STABILITY OF THE SINGLE-MODE REGIME AND FLUCTUATIONS IN A PIEZO-SEMICONDUCTING ACOUSTIC GENERATOR

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Equations are obtained describing the multi-mode regime of a piezosemiconductor acoustic generator. The stability conditions of the single-mode regime are determined. The line width connected with the natural fluctuations of the current in the sample and with the fluctuations of the applied voltage is calculated.

WHITE[1] has indicated the possibility of generating acoustic waves in a plane-parallel layer of a piezoelectric semiconductor placed in a constant electric field. The first such generator was constructed by White and Wang[2] and later by Maines and Paige[3], using single-crystal CdS. It is noted in these papers that in the case of small supercriticality (i.e. at a small excess of the drift velocity of the electrons over the critical value at which generation begins), the generator operates only on one mode, although the condition for the excitation is satisfied already for several modes. It was also noted that the generated sound has a small spectral width. According to the data of White and Wang[2], the relative line width was smaller than one millionth.

The purpose of the present paper is to investigate the stability of the single-mode generation regime and to estimate the line broadening resulting from natural fluctuations.

It follows from the linear theory[1] that the sound gain is maximal for waves with wave number q close to the reciprocal Debye radius $\kappa = (4\pi e^2 n/\epsilon T)^{1/2}$ (n—average electron concentration, e—their charge, ϵ—dielectric constant, T—temperature in energy units). The amplification of a sound wave propagating along the electron drift exceeds the absorption produced when this wave propagates in the opposite direction, when the drift velocity V exceeds the value V_{c0} defined by the equation[4]

$$V_{c0}^2 = w^2 + 4/\varkappa^2\tau^2 \tag{1}$$

(here $\tau = \epsilon/4\pi en\mu$ is the Maxwellian relaxation time, μ is the mobility, and w is the speed of sound). In a sufficiently long sample, when $\kappa L \gg 1$ and $L \gg V\tau$ (L is the length of the sample), the reflection of the sound from opposite walls can be described with the aid of the reflection coefficients r_1 and r_2, which, by virtue of the weakness of the electromagnetic coupling, are calculated from the boundary conditions of elasticity theory[5]. The critical value of the drift velocity V_c, at which the amplification of the wave with wave vector κ becomes comparable with the absorption and loss to reflection, usually differs little from V_{c0}:

$$\frac{V_c - V_{c0}}{V_{c0}} = \frac{8\ln(1/r_1r_2)}{\chi\varkappa^2 Lw\tau} \ll 1 \tag{2}$$

(here χ is the square of the coefficient of electromagnetic coupling). For example, for a CdS crystal of 1 cm length at room temperature we have $\chi\kappa^2 Lw\tau = \chi Lwe/\mu T = 6 \cdot 10^2$ [1]).

If the drift velocity does not greatly exceed the critical value

$$(V - V_c)/V_{c0} \ll 1/V_{c0}^2\varkappa^2\tau^2, \tag{3}$$

then the linear theory gives for the growth increment of the sound with wave vector κ the following value:

$${}^1/_2\gamma_0 = {}^1/_{16}\chi w^2\varkappa^2\tau(V - V_c)/V_{c0}. \tag{4}$$

The frequencies of the natural modes of the generator should satisfy the interference conditions. Therefore the maximum possible growth increment of the natural modes, generally speaking, may turn out to be smaller than $\gamma_0/2$. With increasing drift velocity, the condition for the excitation is first satisfied for one mode with the maximum gain. The theory of a generator operating in such a regime was developed by V. L. Gurevich and the author[5]. However, the supercriticality necessary to realize such a regime is so small that it is difficult to attain in experiment. Thus, at L = 1 mm the value of $(V - V_c)/V_{c0}$ should not exceed 2×10^{-6}. In practice the excitation condition is always satisfied for a large number of modes. This raises the question of the stability of the single-mode stationary regime.

To answer this question we must have a theory capable of taking into account the interaction between the different generator modes. A similar problem arises for microwave and optical generators. It is usually solved by expanding the field in the resonator in the natural modes of the resonator[6]. In the present paper we use the same method. As a result, the problem reduces to a study of a system with a discrete number of degrees of freedom. In such a formulation, it is possible to solve the problem of stability of the single-mode regime. It turns out that if the generator operates at a mode whose growth increment is close to $\gamma_0/2$, then the remaining modes are suppressed as a result of the nonlinear interaction.

The smallness of the line width in optical and micro-

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1) In this and subsequent estimates we use the following constants corresponding to a CdS crystal at room temperature: $\chi \sim 3 \times 10^{-2}$, $w \sim 2 \times 10^5$ cm/sec, $\mu \sim 300$ cm²/V-sec, and $\epsilon \sim 10$.

wave generators raises the question of the line width in an analogous acoustic generator. The minimum possible values of the line width is governed by the natural fluctuations in the conductor. The low intensity of the sound in the generator makes it possible to use the Langevin method for the study of the fluctuations. In the single-mode regime this problem reduces to a study of the fluctuations in an ordinary Thompson generator. At an electron concentration not exceeding 10^{13} cm^{-3}, the natural width turns out to be exceedingly small. However, with increasing concentration the natural width increases exponentially, since the noise is amplified together with the signal[4], the gain being proportional to the concentration. At a low natural width, the observed broadening of the line may be connected with other mechanisms. We consider here one such mechanism—the fluctuations of the external constant field[2]. In this case the problem reduces to a study of the fluctuations of the generator parameters.

1. FUNDAMENTAL EQUATIONS OF THE PROBLEM

The papers published to date[2,3,7,8] report observation of sound generation with CdS crystals at room temperature. The only mechanism of nonlinear interaction of sound waves is the concentration nonlinearity. In the present paper we consider only this case[3].

