PLASMA OSCILLATIONS AND METASTABLE STATES IN JOSEPHSON TUNNEL JUNCTIONS

A. E. GORBONOSOV and I. O. KULIK

Physico-technical Institute of Low Temperatures, Ukrainian Academy of Sciences

Submitted July 13, 1970

Zh. Eksp. Teor. Fiz. 60, 688–694 (March, 1971)

The electrodynamics of weak superconductivity in finite-width tunnel junctions is investigated. When account is taken of the boundary conditions of the junction borders, phenomena of "magnetic overheating" type arise in both the Meissner and mixed states. A diagram of state is set up for a finite-width tunnel junction in an arbitrary magnetic field. In contrast to the situation encountered in ordinary superconductors, the absolute stability boundary of the metastable superconducting state coincides with the upper boundary for the existence of the Meissner solutions. Josephson plasma oscillations in a semi-infinite junction are studied in the Meissner and mixed states.

An elementary analysis of the influence of the boundary of the metal on the magnetization curve of a bulky superconductor[1] has shown that although a transition to the mixed state becomes thermodynamically favored in a field $H > H_{c1}$, in order for the vortex to penetrate into the superconductor the latter must overcome a certain potential barrier, the magnitude of which (in the simplest case) is determined by the elastic properties of the vortex filament and its interaction with the magnetic field.[1,2] A thermodynamic consideration of the stability limits of the Meissner state in the limit of large κ [2] yielded for the field H_{s1} the value $H_{s1} \approx 0.75H_c$ (at $T = 0$) and $H_{s1} \approx 0.8H_c$ (at $T = T_c$), which thus turned out to be lower than the thermodynamic critical field H_c. Subsequently[3] it was suggested that the superheat field exceeds H_{s1}. The equations of electrodynamics admit formally of a solution with $H = 0$ in the volume of the superconductor, up to a field H_{s2} that coincides with H_c: $H_{s2} = H_c$. In the present paper we discuss similar problems of the electrodynamics of weak superconductivity. A phase diagram is constructed for a tunnel junction of finite width in an arbitrary magnetic field. It turns out that, unlike the situation that takes place in ordinary superconductors, the field H_{s1} is the absolute limit of stability, in the small, of the metastable superconducting state, and the field H_{s2} is the upper limit for the existence of Meissner solutions (the Ferrel-Prange field[4]); in the limit of very broad junctions, the fields are equal and are characterized by the quantity

$$H_* = \pi H_{c1}/2 = \Phi_0 / 2\pi\lambda_L\lambda_J$$

($\Phi_0 = \pi c\hbar/e$ is the quantum of the magnetic flux).

Since in a field $H = H_s$ the instability against the entry of vortices first arises near the edge of the junction, the question of the stability limits is closely connected with the investigation of the spectrum of the "plasma oscillations" of small perturbations of the phase φ_1 against the background of the equilibrium distribution $\varphi = \varphi_0(x)$ (the amplitude φ_1 is localized near the surface and attenuates in the interior of the junction.[5] If the frequency of such oscillations is imaginary ($\mathrm{Im}\,\omega > 0$), this means the appearance of instability.

The phase distribution in the contact is determined by the nonstationary Ferrel-Prange equation[4,6]

$$\Delta\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{\lambda_J^2}\sin\varphi. \tag{1}$$

Here $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator in the junction plane, $c_0 = (c^2 l/2\epsilon_0\lambda_L)^{1/2}$ is the velocity of the electromagnetic waves in the barrier (ϵ_0 and l are the dielectric constant and the thickness of the layer of dielectric between the superconductors), and λ_L and λ_J are the London and Josephson depths of penetration.

On the basis of (1), the equation for small oscillations takes the form

$$\Delta\varphi_1 - \frac{1}{c_0^2}\ddot{\varphi}_1 = \lambda_J^{-2}\varphi_1\cos\varphi_0 \tag{2}$$

where $\varphi_0 = \varphi_0(x)$ is the "equilibrium" solution corresponding to the unperturbed state (Meissner or mixed).[7]

The boundary conditions in the case of a junction of finite width have the form (in dimensionless variables $x' = x/\lambda_J$, $H' = 4e\lambda_L\lambda_J H/\hbar c$

$$\left.\frac{d\varphi_0}{dx}\right|_{x=0} = \left.\frac{d\varphi_0}{dx}\right|_{x=L} = H_0. \tag{3}$$

Accordingly, the boundary conditions for the nonequilibrium increment φ_1 are

$$\left.\frac{\partial\varphi_1}{\partial x}\right|_{x=0} = \left.\frac{\partial\varphi_1}{\partial x}\right|_{x=L} = 0. \tag{4}$$

