SOVIET PHYSICS JETP VOLUME 33, NUMBER 1 JULY, 1971

SINGULARITIES OF THE FIELD OF AN ELECTROMAGNETIC WAVE IN A COLD ANISOTROPIC PLASMA WITH TWO-DIMENSIONAL INHOMOGENEITY

A. D. PILIYA and V. I. FEDOROV

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences

Submitted July 3, 1970

Zh. Eksp. Teor. Fiz. 60, 389–399 (January, 1971)

The singularities of the field in a cold anisotropic plasma with two-dimensional inhomogeneity are considered in connection with the problem of transformation of electromagnetic waves into plasma waves. It is shown that these features are connected with singular points of the characteristics of the equation div D = 0, $D_i = \epsilon_{ik}E_k$, $E = -\nabla\varphi$. The character of the field singularities is determined by the type of the singular point of the characteristic (a saddle point leads to a singularity on the line, and a node leads to a singularity at a point).

THE mutual linear transformation of electromagnetic and plasma waves in an inhomogeneous plasma is of considerable interest and has been investigated by many, see, for example[1-5]. In all cases, however, they considered only a plane-layered medium, i.e., a medium whose parameters depend on one spatial coordinate. Since such conditions are never encountered in practice, it is desirable to investigate the process of linear transformation under more realistic assumptions.

As is well known, linear transformation of electromagnetic waves into plasma waves in a plane layer is closely connected with the singularities of the field in a cold plasma[2,5,6]. It is just the presence of such singularities that is the necessary condition for the transformation, and the energy carried away by the plasma wave is equal to the energy absorbed in the cold plasma. This circumstance makes it in essence unnecessary to solve the transformation problem if the simpler problem is solved for the cold plasma. The simple physical consideration[2,6] from which the indicated connection of the transformation and absorption follow can be directly generalized to the case of an arbitrary inhomogeneity. It can therefore be stated that in the general case greatest interest attaches to the question of the singularities of the field in a cold plasma. The present article is devoted to investigations of the singularities of the field in the case when the parameters of the medium depend on two spatial coordinates. The model of two-dimensional inhomogeneity covers a large number of plasma configurations, including the toroidal system of the "Tokamak" type.

1. QUALITATIVE INVESTIGATION OF THE FIELD SINGULARITIES

For a plane medium, whose parameters depend on the coordinate X, the question of the position and type of singularities of the field is solved essentially in a very simple manner. The field of the wave near the singularity can be regarded as potential, $\mathbf{E} = -\nabla\Phi$, $\Phi = \varphi(X)\exp[ik_y y + ik_z Z]$. Using the equation

$$\operatorname{div}\mathbf{D} = 0, \quad D_i = \varepsilon_{ik}D_k,$$

where ϵ_{ik} is the dielectric tensor, and retaining in it only the terms containing derivatives of the rapidly-alternating function φ, we obtain

$$\varepsilon_{xx}\frac{d^2\varphi}{dX^2} + \left(\frac{d\varepsilon_{xx}}{dX} + i\sigma_0\right)\frac{d\varphi}{dX} = 0, \tag{1}$$

where $\sigma_0 = k_z(\epsilon_{xz} + \epsilon_{zx}) + k_y(\epsilon_{xy} + \epsilon_{yx})$. It follows from (1) that $d\varphi/dX$ can have a singularity only at the point $X = X_0$ at which $\epsilon_{xx} = 0$. In the vicinity of this point we can put $\epsilon_{xx} = (X_0 - X)/a$, and the remaining components of the tensor ϵ_{ik} can be regarded as constant. Then as $X \to X_0$ we have $\varphi = (X - X_0)^{i\sigma_1}$, $\sigma_1 = a\sigma_0(X_0)$ and $E_x = (X - X_0)^{-1+i\sigma_1}$[7].