The smallness of the electromechanical-coupling constant causes all the electronic effects to play a role only at distances that are large compared with the wavelength, and at times large compared with the period of the sound. This makes it possible to reduce the complete system of equations obtained in[12] to a system of abrreviated equations for slowly varying amplitudes. In the case of small supercriticality (3) the amplitude of the sound is small, and allowance for the nonlinearity reduces to allowance for the interaction of the first and second harmonics. For the same reason, the frequency interval within which the excitation condition is satisfied is small compared with the average frequency $w\kappa$, and the elastic displacement can be written in the form

$$u=\frac{\varepsilon T}{4\pi\beta e}[a^{+}e^{i\kappa(x-wt)}+a^{-}e^{-i\kappa(x+wt)}+b^{+}e^{2i\kappa(x-wt)}+b^{-}e^{-2i\kappa(x+wt)}+\text{к. с.}]. \tag{1.1}$$

Here β is the piezoelectric modulus, the x axis is chosen in the direction of the drift velocity. The indices + and – pertain respectively to waves propagating in the direction of the drift velocity (forward waves) and in the opposite direction (backward waves). By velocity of sound is meant here throughout the velocity of sound in the piezoelectric dielectric $w=\sqrt{c/\rho}(1+\chi/2)$ (c is the modulus of elasticity and ρ is the density of the crystal).

The abbreviated equations for the amplitudes are derived by a well known method, and it is therefore meaningful here only to denote the peculiarities of the calculation. The concentration nonlinearity is quadratic, and for the abbreviated equations it is necessary to perform two iterations in the amplitude—first to second order and then to third[12]. The main feature of the calculation is allowance for the interaction of opposing waves (see[13]). In second order in the amplitude, this interaction leads to the appearance of harmonics of the type $\exp(2i\kappa x)$ and $\exp(-2i\kappa wt)$. These harmonics do not describe the propagation of waves with sound velocity, and by virtue of the smallness of the electromechanical-coupling constant we can neglect the elastic displacement corresponding to them. From the charge conservation law it follows that the concentration of the electrons does not contain the harmonic $\exp(-2i\kappa wt)$. To determine the amplitudes of the current density and of the potential of the electric field at this harmonic, we use the law of conservation of the total current, and for simplicity we assume that the total current in the external circuit at the frequency $2w\kappa$ is equal to zero.

The wave amplitudes are assumed to be independent of the transverse coordinates y and z, i.e., we disregard waves propagating at an angle to the x axis. The generation threshold for such waves is obviously higher than for waves propagating along the x axis, since the coefficient of reflection from the unfinished side surfaces of the crystal is small, and the end surfaces are metallized and produce additional absorption.

The excitation condition can be satisfied simultaneously for several modes. In this connection, it is necessary to take into account in the abbreviated equations the weak spatial dispersion, i.e., it is necessary to retain second derivatives of the amplitudes with respect to x (see[14,15]). These terms are small compared with the remaining ones by a factor equal to the ratio of the width of the frequency interval within which the excitation condition is satisfied to the average frequency $w\kappa$. They must be compared, however, with the supercriticality, and if they are discarded then it turns out that the gains for all the modes are equal.

Thermal fluctuations in the sample are taken into account by introducing random currents[10][4]. This is permissible if the drift field is not heating, which is well satisfied in CdS at room temperature, and if the electron-density change connected with the sound is small compared with the average density, a condition that coincides with the condition for the smallness of the amplitude a^+ and a^- compared with unity. The contribution of the fluctuations to the abbreviated equations are calculated in the same manner as used in the design of radio oscillators (see[17]). Only the terms linear in the fluctuations are taken into account, since inclusion of the nonlinear terms would be an exaggeration of the accuracy in the Langevin approach.

As a result we obtain the following system of equations for the dimensionless amplitudes:

$$\frac{\partial a^{+}}{\partial t}+w\frac{\partial a^{+}}{\partial x}-\mathscr{L}_{1}^{+}a^{+}-\frac{\Lambda^{+}}{\varkappa^{2}}\frac{\partial^{2}a^{+}}{\partial x^{2}}=F^{+},$$
$$\frac{\partial a^{-}}{\partial t}-w\frac{\partial a^{-}}{\partial x}-\mathscr{L}_{1}^{-}a^{-}-\frac{\Lambda^{-}}{\varkappa^{2}}\frac{\partial^{2}a^{-}}{\partial x^{2}}=F^{-}; \tag{1.2}$$

$$\frac{\partial b^{+}}{dt}+w\frac{\partial b^{+}}{\partial x}-\mathscr{L}_{2}^{+}b^{+}=K^{+}(a^{+})^{2},$$
$$\frac{\partial b^{-}}{dt}-w\frac{\partial b^{-}}{\partial x}-\mathscr{L}_{2}^{-}b^{-}=K^{-}(a^{-})^{2}. \tag{1.3}$$

[2]I am grateful to A. M. D'yakonov who pointed out this mechanism to me.

[3]Generally speaking, nonlinear effects of another type [9-11] can play a role in the case of sound in piezoelectric semiconductors.

[4]The fluctuations of the elastic stresses are disregarded, since they are usually small in comparison with the fluctuations of the current, by a ratio $\eta w\kappa/c\chi$, where η is the viscosity coefficient. The viscous absorption is small compared with the electronic absorption by the same ratio.

We have introduced here the notation

$$F^+ = P^+|a^+|^2a^+ + S^+|a^-|^2a^+ + Q^+b^+a^{+*} + G^+(x,t) + R^+\Big[|a^+|^2 - \frac{1}{L}\int_{-L/2}^{L/2}|a^+|^2dx\Big]a^+ + T^+\Big[|a^-|^2 - \frac{1}{L}\int_{-L/2}^{L/2}|a^-|^2dx\Big]a^+,$$

$$F^- = P^-|a^-|^2a^- + S^-|a^+|^2a^- + Q^-b^-a^{-*} + G^-(x,t) + R^-\Big[|a^-|^2 - \frac{1}{L}\int_{-L/2}^{L/2}|a^-|^2dx\Big]a^- + T^-\Big[|a^+|^2 - \frac{1}{L}\int_{-L/2}^{L/2}|a^+|^2dx\Big]a^-. \quad (1.4)$$

