type $\Gamma^{-1}(k')$, which arise as a result of the fact that the function $\lambda(\tau')$ can assume a value $-\infty$. Therefore in formula (3.7), in the integrals with respect to τ and τ', it is necessary to separate those integrals where the function $\lambda(\tau')$ has no jumps. This separates in explicit form all the singularities of the type $\Gamma^{-1}(k')$, after which the remaining smooth functions of γ and Φ can be replaced by their values at $\gamma=\Phi=0$. As a result of these essentially simple but very laborious calculations, we can obtain the following approximate formula:

$$P_{\mu\nu}(k,k')=P_{\mu\nu}^{(1)}(k,k')+P_{\mu\nu}^{(2)}(k,k'), \qquad (3.8)$$

$$P_{\mu\nu}^{(1)}(k,k')=\frac{\sigma_H}{2\pi i(k-k')}\int_0^{\pi/2} d\theta\frac{\sin\theta}{\Gamma(k)}\left\{F_\mu(-k',-k)\left[\frac{\Gamma(k)}{\Gamma(k')}F_\nu(k,k')\right.\right.$$

$$-F_\nu(k', k)\Big] - F_\mu(k', -k)\left[F_\nu(-k, -k') - F_\nu(-k', k)\right]\Big\},$$

$$P^{(2)}_{\mu\nu}(k, k') = \frac{\sigma_H}{2\pi i(k-k')}\int_{\pi/2}^{\pi} d\theta \frac{\sin\theta}{\Gamma(k)}\Big\{F_\mu(-k', -k)\left[F_\nu(k, k')\right. \quad (3.9)$$

$$\left. - F_\nu(k', k)\right] - F_\mu(k', -k)\left[\frac{\Gamma(k)}{\Gamma(k')}F_\nu(-k, k') - F_\nu(-k', k)\right]\Big\}.$$

Formulas (3.3), (3.8), and (3.9) can be further simplified by using the asymptotic forms of the Bessel functions that are contained in $F_\mu(k, k')$ for large values of the argument. The transition to this asymptotic form is connected with the inequality (2.11), and, as indicated above, is determined by the stationary-phase points $q(\tau)$, which satisfy Eq. (2.14). Owing to (2.14), the coupling of the tangential components of the electric field $\mathscr{E}_\xi$ and $\mathscr{E}_\zeta$ with the longitudinal component $\mathscr{E}_\eta$ turns out to be negligibly small. In other words, in the expression for the tangential components of the current density one can disregard the terms with $\mathscr{E}_\eta$, and Eq. (2.2) need not be considered at all in the calculation of $\mathscr{E}_\zeta$, $\mathscr{E}_\xi$, and the surface impedance.

We shall investigate below the case of linear polarization of the current along the ξ axis and formulate a criterion of conductivity along this direction much larger than conductivity in the ζ direction. Omitting the vector indices, we write down the asymptotic expression for $j_\xi(k) = j(k)$ in the form

$$j(k) = K(k)\mathscr{E}(k) - \frac{1}{\pi}\int_0^\infty dk' B(k, k')\mathscr{E}(k'), \qquad (3.10)$$

where

$$K(k) = \frac{\sigma_H}{kR}\int_{-1}^{1} dx \frac{(1-x^2)^{1/2}}{\Gamma(k)}[1 - \sin kD]; \qquad (3.11)$$

$$B(k, k') = \frac{\sigma_H}{2(kk')^{1/2}R}\Big\{\frac{1}{k-k'}\int_{-1}^{1} dx (1-x^2)^{1/2}\cos kD[1 - \cos(k-k')D]$$

$$\times(\Gamma^{-1}(k) + \Gamma^{-1}(k')) - \frac{i}{k-k'}\int_{-1}^{1} dx \,\mathrm{sign}\, x(1-x^2)^{1/2}[1 - \sin kD$$

$$-\sin k'D + \cos(k-k')D](\Gamma^{-1}(k') - \Gamma^{-1}(k)) + \frac{1}{k+k'}\int_{-1}^{1} dx(1-x^2)^{1/2}\cdot$$

$$\times[1 - \sin kD - \sin k'D + \cos(k+k')D](\Gamma^{-1}(k') + \Gamma^{-1}(k))$$

$$-\frac{i}{k+k'}\int_{-1}^{1} dx\,\mathrm{sign}\, x(1-x^2)^{1/2}[\cos kD + \cos k'D - \sin(k+k')D]$$

$$\times(\Gamma^{-1}(k) + \Gamma^{-1}(k'))\Big\}; \qquad (3.12)$$

$$D = 2R(1-x^2)^{1/2}, \quad x = \cos\theta. \qquad (3.13)$$

Formulas (3.10)–(3.12) represent an asymptotically correct expression for the ξ component of the current density in the region of the anomalous skin effect (2.11), strong magnetic fields (2.12), and small inclination angles (2.13). However, even these asymptotic expressions are still sufficiently complicated. In order to simplify formulas (3.11) and (3.12), we stipulate the satisfaction of one more inequality

$$w = \frac{|\gamma|}{kR\Phi}\sqrt{kR} \gg 1, \qquad \omega \lesssim \nu. \qquad (3.14)$$