If the plasma parameters depend on two coordinates, X and Z, then the situation becomes much more complicated, since it is necessary to consider not the ordinary differential equation (1) but the partial differential equation

$$\varepsilon_{xx}\frac{\partial^2\varphi}{\partial X^2} + 2\sigma\frac{\partial^2\varphi}{\partial X\partial Z} + \varepsilon_{zz}\frac{\partial^2\varphi}{\partial Z^2} + e_1\frac{\partial\varphi}{\partial X} + e_2\frac{\partial\varphi}{\partial Z} = 0, \tag{2}$$

where

$$2\sigma = \varepsilon_{xz} + \varepsilon_{zx}, \quad e_1 = \frac{\partial\varepsilon_{xx}}{\partial X} + ik_y(\varepsilon_{xy} + \varepsilon_{yx}) + \frac{\partial\varepsilon_{zx}}{\partial Z},$$

$$e_2 = \frac{\partial\varepsilon_{xz}}{\partial X} + ik_y(\varepsilon_{yz} + \varepsilon_{zy}) + \frac{\partial\varepsilon_{zz}}{\partial Z},$$

and, just as in (2), terms containing no derivatives of φ have been discarded.

We investigate first the question of the possible types of singularities of the solutions of Eq. (2) qualitatively. We assume that certain solutions $\varphi(X, Z)$ of Eq. (2) have a singularity on the smooth curve $X = X_0(Z)$, such that $|\partial\varphi/\partial n| \to \infty$ (n is the normal to the curve $X = X_0(Z)$), and attempt to determine the form of the equipotentials near this curve. The singular curve $X_0(Z)$ is also an equipotential, and since by assumption it is smooth, the same property should be possessed also by all the equipotentials in its vicinity. Therefore the quantity $\mu = (\partial\varphi/\partial Z)/(\partial\varphi/\partial X)$ should be a finite and continuous function of the coordinates in the entire region under consideration. Differentiating the entity $\partial\varphi/\partial Z = \mu\partial\varphi/\partial Z$ with respect to X and Z and eliminating with the aid of the thus-obtained equations $\partial^2\varphi/\partial X\partial Z$ and $\partial^2\varphi/\partial Z^2$ from Eq. (2), we obtain approximately (neglecting the quantities $(\partial\mu/\partial X)(\partial\varphi/\partial X)$ and $(\partial\mu/\partial Z)(\partial\varphi/\partial X)$ compared with $\mu\partial^2\varphi/\partial X^2$)

$$\varepsilon_{xx} + 2\sigma\mu + \varepsilon_{zz}\mu^2 = 0. \quad (3)$$

Consequently, the equipotentials should satisfy one of the ordinary differential equations

$$dX/dZ = -\mu_1, \quad dX/dZ = -\mu_2, \quad (4)$$

where $\mu_{1,2} = A \mp \sqrt{B}$ are the roots of the equation (3), $A = -\sigma/\epsilon_{zz}$, $B = (\sigma^2 - \epsilon_{xx}\epsilon_{zz})/\epsilon_{zz}^2$.

Thus, a singularity of the type in question is possible only in that region of the plasma where the quantities $\mu_{1,2}$ are real (i.e., Eq. (2) is hyperbolic), and the singular curve itself, as well as the equipotentials in its vicinity, coincide with one of the families of the characteristics of this equation. (Equation (2) is in general of the mixed type, i.e., in one part of the layer it is elliptic and in another it is hyperbolic.)