It should be noted that an additional condition to (4) is a sufficiently rapid decrease of φ_1 with increasing distance from the edges of the junction (see footnote 3 below). For an investigation of the spectrum of the surface waves it suffices therefore to consider a semi-infinite junction with boundary conditions

$$\left.\frac{\partial\varphi_1}{\partial x}\right|_{x=0} = 0, \quad \varphi_1(\infty) = 0. \tag{5}$$

In the Meissner state, the distribution of the phase $\varphi_0(x)$ is expressed by the formula (the solution for the isolated vortex in the stationary case)

$$\varphi_0^{1,2}(x) = 2\arcsin\{\mathrm{ch}^{-1}[x \pm C(h)]\}, \tag{6}$$

C(h) is the distance from the nearest maximum on the field to the edge of the transition and is connected with the applied field H_0 by the relation ($h = H_0/H_S$)

$$C(h) = \mathrm{arcch}(h^{-1}). \tag{7}$$

The two signs in formula (6) correspond to an isolated vortex, the maximum distribution of the field of which is located outside and inside the junction. Formula (6) was obtained as the limiting transition, as $\gamma \to 1$, from the general distribution of the phase[7] for the mixed state, determined by the expression

$$\sin\frac{\varphi_0 - \varphi_0(0)}{2} = \mathrm{sn}(x/\gamma, \gamma), \quad 0 \leqslant \gamma \leqslant 1. \tag{8}$$

The quantity $\varphi_0(0)$ in (8) (the value of the phase $\varphi_0(x)$ at $x = 0$) is determined by the applied field

$$\varphi_0(0) = \pm 2\arccos(\gamma^{-2} - h^2)^{1/2} \tag{9}$$

(the coordinate x is measured in units of λ_J). We represent the solution (8) in a form more convenient for the subsequent analysis:

$$\cos\frac{\varphi_0(x)}{2} = \mathrm{sn}\left(\frac{x \pm A(\gamma)}{\gamma}, \gamma\right). \tag{10}$$

The quantity $A(\gamma)$ is determined by an incomplete elliptic integral of the first kind:

$$A = \gamma F\left(\frac{\pi - \varphi_0(0)}{2}, \gamma\right) \tag{11}$$

($A(\gamma)$ determines the position of the junction point closest to the edge, where the field is maximal).

The solution of Eq. (2) is

$$\varphi_1(x, y, t) = \Psi(x)\exp(iky - i\omega t), \tag{12}$$

where $\Psi(x)$ satisfies the "Schrödinger equation":

$$\Psi'' + 2[E - U(x)]\Psi = 0 \tag{13}$$

with periodic potential

$$U(x) = -\gamma^2 \mathrm{cn}^2(x \pm A) \tag{14}$$

and boundary conditions $\Psi'(0) = 0$ and $\Psi(\infty) = 0$.[1)] In the foregoing formulas we introduce the quantities $E = \epsilon\gamma^2$, $\epsilon = (\omega^2 - k^2 - 1)/2$ (the wave vector is measured in units of λ_J^{-1} and the frequency ω in units of the Josephson plasma frequency $\omega_0 = c_0\lambda_J^{-1}$). Equation (13) is the Lame equation (see [8,9]):

$$\Psi'' = [n(n+1)\gamma^2 \mathrm{sn}^2(x \pm A) + B]\Psi \tag{15}$$

with $n = 1$, $B = -2(\epsilon + 1)\gamma^2$, the exact solution of which (which decreases at infinity) is of the form

$$\Psi(x) = \frac{\mathrm{H}(x - \alpha \pm A)}{\Theta(x \pm A)} e^{-q(x\pm A)}, \tag{16}$$

$\mathrm{H}(x)$ and $\Theta(x)$ are the Eta and Theta functions, q plays the role of the wave vector in the direction normal to the boundary. The value of q is fixed by the condition $\Psi'(0) = 0$. As expected, $\mathrm{Re}\, q > 0$, corresponding to surface waves that attenuate in the interior of the junction.