In this case we can neglect the oscillating terms in the integrands of (3.11) and (3.12), since their contribution to the conductivity is smaller by a factor w than the contribution of the non-oscillating terms. This result is due to the fact that when the condition (3.14) is satisfied the interval of the characteristic variation of the oscillating functions ($|x| \sim (kR)^{-1/2}$) near the point of the stationary phase x = 0 turns out to be much smaller than the width of the resonant denominator ($|x| \sim |\gamma|/kR\Phi$).

The physical meaning of the inequality (3.14) can be understood by starting from the following simple considerations, (see [1,2]). The drift motion of the electrons along the magnetic field, and consequently into the depth of the metal, causes the electrons near the central cross section of the Fermi surface to turn out to be in a special position relative to those electrons whose drift velocity along the vector **H** is of the order of the Fermi velocity. Since the electrons that drift slowly in the interior of the metal return many times to the skin layer, their conductivity is much larger than the conductivity of the remaining electrons. This fact is described by the resonant denominators $\Gamma^{-1}(k)$ or $\Gamma^{-1}(k')$, and the relative number of the slow drifting electrons in the quasistatic region of frequencies $\omega \lesssim \nu$ is of the order of $|\gamma|/kR\Phi$. On the other hand, the interaction of such resonant electrons with the electromagnetic field depends on the phase relations, namely on the number of wavelengths subtended by the diameter of the electron trajectory. The inequality (3.14) represents the condition that the scatter of the diameters of the resonant electrons $\delta D \sim R(|\gamma|/kR\Phi)^2$ is much larger than the wavelength. Owing to the interference of the contributions of different resonant electrons, averaging takes place over the phase of the interaction. This averaging corresponds to neglect of the oscillating terms in the current density. As a result we obtain

$$K(k) = \frac{\sigma_H}{kR}\int_{-1}^{1} dx \frac{(1-x^2)^{1/2}}{\Gamma(k)}, \qquad (3.15)$$

$$B(k, k') = \frac{\sigma_H}{2(kk')^{1/2}R}\Big\{\frac{i}{k-k'}\int_{-1}^{1} dx\,\mathrm{sign}\, x(1-x^2)^{1/2}(\Gamma^{-1}(k) - \Gamma^{-1}(k'))$$

$$+\frac{1}{k+k'}\int_{-1}^{1} dx(1-x^2)^{1/2}(\Gamma^{-1}(k) + \Gamma^{-1}(k'))\Big\}. \qquad (3.16)$$

Formulas (3.15) and (3.16) admit of one more limiting transition to the case of parallel orientation of the magnetic field ($\Phi = 0$). In this case one obtains the well-known expression (see [5]) for the Fourier component of the current density. Recognizing that in the case considered by us the relative number of resonant electrons is small and

$$|\gamma|/kR \ll \Phi, \qquad (3.17)$$

it is easy to calculate the integrals with respect to x and to obtain the following asymptotic formula for the Fourier component of the current density:

$$j(k) = \frac{\pi\sigma_H}{(kR)^2\Phi}\Big\{\mathscr{E}(k) + \frac{1}{\pi}\int_0^\infty dk'\left(\frac{k}{k'}\right)^{3/2}\left[\frac{\ln kL}{k} + \frac{\ln(k'/k)}{k-k'}\right]\mathscr{E}(k')\Big\}, \qquad (3.18)$$

where

$$L = 2\exp\left(-1-\frac{\pi}{2}\right)\frac{R\Phi}{\gamma} \quad (3.19)$$

If we solve (2.1) jointly with (3.18), then we can calculate the distribution of the electric field in the metal and the surface impedance

$$Z = \frac{4i\omega}{c^2 E'(0)}\int_0^\infty dk\,\mathscr{E}(k). \quad (3.20)$$

It should be noted that the inequality (3.17) ensures the smallness of the conductivity along the ζ axis compared with the conductivity in the direction of the ξ axis, and the condition (3.14) leads to a small coupling between the tangential components of the electric field.