The interaction of the first harmonics in (1.4) is the result of the interaction of the sound with the strongly damped electron-density waves[16]. The last two terms are connected with the acousto-electric current[12]. Further

$$S^+ = -\frac{i\chi w\varkappa}{(4+\nu_-^2)(2+i\nu_+)^2}\Big[\frac{tV\varkappa\tau}{1-2iw\varkappa\tau} + \frac{(3+2i\nu_-)(1+iV\varkappa\tau)}{5+2iV\varkappa\tau}\Big],$$

$$S^- = -\frac{i\chi w\varkappa}{(4+\nu_+^2)(2+i\nu_-)^2}\Big[-\frac{iV\varkappa\tau}{1-2iw\varkappa\tau} + \frac{(3+2i\nu_+)(1-iV\varkappa\tau)}{5-2iV\varkappa\tau}\Big]; \quad (1.5)$$

$$R^+ = \frac{\partial\mathscr{L}_1^+}{\partial V}\frac{V-w}{4+(V-w)^2\varkappa^2\tau^2}, \quad R^- = \frac{\partial\mathscr{L}_1^-}{\partial V}\frac{V+w}{4+(V+w)^2\varkappa^2\tau^2}, \quad (1.6)$$

$$T^+ = \frac{\partial\mathscr{L}_1^+}{\partial V}\frac{V+w}{4+(V+w)^2\varkappa^2\tau^2}, \quad T^- = \frac{\partial\mathscr{L}_1^-}{\partial V}\frac{V-w}{4+(V-w)^2\varkappa^2\tau^2}. \quad (1.7)$$

The remaining coefficients are given by the same expressions for both indices + and –[5]:

$$\mathscr{L}_1 = \frac{\chi w\varkappa}{2}\frac{i}{2+i\nu}, \quad \mathscr{L}_2 = \chi w\varkappa\frac{i}{5+2i\nu}, \quad \Lambda = -\frac{\chi w\varkappa}{2}\frac{i}{(2+i\nu)^2},$$

$$P = -\frac{i\chi w\varkappa(1+i\nu)(3-2i\nu)}{2(4+\nu^2)(2+i\nu)^2(5+2i\nu)}, \quad Q = \frac{2i\chi w\varkappa(1+2i\nu)}{(4+\nu^2)(5+2i\nu)},$$

$$K = \frac{i\chi w\varkappa(1+i\nu)}{2(2+i\nu)^2(5+2i\nu)}, \quad (1.8)$$

where ν should be replaced respectively by

$$\nu_+ = (V-w)\varkappa\tau \quad \text{or} \quad \nu_- = -(V+w)\varkappa\tau.$$

The functions G^+ and G^- are connected by the following correlation relations:

$$\langle G_+^*(x,t)G_+(x',t')\rangle = \frac{\chi^2w^2\varkappa^2\tau}{2nS}\frac{\delta(x-x')\delta(t-t')}{4+(V-w)^2\varkappa^2\tau^2},$$

$$\langle G_-^*(x,t)G_-(x',t')\rangle = \frac{\chi^2w^2\varkappa^2\tau}{2nS}\frac{\delta(x-x')\delta(t-t')}{4+(V+w)^2\varkappa^2\tau^2}, \quad (1.9)$$

where S is the area of the transverse cross section of the sample. The remaining correlations are equal to zero.

Equations (1.2) and (1.3) were derived assuming small supercriticality. Therefore the difference between V and V_c is taken into account only in the calculation of the linear gain. In all the remaining cases it can be assumed that $V = V_c$. In the calculation of the nonlinear terms it is necessary, in addition, to use the inequality (2) and to put $V = V_{c0}$.

[5]In those cases where the formulas have the same form for the forward and backward waves, the indices + and – will be omitted for brevity.

The system (1.2) and (1.3) must be supplemented by the boundary conditions on the end faces of the sample

$$a^+ - r_1a^- = 0, \quad b^+ - r_1b^- = 0 \quad \text{for} \quad x = -L/2,$$
$$a^- - r_2a^+ = 0, \quad b^- - r_2b^+ = 0 \quad \text{for} \quad x = L/2. \quad (1.10)$$

These boundary conditions are sufficient, in spite of the fact that Eq. (1.2) is of second order in the coordinate. It is necessary to choose those solutions of the equations, which describe sound waves and for which the second derivatives are only small corrections. Other solutions of Eqs. (1.2) describe electron-density waves, which play no role in our problem[12].

2. MULTIMODE GENERATION EQUATIONS

In the linear approximation, the solution of the system (1.2) is a superposition of the sound distributions corresponding to the resonator modes:

$$\begin{pmatrix} a^+(x,t) \\ a^-(x,t) \end{pmatrix} = \sum_k \alpha_k(t)\begin{pmatrix} \sqrt{r_1}e^{i\pi k/2}\psi_k^+(x) \\ \sqrt{r_2}e^{-i\pi k/2}\psi_k^-(x) \end{pmatrix}, \quad (2.1)$$

where

$$\psi_k^+(x) = \exp\Big(\frac{\mathscr{L}_1^+ - p_k}{w}x\Big), \quad \psi_k^-(x) = \exp\Big(-\frac{\mathscr{L}_1^- - p_k}{w}x\Big); \quad (2.2)$$

$$p_k = \frac{\gamma_0}{2} + \frac{1}{8}i\chi w\varkappa - \frac{w}{L}k\pi i - \frac{\chi w^2}{16\varkappa^2L^2V_{c0}^2}\frac{2+iw\varkappa\tau}{\tau}\pi^2(k-k_0)^2,$$

$$k_0 = -\frac{\chi\varkappa L}{4\pi}, \quad (2.3)$$

with $\alpha_k(t) \sim \exp(p_kt)$. The quantity Re p_k determines the growth increment of the k-th mode. It is maximal for the mode for which $(k-k_0)^2$ is minimal. The excitation condition for the k-th mode is Re $p_k > 0$. Its frequency is determined by the quantity Im p_k.

The terms describing the dispersion in (1.2), in view of their smallness, were taken into account only in the calculation of the eigenvalues p_k.