The dispersion relations are determined in parametric form (α is the parameter):

$$q = Z(\alpha, \gamma), \quad \omega^2 = k^2 + \Omega^2(\alpha, \gamma). \tag{17}$$

Here $\Omega(\alpha, \gamma) = \gamma^{-1}\mathrm{dn}\alpha$ (dn is the delta-amplitude) is the threshold frequency of the plasma oscillations and $Z(\alpha, \gamma)$ is the Jacobi Zeta function, which can be expressed in simple form in terms of the Euler integrals of the first and second kind:

$$Z(\alpha, \gamma) = E(\alpha, \gamma) - \alpha\mathbf{E}(\gamma)/\mathbf{K}(\gamma). \tag{18}$$

The value of q, and consequently also the form of the threshold function $\Omega^2(\gamma)$ is determined from the boundary condition $\Psi'(0) = 0$. Differentiating (16) and using the periodicity properties of elliptic functions[9] we obtain after a number of transformations ($x = 0$)

$$\mathrm{cn}(\alpha \pm A)\mathrm{dn}(\alpha \pm A) \mp \gamma^2 \mathrm{sn}\,\alpha\, \mathrm{sn}\, A\, \mathrm{sn}^2(\alpha \pm A) = 0. \tag{19}$$

Using the relations between the functions $\mathrm{cn}\,\alpha$, $\mathrm{sn}\,\alpha$, $\mathrm{dn}\,\alpha$, we can easily reduce (19) to the following equation with respect to the threshold function $\Omega^2(h)$:

$$F(\Omega, h) = 0, \tag{20}$$

where

$$F(\Omega, h) = \eta\eta_2(1 - \Gamma\eta_1^2) \mp \eta_1(\eta_2^2 + \Gamma\eta^2) \mp \\ \mp \Gamma\eta_1^2[(hh_2\Omega_1)^2 + (h_1\Omega\Omega_2)^2]. \tag{21}$$

In the last expression we have introduced the notation:

$$\eta = h\Omega, \quad h_1 = \sqrt{\gamma^{-2} - h^2}, \quad \Omega_1 = \sqrt{\gamma^{-2} - \Omega^2}, \\ \eta_{1,2} = h_{1,2}\Omega_{1,2}, \quad h_2 = \sqrt{1 - h_1^2}, \quad \Omega_2 = \sqrt{1 - \Omega_1^2}, \quad \Gamma = \gamma^2, \tag{22}$$

$h = H_0/H_S$ is the dimensionless magnetic field, and $\Omega = \omega/\omega_0$ is the dimensionless threshold frequency of the small oscillations. Equation (21) can be readily reduced to an equation of fourth degree in the sought function $\Omega^2(\gamma)$ ($u = \sqrt{\gamma^{-2} - h^2}$, $t = \sqrt{\gamma^{-2} - \Omega^2}$):

$$[(1 - u^2)(1 - \gamma^2 u^2)(1 - t^2)(1 - \gamma^2 t^2)]^{1/2} \\ = \pm ut[1 + \gamma^2(1 - u^2 - t^2)]. \tag{23}$$

One root of this equation (of second multiplicity), negative for all h, is[2)]

$$\Omega^2 = \gamma^{-2}(h^2 - h_{min}^2)/(h^2 - h_{max}^2), \tag{24}$$

where $h_{min} = \sqrt{1 - \gamma^2}/\gamma$, $h_{max} = \gamma^{-1}$ (see below), and the remaining two are given by

$$\Omega_{1,2}^2 = \frac{1}{2\gamma^2}\{2 - \gamma^2(1 + h^2) \\ \pm [\gamma^2(1 - h^2)^2 + 4h^2(1 - \gamma^2 h^2)]^{1/2}\}. \tag{25}$$

Here $\Omega_1^2 > 0$ and $\Omega_2^2 < 0$ for all γ. For branch 2, the frequency turns out to be imaginary (at sufficiently small k, see (17)), and therefore the corresponding solution is unstable.

In broad junctions (without allowance for the influence of the boundaries of the contact), the solution of Eq. (1) for the equilibrium phase $\varphi_0(x)$ is parametrized by the quantity γ[7] (the value of γ determines the period of the vortex structure $a(\gamma) = 2\lambda_J\gamma\mathbf{K}(\gamma)$). The

1)The boundary conditions differ from the corresponding conditions of quantum mechanics.

2)Actually the root Ω^2, defined by formula (24), is extraneous and has no bearing on the problem in question, as follows from an analysis carried out in the calculation of (21), and also from an investigation of the asymptotic forms of Eq. (13).

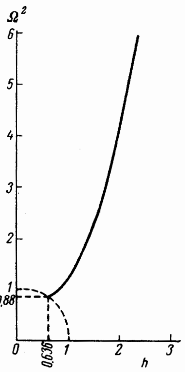

FIG. 1. Dependence of the threshold $\Omega^2(h)$ on the field in the mixed state for very broad junctions (solid curve). The dashed curve shows the dependence of the threshold frequency on h in the Meissner state (see also Fig. 3).