The general integral of Eq. (2) depends on two arbitrary functions, i.e., it includes a very large degree of leeway. It is clear that by using this leeway it is possible to make any characteristic singular. Such solutions, however, are not the analog of the singular solutions considered in the plane case, since they require very specific boundary conditions, which apparently are not realized in physical problems. On the other hand, if intersection of the characteristics belonging to one family (i.e., of the equipotentials) takes place, then this should lead to the occurrence of a singularity for a wide class of boundary conditions. Such an intersection or approach of the characteristics is possible if the solutions of (4) have singular points or asymptotes. The singularity considered above, in the case of plane geometry, pertains to this type, since the singular line $X = X_0$ is, as can be readily verified, the asymptote of one of the families of the characteristics. In the case of a bounded region, the asymptote can be only a closed curve[8]. However, the characteristics of Eq. (2) cannot be closed or spiral-like curves under reasonable assumptions concerning the distribution of the plasma concentration and of the magnetic field (e.g., in the case of a homogeneous magnetic field they are monotonic curves). In exactly the same way, there can exist no singular points of Eq. (4) inside the hyperbolic region (if the components of the tensor ϵ_{ik} have no singularity). It is therefore necessary to investigate the line $B(X, Z) = 0$ (see (4)), which separates the elliptic and hyperbolic regions. With the aid of Eq. (4) it is easy to establish that at an arbitrary point of this line there terminates only one characteristic of each family, and consequently this point is not singular. The only exceptions are the points at which the characteristics are tangent to the curve $B = 0$, i.e., points defined by the conditions[1)]

$$B = 0, \quad \frac{\partial B}{\partial X} A = \frac{\partial B}{\partial Z}. \quad (5)$$

Recognizing that in the coordinate system with a Z axis directed along the magnetic field the dielectric tensor is of the form $\epsilon_{xx} = \epsilon_{yy} = \epsilon$, $\epsilon_{xy} = -\epsilon_{yz} = ig$, $\epsilon_{zz} = \eta$ and the remaining components are equal to zero, we can show that the condition $B = 0$ is equivalent either to $\epsilon = 0$ or to $\epsilon \sin^2\alpha + \eta\cos^2\alpha = 0$, where $\alpha \equiv \alpha(X, Z)$ is the angle between the direction of the magnetic field and the XZ plane. The singular points on the curve $\epsilon = 0$ are those where this curve is tangent to the projection of the force lines of the magnetic field on the XZ plane, and the singular points on the curve $\epsilon \sin^2\alpha + \eta\cos^2\alpha = 0$ are those where this projection is perpendicular to the curve.

We locate the origin at one such point, and direct the Z axis along the tangent and the X axis along the inward normal (relative to the hyperbolic region) to the curve $B = 0$. At $X = Z = 0$ both ϵ_{xx} and σ vanish simultaneously. Therefore at small X and Z we can put

$$\varepsilon_{xx} = \frac{X}{a} + \frac{Z^2}{b}, \quad \sigma = \frac{Z}{a_\sigma}, \quad \varepsilon_{zz} = -\eta_0, \quad 0 < \eta_0 = \text{const}$$

(the discarded terms of the expansions lead to quantities of higher order in the equation of the characteristics).

Substituting these expressions in (4) and introducing the dimensionless variables $x = X/a$, $z = Z/a\sqrt{\eta_0}$, we obtain

$$dx/dz = \alpha_0 z \pm \sqrt{x + \alpha_1 z^2}, \qquad \alpha_0 = -\frac{a}{a_\sigma}, \quad \alpha_1 = \frac{a^2}{a_\sigma^2} + \frac{\eta_0 a^2}{b}. \quad (6)$$

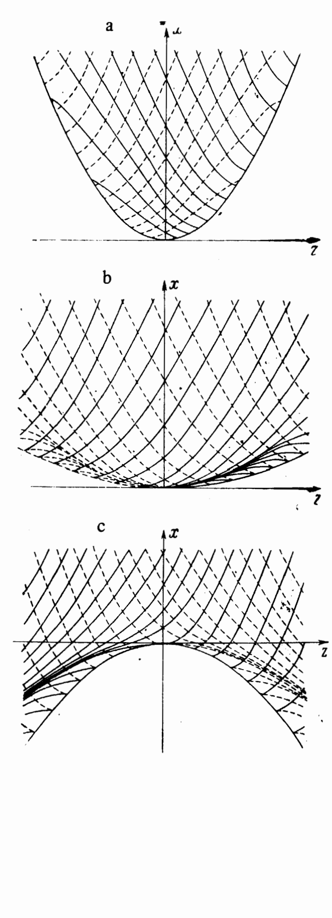
The general integral of (6) is of the form

$$u_1(x, z) = \text{const}, \quad u_2(x, z) = \text{const},$$

$$u_{1,2} = (\sqrt{x + \alpha_1 z^2} \pm \beta z)^{\beta} [\sqrt{x + \alpha_1 z^2} \mp (1/2 + \beta)]^{1/2 + \beta}, \quad (7)$$
$$\beta = 1/4(\sqrt{1 + 16\alpha} - 1), \quad \alpha = 1/2\alpha_0 + \alpha_1.$$

The function u_1 corresponds to the plus sign in (6).