To obtain the system of equations describing the behavior of $\alpha_k(t)$ with allowance for the nonlinear effects, it is necessary to multiply Eqs. (1.2) by the solution of the conjugate boundary-value problem[18] which can be readily verified to be of the form

$$\Big(\frac{e^{-i\pi k/2}}{\sqrt{r_1}\psi_k^+(x)}, \frac{e^{i\pi k/2}}{\sqrt{r_2}\psi_k^-(x)}\Big), \quad (2.4)$$

and to integrate with respect to x from $-L/2$ to $L/2$. As a result we obtain a system

$$d\alpha_k/dt - p_k\alpha_k = F_k(t), \quad (2.5)$$

where

$$F_k(t) = \frac{1}{2L}\Big\{\frac{e^{-i\pi k/2}}{\sqrt{r_1}}\int_{-L/2}^{L/2}\frac{F^+(x,t)}{\psi_k^+(x)}dx + \frac{e^{i\pi k/2}}{\sqrt{r_2}}\int_{-L/2}^{L/2}\frac{F^-(x,t)}{\psi_k^-(x)}dx\Big\}. \quad (2.6)$$

To calculate $F_k(t)$ it is necessary to solve Eqs. (1.3) for the amplitude of the second harmonics. It must be borne in mind here that the conditions for the amplification of the first harmonics are close to the threshold values, and their amplitudes change slowly. For the second harmonics this is not so, and at sufficiently low supercriticality, when the condition

$$\gamma_0 \ll \Big|\mathrm{Re}(\mathscr{L}_2^+ + \mathscr{L}_2^-) - \frac{w}{L}\ln\frac{1}{r_1r_2}\Big|$$

is satisfied, or, equivalently,

$$\frac{V-V_c}{V_{c0}} \ll \frac{288}{1681+400w^2\varkappa^2\tau^2}+\frac{8\ln(1/r_1r_2)}{\chi w\varkappa^2 L\tau}, \qquad (2.7)$$

the amplitudes of the second harmonics assume their steady-state values before the amplitudes of the first harmonics have time to change. Then the solution of Eqs. (1.3) with boundary conditions (1.10) is given by

$$b^+(x,t)=\sum_{l,m}\exp\left(i\pi\frac{l+m}{2}-\frac{p_l+p_m}{w}x\right)\alpha_l(t)\alpha_m(t)\beta^+(x),$$
$$b^-(x,t)=\sum_{l,m}\exp\left(-i\pi\frac{l+m}{2}+\frac{p_l+p_m}{w}x\right)\alpha_l(t)\alpha_m(t)\beta^-(x), \qquad (2.8)$$

where

$$\beta^+(x)=\left[C_+-\frac{r_1K^+}{w}\int_x^{L/2}\frac{(\psi^+)^2}{\varphi^+}dx\right]\varphi^+(x), \qquad (2.9)$$

$$\beta^-(x)=\left[C_-+\frac{r_2K^-}{w}\int_x^{-L/2}\frac{(\psi^-)^2}{\varphi^-}dx\right]\varphi^-(x);$$

$$\varphi^+(x)=\exp\left(\frac{x}{w}\mathscr{L}_2^+\right),\quad \varphi^-(x)=\exp\left(-\frac{x}{w}\mathscr{L}_2^-\right); \qquad (2.10)$$

$$C_+=-\frac{1}{D_2}\left\{\frac{r_1K^+}{w}\varphi^+\left(-\frac{L}{2}\right)\varphi^-\left(\frac{L}{2}\right)\int_{-L/2}^{L/2}\frac{(\psi^+)^2}{\varphi^+}dx\right.$$
$$\left.+\frac{r_1r_2K^-}{w}\exp\left[-i\frac{L}{w}\mathrm{Im}(\mathscr{L}_1^++\mathscr{L}_1^-)\right]\int_{-L/2}^{L/2}\frac{(\psi^-)^2}{\varphi^-}dx\right\},$$

$$C_-=-\frac{1}{D_2}\left\{\frac{r_1r_2K^-}{w}\exp\left[-i\frac{L}{w}\mathrm{Im}(\mathscr{L}_1^++\mathscr{L}_1^-)\right]\int_{-L/2}^{L/2}\frac{(\psi^+)^2}{\varphi^+}dx\right. \qquad (2.11)$$
$$\left.+\frac{r_2K^-}{w}\varphi^+\left(-\frac{L}{2}\right)\varphi^-\left(\frac{L}{2}\right)\int_{-L/2}^{L/2}\frac{(\psi^-)^2}{\varphi^-}dx\right\};$$

$$D_2=r_1r_2\varphi^+\left(\frac{L}{2}\right)\varphi^-\left(-\frac{L}{2}\right)\exp\left[-2i\frac{L}{w}\mathrm{Im}(\mathscr{L}_1^++\mathscr{L}_1^-)\right]$$
$$-\varphi^+\left(-\frac{L}{2}\right)\varphi^-\left(\frac{L}{2}\right). \qquad (2.12)$$

Using these results, we can rewrite $F_k(t)$ in the form

$$F_k(t)=\sum_{lmn}F_{lm,n}^k\alpha_l(t)\alpha_m(t)\alpha_n^*(t)+f_k(t), \qquad (2.13)$$

where

$$f_k(t)=\frac{1}{2L}\left\{\frac{e^{-i\pi k/2}}{\sqrt{r_1}}\int_{-L/2}^{L/2}\frac{G^+(x,t)}{\psi_k^+(x)}dx+\frac{e^{i\pi k/2}}{\sqrt{r_2}}\int_{-L/2}^{L/2}\frac{G^-(x,t)}{\psi_k^-(x)}dx\right\}. \qquad (2.14)$$

Because of their complexity, the explicit expressions for the matrix elements are given in the Appendix.