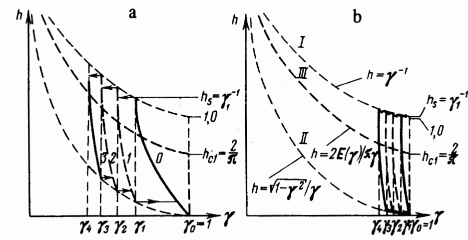

FIG. 2. Phase diagram of a tunnel contact of finite width in a magnetic field: a—case of relatively narrow contacts, $L = 5\lambda_J$; b—case of broad contacts, $L = 50\lambda_J$. The values of γ_n are the roots of the equation $L = na(\gamma)$. The curves I and II determine the values of the maximum and the minimum of the field at a specified value of $n\Phi_0$—the magnetic flux in the contacts. Curve III represents the thermodynamic-equilibrium connection between the magnetic field h and the parameter γ. Curves 0, 1, 2, and 3 describe the distribution of the field in the presence in the barrier of respectively 0, 1, 2, and 3 complete quanta of the magnetic flux.

choice of γ is based on considerations that the energy be a minimum, which leads to a connection between the "equilibrium" field and γ (see [7]):

$$h(\gamma) = 2\mathbf{E}(\gamma) / \pi\gamma. \tag{26}$$

The dependence of the threshold on the field for this case is shown in Fig. 1.

When account is taken of the influence of the boundaries of the contact, the value of γ is determined by the boundary conditions. In dimensionless variables $(d\varphi_0/dx)_b = h_0$, which leads to the equation

$$2b(\gamma, h) + na(\gamma) = L, \tag{27}$$

here L is the width of the junction, $b(\gamma, h) = \gamma\mathbf{K}(\gamma) - A(\gamma, h)$ is the distance to the minimum-field point closest to the edge of the junction, and n is the number of vortices in the contact.

Using the properties of elliptic functions, we can rewrite (27) in the form

$$F\left(\frac{\pi+\varphi_0(0)}{2}, \gamma\right) - F\left(\frac{\pi-\varphi_0(0)}{2}, \gamma\right) = \frac{L - na(\gamma)}{\gamma}. \tag{28}$$

Equation (28) admits of an exact analytic solution for the function $h(\gamma)$:

$$h(\gamma) = \gamma^{-1}(1-\gamma^2)^{1/2}\,\mathrm{dn}^{-1}\,[(L - na(\gamma))/2\gamma]. \tag{29}$$

An analysis of the formula (29) leads to the conclusion that for even n (n = 2k) the behavior of $h(\gamma)$ is determined by the expression

$$h_{2k}(\gamma) = (1-\gamma^2)^{1/2}/\gamma\,\mathrm{dn}\,(L/2\gamma),$$

and for odd n $(n \pm 2k + 1)$

$$h_{2k+1}(\gamma) = \gamma^{-1}\,\mathrm{dn}\,(L/2\gamma).$$

An investigation of these formulas makes it possible to reconstruct completely the picture of the distribution of the vortices (the phase diagram) in an arbitrary magnetic field (see Fig. 2). The point $\gamma_0 = 1$ corresponds to the absence of a field in the junction. At the point γ_1 there is only one vortex in the junction, and the field is equal to $h = \gamma_1^{-1}$. In this case the curve 0, as can be easily seen, corresponds to the Meissner (vortex-free) state, the upper limit of which is determined thus by the field:

$$h_s(L) = \gamma_1^{-1}(L), \quad \gamma_1 < 1, \tag{30}$$

where γ_1 is the root of the equation (the condition of the presence of one vortex in the junction)

$$L = a(\gamma). \tag{31}$$

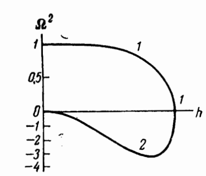

FIG. 3. Dependence of the threshold $\Omega^2(h)$ on the field in the Meissner state in semi-infinite junctions. Ω^2 has different scales along the positive and negative axes.

In very broad junctions $(L \gg \lambda_J)$, γ_1 is close to unity, and thus in this case $h_s = 1$ or in dimensional variables

$$H_s = \Phi_0 / 2\pi\lambda_L\lambda_J, \tag{32}$$

i.e., it coincides with the Ferrel-Prange critical field.[4] Further investigation of the phase diagram leads to the following result: to each value of the field h (on moving from weak fields) there corresponds a minimum possible number of vortices. When the field decreases, the junction will tend to "retain" the vortices, as a result of which, as seen from Fig. 2, a hysteresis of special type will occur. Curve I shows the behavior of h with variation of γ when the field increases, and curve II the same for a decreasing field. The dashed curve III represents the thermodynamic equilibrium. In Fig. 2a is shown the case of relatively narrow junctions,[3)] and Fig. 2b shows the opposite case. $\gamma_n(L)$ is the root of the equation $L = na(\gamma)$. At the points γ_n we have

$$h_{min}(L) = \gamma_n^{-1}\sqrt{1-\gamma_n^2}, \quad h_{max}(L) = \gamma_n^{-1}.$$