From (7) we see that the singularity is determined



[1)]It should be noted that in the immediate vicinity of such points the equipotentials and the characteristics do not coincide, since near these points the terms of the type $(\partial\mu/\partial X)(\partial\varphi/\partial X)$ which were discarded in the derivation of (3), are not small compared with $\epsilon\partial^2\varphi/\partial X^2$. A rigorous analysis of the solutions of (2) will be presented in the next section.

by the value of the parameter α. Namely, at $\alpha < -1/16$ not a single characteristic passes through the point $x = z = 0$, and the point in the focus (see the figure, a). When $-1/16 < \alpha < 0$, an infinite number of characteristics terminate at the point $x = z = 0$, namely all the characteristics from the region lying between the parabolas $x = -\alpha_1 z^2$ and $x = [(\beta + 1/2)^2 - \alpha_1] z^2$ (see the figure, b). The singular point is the node. When $\alpha > 0$ one characteristic from each family passes through the point $x = z = 0$. These characteristics consist of the corresponding segments of the parabolas $x = [(\beta + 1/2)^2 - \alpha_1] z^2$ and $x = (\beta^2 - \alpha_1) z^2$ (see the figure, c). The singular point is the saddle.

With the aid of (7) it is easy to verify that if a certain characteristic passes at an infinitesimally small distance from the first of the indicated parabolas, then its distance from the second parabola will be an infinitesimally small quantity of higher order. Consequently, in the vicinity of the saddle point the characteristics, approach each other, and this should lead to the appearance of a singularity on the parabola $x + (\alpha_1 - \beta^2) z^2 = 0$ and on the entire length of the characteristic which is its continuation. Like any characteristic, this singular curve should reach, when continued, the boundary of the hyperbolic region $B = 0$. If the point of encounter of the singular characteristic with the boundary is not a singular point (node), then "reflection" of the singularity should take place, i.e., the second characteristic emerging from the point of encounter should also be singular. In addition to the singular line passing through the singular saddle point, the field of the wave should obviously have a singularity at the nodes; a singular point of the focus type should not lead to singularities of the field.

The foregoing cases cover all the situations under which a "real" singularity takes place in a bounded plasma, i.e., the field becomes infinite. In addition, there can exist apparently regions in which the field, while not becoming infinite, reaches anomalously large values and becomes sharply inhomogeneous because of the condensation of the characteristics. With the aid of Eq. (4) it is easy to establish that such a condensation can occur only in the case when the dependence of the plasma parameters on one coordinate is much weaker the dependence on the other, and under the condition when the characteristics make a small angle to the lines μ = const. This angle is equal to zero (the characteristics are tangent to the lines μ = const) at points satisfying the condition

$$\mu \partial\mu / \partial X - \partial\mu / \partial Z = 0.$$

The structure of the characteristics in the vicinity of the curve $X = X_1(Z)$, defined by this equation (this curve can be called the line of hybrid resonance), is best investigated by changing over in (4) to curvilinear coordinates S and q, where q is the line of the segment of the normal dropped from the point in question on the curve $X_1(Z)$, and S is the length of this curve from the origin to the base of the normal. If it is assumed that along the hybrid-resonance line the conditions $\rho \gg l$ and $q \ll l$ are satisfied, where ρ is the curvature radius of the lines μ = const, and $l^{-1} = |\nabla \tan^{-1}\mu|$, then Eq. (4) takes the form