4. DISTRIBUTION OF THE FIELD AND SURFACE IMPEDANCE

We write down the integral equation (2.1), in which the current is determined by formula (3.18), in terms of dimensionless variables[1)]

$$\left(\xi^2 - \frac{\alpha\pi}{\xi^2}\right)F(\xi) + \frac{\alpha}{\pi}\int_0^\infty \frac{d\xi'}{(\xi\xi')^{3/2}}\left[\xi\frac{\ln(\xi/\xi')}{\xi-\xi'} - \ln\xi\right]F(\xi') = 1. \quad (4.1)$$

Here $\xi = kL$ is the dimensionless wave number,

$$\mathscr{E}(\xi) = -2L^2E'(0)F(\xi), \qquad \alpha = i\frac{4\pi\sigma_H\omega L^4}{c^2R^2\Phi}. \quad (4.2)$$

The surface impedance (3.20) is expressed in terms of $F(\xi)$ by means of the following formula:

$$Z = -\frac{8i\omega L}{c^2}\int_0^\infty F(\xi)\,d\xi. \quad (4.3)$$

The integral equation (4.1) can be solved exactly with the aid of a bilateral Laplace transformation. The method of solution is close to that proposed by Hartmann and Luttinger[6] for an integral equation with a homogeneous singular kernel. Unlike the equations considered in [6], the kernel of the integral operator (4.1) is not homogeneous: it contains a term with $\ln\xi$, the presence of which changes the form of the integral operator under similarity transformations of the independent variable. This leads to the need for modifying the method of Hartmann and Luttinger.

We make the change of variables

$$\xi = e^t, \quad \xi' = e^\tau, \quad F(e^t) = g(t). \quad (4.4)$$

Then (4.1) takes the form

$$(e^{4t} - \alpha\pi)g(t) + \frac{\alpha}{\pi}\int_{-\infty}^{\infty} d\tau\Lambda(t-\tau)g(\tau) - \frac{\alpha}{\pi}te^{t/2}\int_{-\infty}^{\infty} d\tau e^{-\tau/2}g(\tau) = e^{2t}, \quad (4.5)$$

$$\Lambda(x) = xe^{x/2}[1-e^{-x}]^{-1}. \quad (4.6)$$

The kernel of the integral equation (4.5) does not make it possible to use directly the method described in [6]. The difficulty lies in the fact that after the change of variables (4.4) the kernel of the integral operator (4.5) is not of the difference type, but has the form of a sum of a difference kernel and a degenerate kernel.

We introduce the Laplace transformation M(z):

$$M(z) = \int_{-\infty}^{\infty} g(t)e^{-tz}dt, \quad g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dzM(z)e^{zt}, \quad c = \operatorname{Re} z. \quad (4.7)$$

The real number c is arbitrary and is chosen inside the band within which M(z) is a regular function. In the case when the sought function g(t) is an exponential-growth function, with $g(t) \sim e^{at}$ as $t \to +\infty$ and $g(t) \sim e^{bt}$ as $t \to -\infty$, the Laplace transformation M(z) is regular in the strip $a < \operatorname{Re} z < b$. From (4.5) it follows that as $t \to +\infty$ the function $g(t) \sim e^{-2t}$, and as $t \to -\infty$ the asymptotic form is $g(t) \sim e^{2t}$. Consequently, the strip in which M(z) is regular is $-2 < \operatorname{Re} z < 2$, i.e., $a = -2$ and $b = 2$. It will be convenient for us to seek a solution M(z) that is regular in a displaced and somewhat broader strip, overlapping the strip $-2 < \operatorname{Re} z < 2$. The transition to the region $-2 < \operatorname{Re} z < 2$ is realized with the aid of analytic continuation.

We stipulate that the function M(z) satisfy the following conditions:

1) M(z) is regular in the strip $-4 + \Delta_1 < \operatorname{Re} z < \Delta_2$, $0 < \Delta_1 < \Delta_2 < \frac{1}{2}$, with the exception of certain points;

2) at the point $z = -2$ the function M(z) has a simple pole with unity residue;

3) at the point $z = -\frac{7}{2}$ the function M(z) has a second-order pole and near it $M(z) = (\alpha/\pi)M(\frac{1}{2})(z+\frac{7}{2})^2$;

4) M(z) should decrease when $z \to \pm i\infty$ and, finally,

5) M(z) satisfies the difference equation

$$M(z-4) - \Theta(z)M(z) = 0, \quad (4.8)$$

where

$$\Theta(z) = -\alpha\pi\operatorname{tg}^2\pi z. \quad (4.9)$$

The transform M(z) satisfying all these conditions determines a unique and single-valued solution of Eq. (4.5). Conditions 2 and 3 make it possible to obtain instead of the inhomogeneous equation (4.5) the homogeneous equation (4.8) for the transform M(z). The presence of condition 3 is that additional requirement which is missing from [6] and results from the degenerate part in the kernel of the integral operator of Eq. (4.5). In spite of this additional requirement, it is possible, nonetheless, to find the unique function M(z) satisfying all the conditions 1–5. To prove this statement and to find M(z) in explicit form, we substitute g(t) from (4.7) in (4.5). We obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dzM(z)\left[e^{(z+4)t} - \Theta(z)e^{zt}\right] = e^{2t} + \frac{\alpha}{\pi}te^{t/2}M(1/2); \quad \Delta_1 < c < \Delta_2. \quad (4.10)$$