When substituting (2.13) in (2.5) it must be borne in mind that by virtue of the smallness of the nonlinear terms in (2.13), the only terms that matter in the sum are those that may be resonant. Since the dispersion corrections to the frequencies in (2.3) are small, this limitation of the summation has an exceedingly simple form:

$$\frac{d\alpha_k}{dt}=p_k\alpha_k+\sum_{l+m-n=k}F_{lm,n}^k\alpha_l(t)\alpha_m(t)\alpha_n^*(t)+f_k(t). \qquad (2.15)$$

For the functions $f_k(t)$ we obtain from (2.14) and (1.9) the following correlation relations:

$$\langle f_k(t_1)f_l^*(t_2)\rangle=\frac{\chi^2w^2\varkappa^2\tau}{8NL}\left\{\frac{\exp[-i\pi(k-l)/2]}{r_1|2+i\nu_+|^2}\int_{-L/2}^{L/2}\frac{dx}{\psi_k^+\psi_l^{+*}}\right. \qquad (2.16)$$
$$\left.\times\frac{\exp[i\pi(k-l)/2]}{r_2|2+i\nu_-|^2}\int_{-L/2}^{L/2}\frac{dx}{\psi_k^-\psi_l^{-*}}\right\}\delta(t_1-t_2).$$

Here N = nLS is the number of electrons in the sample.

The problem is thus reduced to an investigation of a system with a denumerable number of degrees of freedom, the dynamics equations of which are of the form (2.15). Equations of the type (2.15) are encountered in the theory of multimode laser generation[6].

3. STABILITY OF SINGLE-MODE GENERATION REGIME

The simplest generator operating regime is one in which the generator operates in the single mode. In this case

$$d\alpha_k/dt=p_k\alpha_k+F_1|\alpha_k|^2\alpha_k, \qquad (3.1)$$

where $F_1 \equiv F_{kk,k}^k$. We put

$$\alpha_k=|\alpha_k|\exp(i\vartheta_k), \qquad (3.2)$$

then

$$d|\alpha_k|^2/dt=2\mathrm{Re}\,(p_k+F_1|\alpha_k|^2)|\alpha_k|^2. \qquad (3.3)$$

The solution of this equation is

$$\frac{1}{|\alpha_k|^2}=-\frac{\mathrm{Re}\,F}{\mathrm{Re}\,p_k}+\left(\frac{1}{A_{k0}^2}+\frac{\mathrm{Re}\,F_1}{\mathrm{Re}\,p_k}\right)\exp(-2\mathrm{Re}\,p_kt), \qquad (3.4)$$

where $A_{k_0}^2$ is the square of the modulus of the amplitude at the initial instant of time. When Re $F_1 > 0$, the amplitude of the sound increases from an initial value to infinity, i.e., hard excitation takes place. When Re $F_1 < 0$, the amplitude tends to the stationary value

$$|\alpha_k|^2|_{t\to\infty}\equiv A_k^2=-\mathrm{Re}\,p_k/\mathrm{Re}\,F_1. \qquad (3.5)$$

This is the case of soft excitation.

In view of the complexity of the formulas for the matrix elements, the dependence of the character of excitation of the generation on the parameters can be established only in the following limiting cases.

Long generator:

$$\frac{L}{w}|\mathrm{Re}\,\mathscr{L}_1|\gg 1,\quad \frac{L}{w}|\mathrm{Re}\,\mathscr{L}_2|\gg 1. \qquad (3.6)$$

At large concentration, $w\kappa\tau \ll 1$ ($n \gg 10^{14}$ cm^{-3}) we have

$$\mathrm{Re}\,F_1=r_1(1-r_2)\frac{367+1437r_2}{34112}\frac{w}{L}\exp\left(\frac{L}{w}\mathrm{Re}\,\mathscr{L}_1^+\right),$$
$$\mathrm{Im}\,F_1=r_1\frac{155-246r_2-91r_2^2}{17056}\frac{w}{L}\exp\left(\frac{L}{w}\mathrm{Re}\,\mathscr{L}_1^+\right). \qquad (3.7)$$

At low concentration, $w\kappa\tau \gg 1$ ($n \ll 10^{14}$ cm^{-3}) we obtain

$$\mathrm{Re}\,F_1=-\frac{r_1}{72}(5+9r_2^2)\frac{w}{L}\exp\left(\frac{L}{w}\mathrm{Re}\,\mathscr{L}_1^+\right), \qquad (3.8)$$
$$\mathrm{Im}\,F_1=-\frac{r_1}{96}\frac{w}{L}w\varkappa\tau\exp\left(\frac{L}{w}\mathrm{Re}\,\mathscr{L}_1^+\right).$$

Short generator:

$$\frac{L}{w}|\mathrm{Re}\,\mathscr{L}_1|\ll 1,\quad \frac{L}{w}|\mathrm{Re}\,\mathscr{L}_2|\ll 1. \qquad (3.9)$$

In this case the general formula is

$$F_1={}^1/_2(r_1P^++r_2P^-), \qquad (3.10)$$

whence

$$F_1 = \frac{\chi w\varkappa}{10496}[63(r_1 - r_2) - 34i(r_1 + r_2)] \text{ for } w\varkappa\tau \ll 1,$$
$$F_1 = \frac{3r_1}{320}\chi w\varkappa\left(i + \frac{32}{15}\frac{1}{w\varkappa\tau}\right) \text{ for } w\varkappa\tau \gg 1. \quad (3.11)$$

Thus, soft excitation occurs in a long generator at low concentration and in a short generator at large concentration, if $r_1 < r_2$.

We shall henceforth consider only these two cases.

Formula (3.5) makes it possible to calculate the acoustic-energy flux density passing from the generator through the boundary $x = L/2$:

$$S_k^{ac} = 2\frac{r_1(1 - r_2^2)}{\chi} nTwA_k^2 \left|\psi^+\left(\frac{L}{2}\right)\right|^2. \quad (3.12)$$

In the case of (3.6) and $w\kappa\tau \gg 1$ we have

$$S_k^{ac} = \frac{9(1 - r_2^2)}{5 + 9r_2^2} Tn \frac{Lw^2}{D}\left[\frac{V_0 - V_c}{V_{c0}} - \frac{2}{w^2\varkappa^2\tau^2}\left(\frac{k\pi}{\varkappa L} + \frac{\chi}{4}\right)^2\right], \quad (3.13)$$

and in the case of (3.9) with $w\kappa\tau \ll 1$ and $r_2 > r_1$ we have

$$S_k^{ac} = \frac{1312}{63}\frac{r_1(1 - r_2^2)}{r_2 - r_1} Tn \frac{w^2}{\chi\varkappa D}\left[\frac{V_0 - V_c}{V_{c0}} - \frac{1}{2}\left(\frac{k\pi}{\varkappa L} + \frac{\chi}{4}\right)^2\right]. \quad (3.14)$$

Here D is the diffusion coefficient.