The question of finding the critical field h (for a semi-infinite contact) can be solved without resorting to the general diagram of Fig. 2. The threshold in the spectrum of the surface waves in the Meissner state has on the basis of (24) the form $(\gamma = 1)$

$$\Omega_{1,2}^2(h) = \frac{1}{2}(1 - h^2 \pm \sqrt{(1-h^2)(1+3h^2)}). \tag{33}$$

The form of the curves (33) is shown in Fig. 3. Curve 1 corresponds to the plus sign in (33), curve 2 to the minus sign (these values of the threshold frequency correspond to distributions of phases 1 and 2 in formula (6)).

[3)] We note that, as seen from (16), the damping of φ_1 occurs at a distance q^{-1} (q is defined in (17). Taking into account the condition $\varphi_1(\infty) = 0$ for a semi-infinite junction, a bounded contact can be regarded as semi-infinite when $L \gg q^{-1}$.

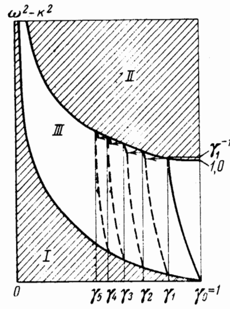

FIG. 4. Electromagnetic oscillations in a tunnel junction. I—thresholdless branch—"oscillations of the vortices"; II—region of "plasma oscillations," see [8]; III—region of surface waves. The solid lines in region III determine the threshold in the spectrum of the surface waves as a function of the parameter γ.

The spectrum of the waves represented by curve 2 is unstable in the entire range of fields. On the other hand, the spectrum of the waves described by curve 1 is stable up to a field $h_s = 1$ (cf. with formula (30)), at which exactly one vortex penetrates into the junction. It is possible to investigate analogously the spectrum of the plasma waves in junctions of limited widths as a function of the parameter γ. Substituting the function $h(\gamma)$, defined by formula (29), into the expression for the threshold (25), we obtain a set of $\Omega^2(\gamma)$ curves. The complete spectrum of the waves in the Josephson junction is shown in Fig. 4. As shown by Lebwohl and Stephen,[8] the spectrum of the "volume" oscillations is of the two-band type. On the other hand, the spectrum of the "surface" waves, as is clear from an investigation of $\Omega^2(\gamma)$, falls entirely inside the forbidden band of the "volume" oscillations.

In conclusion we note that although Eq. (1) with boundary conditions $(d\varphi_0/dx)_b = h_0$ has two solutions, an investigation of the spectrum of the plasma oscillations in the junction makes it possible to conclude that the only stable solution (both in the Meissner and in the mixed states) is the one with $dh_0/dx < 0$ on the edge of the junction (curve 1 on Fig. 3).

[1] C. P. Bean and J. D. Livingston, Phys. Rev. Lett. **12**, 14 (1964).

[2] V. P. Galaĭko, Zh. Eksp. Teor. Fiz. **50**, 717, 1322 (1966) [Sov. Phys.-JETP **23**, 475 (1966)].

[3] L. Kramer, Phys. Rev. **170**, 475 (1968).

[4] R. A. Ferrel and R. E. Prange, Phys. Rev. Lett. **10**, 479 (1963).

[5] A. E. Gorbonosov and I. O. Kulik, Fizika kondensirovannogo sostoyaniya, sb. stateĭ (Physics of the Condensed State, Collection of Articles), FTINT AN USSR **8**, 42 (1970).

[6] I. O. Kulik, ZhETF Pis. Red. **2**, 134 (1965) [JETP Lett. **2**, 84 (1965)].

[7] I. O. Kulik, Zh. Eksp. Teor. Fiz. **51**, 1952 (1966) [Sov. Phys.-JETP **24**, 1307 (1967)]; I. O. Kulik and I. K. Yanson, Éffekt Dzhozefsona v sverkhprovodyashchikh tunnel'nykh strukturakh (Josephson Effect in Superconducting Tunnel Structures), Nauka, 1970.

[8] P. Lebwohl and M. J. Stephen, Phys. Rev. **168**, 475 (1968).

[9] E. T. Whittaker and G. N. Watson, A Cource in Modern Analysis, Cambridge, 1940.

Translated by J. G. Adashko
76