In the integral containing $e^{(z+4)t}$, we displace the contour of integration by Re z = 4 to the left. Using conditions 1–3, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dzM(z)e^{(z+4)t} = e^{2t} + \frac{\alpha}{\pi}te^{t/2}M(1/2) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dzM(z-4)e^{zt}. \quad (4.11)$$

Substituting (4.11) in (4.10), we arrive at the relation

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dze^{zt}[M(z-4) - \Theta(z)M(z)] = 0,$$

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1) Although the integration here is over all ξ, actually the main contribution to the integral is made by the values $\xi' \sim \xi \approx |\alpha\pi|^{1/4}$. It is precisely these values of ξ that should enter in the inequalities given above.

which is equivalent to Eq. (4.8). The general solution of this equation contains as a factor an arbitrary periodic function V(z) with a period 4. Therefore to obtain the general solution it suffices to find some particular solution. Since $\Theta(z)$ is a periodic function with period 1, such a particular solution may be $[\Theta(z)]^{-z/4}$. Consequently, the general solution (4.8) is of the form

$$M(z) = [\Theta(z)]^{-z/4}V(z). \qquad (4.12)$$

The regularity conditions 1–4 make it possible to find the explicit form of V(z). It is easy to verify that the function

$$M(z) = \frac{\pi}{16}\exp\left[-\frac{z+2}{4}\ln(-\pi\alpha)\right](\mathrm{tg}^2\,\pi z)^{-z/4}\left[\frac{\sin{}^1\!/_4\pi(z+{}^1\!/_2)}{\sin{}^3\!/_8\pi}\right]^{1/4}$$
$$\times\left[\frac{\sin{}^1\!/_4\pi(z+{}^3\!/_2)}{\sin{}^1\!/_8\pi}\right]^{3/4}\left[\frac{\sin{}^1\!/_4\pi(z+{}^5\!/_2)}{\sin{}^1\!/_8\pi}\right]^{5/4}\left[\frac{\sin{}^3\!/_8\pi}{\sin{}^1\!/_4\pi(z+{}^7\!/_2)}\right]^{1/4}$$
$$\times\left[\frac{\sin{}^1\!/_4\pi}{\sin{}^1\!/_4\pi(z+1)}\right]^{1/2}\left[\frac{1}{\sin{}^1\!/_4\pi(z+2)}\right]^{2}\left[\frac{\sin{}^1\!/_4\pi}{\sin{}^1\!/_4\pi(z+3)}\right]^{8/2} \qquad (4.13)$$

satisfies all the necessary conditions 1–5. Formula (4.13), together with (4.7), (4.2), and (4.3), gives an asymptotically exact solution for the distribution of the field in the metal. The surface impedance Z is expressed in terms of the value M(−1) and is given by

$$Z = -8i\omega Lc^{-2}M(-1)$$
$$= \left(\frac{\pi}{2}\right)^{1/2}(1+2^{1/2})\frac{1}{c}\left(\Phi\frac{\omega^3}{\sigma_H\Omega}\frac{v^2}{c^2}\right)^{1/4}\exp\left(-i\frac{3\pi}{8}\right). \qquad (4.14)$$

Since σ_H is inversely proportional to Ω, the real and imaginary parts of the surface impedance decrease with the magnetic field like $H^{-1/4}$. The dependence on the frequency is $Z \sim \omega^{3/4}$. The value of the impedance is independent of the mean free path and consequently of the temperature; the ratio of the imaginary part to the real part is equal to $\tan(3\pi/8) \approx 2.41$. The surface impedance was calculated in [1] without allowance for the reflection of the electrons from the separation boundary. The value of Z obtained there turns out to be larger by a factor $(1 + 2^{1/2})/2 \approx 1.21$ than in the case of an unbounded metal.

It should be noted that in an inclined magnetic field, in the limiting case considered here, there should take place a noticeable anisotropy of the surface impedance, even in the case of a spherical Fermi surface. This anisotropy is due to the fact that when the magnetic field is inclined, in the region of (3.14) and (3.17), the conductivity of the metal in the ζ-axis direction turns out to be smaller by a factor $kR\Phi/|\gamma|$ [2] than the high-frequency conductivity in the perpendicular direction (the ξ axis). Accordingly, the surface-impedance component $Z_{\zeta\zeta}$ turns out to be larger than $Z_{\xi\xi}$.

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