In the stationary regime $\vartheta_k = \omega_k t + \vartheta_{k0}$, where

$$\omega_k = -\operatorname{Im} p_k - \operatorname{Im} F_1 A_k^2. \quad (3.15)$$

Formula (3.13) was obtained in[19,5] for the case when the excitation condition $\operatorname{Re} p_k > 0$ is satisfied only for one mode. The theory developed in the present paper makes it possible to determine the possible existence of the single-mode regime in the case when the excitation condition is satisfied immediately for several modes (the question of the method of excitation remains open). To this end it suffices to investigate the stability of the regime against the growth of other modes. If we investigate the stability of the stationary regime at the k-th mode, then, just as in parametric resonance, it is necessary to consider simultaneously two modes between which interference is possible, namely the m-th and the 2k-m-th modes:

$$d\alpha_m/dt = p_m\alpha_m + (F^m_{km,k} + F^m_{mk,k})|\alpha_k|^2\alpha_m + F^m_{kk,2k-m}\alpha_k^2\alpha'_{2k-m},$$
$$d\alpha_{2k-m}/dt = p_{2k-m}\alpha_{2k-m}$$
$$+ (F^{2k-m}_{k,2k-m,k} + F^{2k-m}_{2k-m,k,k})|\alpha_k|^2\alpha_{2k-m} + F^{2k-m}_{kk,m}\alpha_k^2\alpha_m^*. \quad (3.16)$$

From the general formulas (A.1)—(A.4) for the matrix elements it follows that

$$F^m_{km,k} = F^{2k-m}_{k,2k-m,k} = F_1. \quad (3.17)$$

In addition, in the two limiting cases (3.6) and (3.9) we have

$$F^m_{mk,m} = F^m_{kk,2k-m} = F_1 + F_2, \quad F^{2k-m}_{2k-m,k,k} = F^{2k-m}_{kk,m} = F_1 + F_3, \quad (3.18)$$

where $F_2 = F^{(3)m}_{lk,n}$, $F_3 = F^{(3)k}_{lm,n}$.

Using formulas (3.5) and (3.15) and introducing the new variables

$$\alpha_m = A_m \exp[it\operatorname{Im}(p_m + 2F_1A_k^2 + F_2A_k^2)],$$
$$\alpha_{2k-m} = A_{2k-m}\exp[it\operatorname{Im}(p_{2k-m} + 2F_1A_k^2 + F_3A_k^2)], \quad (3.19)$$

we obtain the following system of equations:

$$dA_m/dt = \operatorname{Re}(p_m - p_k + F_1A_k^2 + F_2A_k^2)A_m + (F_1 + F_2)A_k^2A^*_{2k-m}$$
$$\times \exp[-it\operatorname{Im}(p_m + p_{2k-m} - 2p_k + 2F_1A_k^2 + F_2A_k^2 + F_3A_k^2)],$$
$$dA_{2k-m}/dt = \operatorname{Re}(p_{2k-m} - p_k + F_1A_k^2 + F_3A_k^2)A_{2k-m}$$
$$+ (F_1 + F_3)A_k^2A_m^*\exp[-it\operatorname{Im}(p_m + p_{2k-m} - 2p_k + 2F_1A_k^2 + F_2A_k^2 + F_3A_k^2)]. \quad (3.20)$$

For a long generator with $w\kappa\tau \gg 1$

$$F_2 = F_3^* = \frac{w}{16L}r_1(1 + r_2^2)\left[1 + i\frac{\sin[(k-m)\pi/2]}{(k-m)\pi/2}\right]\exp\left(\frac{L}{w}\operatorname{Re}\mathscr{L}_1^+\right). \quad (3.21)$$

Comparing (2.3), (3.8), and (3.21), and also bearing (3.5) in mind, we can easily note that the coefficient in the arguments of the exponentials in (3.20) is larger than the coefficients of the linear terms, by a factor $w\kappa\tau$. This means that the interference terms are rapidly oscillating and they can be discarded. The stability condition then takes the form

$$\operatorname{Re}(p_m - p_k) + \operatorname{Re}(F_1 + F_2)A_k^2 < 0. \quad (3.22)$$

Since

$$\operatorname{Re}(F_1 + F_2) = -\frac{r_1(1 + 9r_2^2)}{144}\frac{w}{L}\exp\left(\frac{L}{w}\operatorname{Re}\mathscr{L}_1^+\right),$$

it follows that if the mode with number k is sufficiently close to the maximum of the linear gain, the stationary regime at this mode will be stable.

For a short generator with $w\kappa\tau \ll 1$

$$F_2 = F_3 = i\frac{\chi w\varkappa}{128}(r_1 + r_2)(1 - \delta_{mk}). \quad (3.23)$$

In addition, we obtain from (2.3) in this case $\operatorname{Im}(p_m + p_{2k-m} - 2p_k) = 0$. Then, putting

$$A_m \sim \exp[-\lambda t - it\operatorname{Im}(F_1 + F_2)A_k^2],$$
$$A_{2k-m} \sim \exp[-\lambda^* t - it\operatorname{Im}(F_1 + F_2)A_k^2], \quad (3.24)$$

we obtain

$$\lambda = -{}^1/_2\operatorname{Re}(p_m + p_{2k-m} - 2p_k + 2F_1A_k^2) \pm$$
$$\pm{}^1/_2\{[\operatorname{Re}(p_m - p_{2k-m}) + 2i\operatorname{Im}(F_1 + F_2)A_k^2]^2 + 4|F_1 + F_2|^2A_k^4\}^{1/2}. \quad (3.25)$$

In the limiting case, when the k-th mode is so close to the maximum of the linear gain that

$$|\operatorname{Re}(p_m - p_{2k-m})| \ll \operatorname{Re} p_k, \quad (3.26)$$

expression (3.25) simplifies and we have

$$(\operatorname{Re}\lambda)_{min} = -\frac{1}{2}\operatorname{Re}(p_m + p_{2k-m} - 2p_k) = \frac{\chi\pi^2}{2\tau\varkappa^2L^2}(k-m)^2 \quad (3.27)$$

i.e., such a regime is stable. It is obvious from (3.25) that when the k-th mode moves away from the maximum of the linear gain, the modulus of the radicand increases and the stability is lost.

4. LINE WIDTH OF SINGLE-MODE GENERATION REGIME

An investigation of the influence of the natural fluctuations on the operation of the generator in the single-mode regime reduces to a solution of the equation

$$d\alpha_k/dt = p_k\alpha_k + F_1|\alpha_k|^2\alpha_k + f_k(t), \quad (4.1)$$

which is obtained by standard methods (see, e.g.,[20]). The spectral density of the mean square of the amplitude

$$(\alpha_k^2)_\Omega \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\Omega\tau}\langle\alpha_k(t+\tau)\alpha_k^*(t)\rangle d\tau \quad (4.2)$$

is given by[6)]

$$(a_k^2)_\Omega = \frac{1}{4\pi}\frac{\langle f_k^2\rangle Z}{(\omega_k-\Omega)^2+\langle f_k^2\rangle^2 Z^2/16A_k^4} + \frac{\langle f_k^2\rangle}{2\pi}\frac{3\,\mathrm{Re}\,p_k+\langle f_k^2\rangle Z/4A_k^2}{(\omega_k-\Omega)^2+(3\,\mathrm{Re}\,p_k+\langle f_k^2\rangle Z/4A_k^2)^2}, \qquad (4.3)$$

where

$$Z = 1+(2\,\mathrm{Im}\,F_1/3\,\mathrm{Re}\,F_1)^2,$$

and $\langle f_k^2\rangle$ is defined by the relation

$$\langle f_k(t) f_k^*(t')\rangle = \langle f_k^2\rangle\delta(t-t'). \qquad (4.4)$$

The first term in (4.3) is connected with the fluctuations of the phase of the oscillation, and the second includes the fluctuations of the phase and the amplitude. The line width is determined by the relation

$$\Delta\Omega = Z\langle f_k^2\rangle/4A_k^2. \qquad (4.5)$$

Taking (2.16) into account, we obtain for a long generator

$$\Delta\Omega = \frac{(w\varkappa\tau)^4}{1152NL^2\varkappa^2\tau}\frac{1}{5+9r_2^2}\left[\frac{V-V_c}{V_{c0}} - \frac{2}{w^2\varkappa^2\tau^2}\left(\frac{k\pi}{\varkappa L}+\frac{\chi}{4}\right)^2\right]^{-1}\exp\left(\frac{2L}{w}\mathrm{Re}\,\mathscr{L}_1^+\right). \qquad (4.6)$$

For a short generator we have

$$\Delta\Omega = \frac{63}{41984}\frac{\chi^2 w\varkappa(r_2^2-r_1^2)}{Nr_1r_2}\left[1+\left(\frac{68}{189}\frac{r_1+r_2}{r_1-r_2}\right)^2\right]\left[\frac{V-V_c}{V_{c0}}-\frac{1}{2}\left(\frac{\pi k}{\varkappa L}+\frac{\chi}{4}\right)^2\right]^{-1}. \qquad (4.7)$$

The closest to the existing experimental data is the case of a long generator. Estimates based on formulas (3.13) and (4.6) give for it the following results. At $L \sim 1$ mm, $S \sim 1\ \mathrm{cm}^2$, $r_1 \sim r_2 \sim 1$, $n \sim 10^{12}\ \mathrm{cm}^{-3}$, $(V-V_c)/V_{c0} \sim 10^{-3}$ we have

$$S^{ac} \sim 10^4\ \mathrm{erg/cm^2sec} \qquad \Delta\Omega \sim 10^{-4}\ \mathrm{sec}^{-1},$$

and at $n \sim 10^{13}\ \mathrm{cm}^{-3}$ and $(V-V_c)/V_{c0} \sim 10^{-3}$ we obtain

$$S^{ac} \sim 10^6\ \mathrm{erg/cm^2sec}, \quad \Delta\Omega \sim 3\cdot 10^{-7}\ \mathrm{sec}^{-1}.$$

When the concentration is increased by one more order of magnitude, the line width increases immediately by ten orders of magnitude owing to the exponential concentration dependence.

In the case of a small natural width, the observed generator line width is determined by other mechanisms. Its estimate in this case can be obtained with the aid of Eq. (4.1). By way of an example we consider the line broadening as a result of fluctuations of voltage on the sample, i.e., fluctuations of the drift velocity. In this case the coefficient p_k in (4.1) acquires an alternating increment[7)]

$$\Delta p_k = {}^1/_{16}\chi w^2\varkappa^2\tau\Delta V(t)/V_{c0}. \qquad (4.8)$$

Let us consider sufficiently strong excitation, when

$$|\Delta p_k/p_k| \ll 1. \qquad (4.9)$$

Then the fluctuations of the amplitude can be regarded as small, and the mean square of the fluctuation of the phase during the time t is given by

$$\langle[\Delta\vartheta(t)]^2\rangle = \left(\frac{\chi w^2\varkappa^2\tau}{16V_{c0}}\frac{\mathrm{Im}\,F_1}{\mathrm{Re}\,F_1}\right)^2\langle(\Delta V)^2\rangle t, \qquad (4.10)$$

where

$$\langle\Delta V(t_1)\Delta V(t_2)\rangle = \langle(\Delta V)^2\rangle\delta(t_1-t_2). \qquad (4.11)$$

In the limiting cases of short and long generators, formula (4.10) gives for the line width the same result:

$$\Delta\Omega \sim \frac{\chi^2}{1024\pi}\frac{w^4e^2\varepsilon}{\mu^4T^3n}\langle(\Delta V)^2\rangle. \qquad (4.12)$$

In samples with large trap concentrations, generation-recombination fluctuations of the electron density may be appreciable. This case, however, calls for a special calculation owing to the nonlinear effects connected with the traps[9].

5. LIMITS OF APPLICABILITY OF THE THEORY

In the present section we compare the limitations assumed during the course of the calculation. The most significant of them is the smallness of the sound amplitude

$$|a^+|^2 \ll 1, \quad |a^-|^2 \ll 1, \qquad (5.1)$$

which made it possible to disregard harmonics higher than the third. With the aid of (3.5) it is easy to verify that this condition coincides with the condition for the smallness of the mean free path of the wave through the sample compared with the time of growth of the amplitude of the individual mode

$$\gamma_0 \ll \frac{w}{L} \quad \text{or} \quad \frac{V-V_c}{V_{c0}} \ll \frac{8}{\chi wL\varkappa^2\tau}. \qquad (5.2)$$

At $L \sim 10$ mm, the right-hand side of the second inequality in (5.2) is of the order of 10^{-1}.

To simplify the algebraic manipulation, we used condition (3), which, in particular, has made it possible to regard the interval of the excited frequencies as smaller than $w\kappa$. For the regime with soft excitation, the inequality (5.2) turns out to be stronger than (3). The inequality (2.7), used to calculate the amplitudes of the second harmonics, is practically equivalent to (3).

For a long generator, the inequality

$$\frac{V-V_c}{V_{c0}} \ll \frac{1}{\chi\varkappa LV_{c0}^2\varkappa^2\tau^2} \qquad (5.3)$$

may turn out to be strong and make it possible to neglect the dispersion in the calculation of the matrix elements.

The principal limitation on the applicability of the calculation is connected with the influence of the fluctuations on the operation of the generator. The line width should be much smaller then the difference between the frequencies of the neighboring modes

$$\Delta\Omega \ll w/L, \qquad (5.4)$$

otherwise the concept of mode itself becomes meaningless.

The condition for small supercriticality does not make it possible to consider a large number of phenomena observed in the generator. Thus, in[8] there was observed a strong distortion of the form of the signal at

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[6)]In this calculation we take into account terms nonlinear in the fluctuation, whereas in the calculation of $f_k(t)$ (Sec. 1) such terms were discarded. In this case we are considering the behavior of a resonant system capable of accumulating the effect of the external action $f_k(t)$ over long time intervals.

[7)]For simplicity we consider a case when the longitudinal waves are not piezoactive and no change takes place in the length of the sample.

large amplitudes. In[2] there was noted appearance of subharmonics with a frequency half the frequency of the excited mode. The reason of the appearance of subharmonics is obviously parametric resonance[21]. As is well known, parametric resonance in the presence of dissipation has a threshold, and therefore there should be no subharmonics in the case of small supercriticality.

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APPENDIX

The expressions for the matrix elements are of the form

$$F_{lm,n}^{k} = F_{lm,n}^{k(1)} + F_{lm,n}^{k(2)} + F_{lm,n}^{k(3)}, \tag{A.1}$$

where

$$F_{lm,n}^{k(1)} = \frac{1}{2L}\Big\{ r_1P^+ \int_{-L/2}^{L/2} |\psi^+|^2\,dx + r_2P^- \int_{-L/2}^{L/2} |\psi^-|^2\,dx + r_2S^+ \exp\left(-i\frac{l-m-n+k}{2}\pi\right) \int_{-L/2}^{L/2} \frac{\psi_l^-\psi_m^+\psi_n^{-*}}{\psi_k^+}\,dx + r_1S^- \exp\left(i\frac{l-m-n+k}{2}\pi\right) \int_{-L/2}^{L/2} \frac{\psi_l^+\psi_m^-\psi_n^{+*}}{\psi_k^-}\,dx \Big\}, \tag{A.2}$$

$$F_{lm,n}^{k(2)} = \frac{1}{2L}\Big\{ Q^+ \int_{-L/2}^{L/2} \frac{\beta^+\psi^{+*}}{\psi^+}\,dx + Q^- \int_{-L/2}^{L/2} \frac{\beta^-\psi^{-*}}{\psi^-}\,dx \Big\}, \tag{A.3}$$

$$F_{lm,n}^{k(3)} = \frac{1}{2L}\Big\{ R^+r_1\Big[\int_{-L/2}^{L/2} |\psi^+|^2\,dx - \frac{\sin[(k-m)\pi/2]}{(k-m)\pi/2} \int_{-L/2}^{L/2} \psi_l^+\psi_n^{+*}\,dx \Big] + R^-r_2\Big[\int_{-L/2}^{L/2} |\psi^-|^2\,dx - \frac{\sin[(k-m)\pi/2]}{(k-m)\pi/2} \int_{-L/2}^{L/2} \psi_l^-\psi_n^{-*}\,dx \Big] + T^+r_2 \exp\left(-i\frac{l-m-n+k}{2}\pi\right) \times \Big[\int_{-L/2}^{L/2} |\psi^-|^2 \exp\left(i\frac{m+n-l-k}{L}\pi x\right) dx - \frac{\sin[\pi(k-m)/2]}{\pi(k-m)/2} \int_{-L/2}^{L/2} \psi_l^-\psi_n^{-*}\,dx \Big] + T^-r_1 \times \exp\left(i\frac{l-m-n+k}{2}\pi\right) \Big[\int_{-L/2}^{L/2} |\psi^+|^2 \exp\left(-i\frac{m+n-l-k}{L}\pi x\right) dx - \frac{\sin[\pi(k-m)/2]}{\pi(k-m)/2} \int_{-L/2}^{L/2} \psi_l^+\psi_n^{+*} \Big]\Big\}. \tag{A.4}$$

Here

$$\psi^+ = \exp(\mathscr{L}_1^+ x/w), \quad \psi^- = \exp(-\mathscr{L}_1^- x/w).$$

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