$$\frac{dq}{dS} = -\frac{l}{\rho} + \frac{q}{l}.$$

It follows from this equation that the distance between the characteristics decreases along the system, roughly speaking, by a factor $e^{\rho/l}$ and accordingly, the amplitude increases and the effective wave-length of the field decreases. This concentration of the field occurs about the curve $q = l^2/\rho$ which passes near the hybrid-resonance line. In the case of a plane medium $\rho \to \infty$ and the field becomes infinite on the hybrid-resonance line. Thus, finite curvature leads to a limitation on the value of the field. If, however, the limitation of the field connected with allowance for spatial dispersion or collisions turns out to be stronger, then it can be assumed that effective transformation or absorption of the wave will take place on the hybrid-resonance line. We note that the singular points considered above also lie on the hybrid-resonance line, and that at these points $l = 0$.

2. SOLUTION NEAR THE SINGULAR POINTS OF THE CHARACTERISTICS

The considerations advanced above are of course purely qualitative. In the present section we present a more rigorous analysis, which, in particular, will make it possible to establish the type of field singularities arising at the indicated "singular" lines and singular points.

Equations of the mixed type, such as (2), have not been exhaustively investigated from the mathematical point of view[9,10]. One of the best investigated is the Tricomi equation

$$z\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0, \tag{8}$$

for which existence and uniqueness theorems have been proved for the solution in regions satisfying definite limitations. The topological structure of the characteristics of Eq. (8) is analogous to the structure of the characteristics of Eq. (2) far from the singular points (5). It can therefore be stated that in this region an analogous theorem is valid also for Eq. (2).

Let us consider the solutions of Eq. (2) in the vicinity of a singular point of the characteristics, assuming first for simplicity that $\epsilon_{XZ} = \epsilon_{ZX} = 0$, and using the same coordinate system and the same notation as in the investigation of the characteristics. Considering for the time being the solutions $\varphi \equiv \varphi(x, z)$, which do not depend on y, and retaining in (2) only terms with the derivatives φ, we obtain at small values of x and z

$$(x + \alpha z^2)\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial z^2} + \frac{\partial\varphi}{\partial x} = 0. \tag{9}$$

If we seek the solution of (9) in the form

$$\varphi = u^{\beta}\xi^{p}v(\zeta), \quad u = x + \frac{\beta}{2}z^2, \quad \xi = u^{2\beta}z, \quad \zeta = \frac{4u}{(1+4\beta)z^2},$$

where p is an arbitrary constant, then we obtain for the function $v(\zeta)$ the hypergeometric equation

$$\zeta(1-\zeta)\frac{d^2v}{d\zeta^2} + \left[1 + \beta(2p+1) - \left(\frac{3}{2} - p\right)\zeta\right]\frac{dv}{d\zeta} - \frac{p(p-1)}{4}v = 0.$$

Consequently, Eq. (9) has two particular solutions:

$$u^{\beta}\xi^{p}F\left[\frac{1-p}{2},\frac{-p}{2},1+\beta(2p+1);\zeta\right], \tag{10}$$

$$z^{p+2\beta(2p+1)}F\{{}^1/{}_2-{}^1/{}_2[p+2\beta(2p+1)],\ -{}^1/{}_2[p+2\beta(2p+1)],\ 1-\beta(2p+1);\ \zeta\}. \tag{11}$$

Particular interest attaches to the solution of the type (10) with $p = n$ and to (11) with $p + 2\beta(2p + 1) = n\,(n = 0, 1, 2\ldots)$, when the hypergeometric series terminates. Then the indicated integrals on the lines u = const

$$v_n = u^{\beta}(u^{2\beta}z)^nF\left(-\frac{n}{2},\frac{1-n}{2},\gamma_1;\zeta\right),$$

$$w_n = z^nF\left(-\frac{n}{2},\frac{1-n}{2},\gamma_2;\zeta\right),\quad \gamma_1 = 1+\beta(2n+1), \tag{12}$$

$$\gamma_2 = 1-\frac{\beta(2n+1)}{1+4\beta}$$

are polynomials of z. We consider now the vicinity of the saddle point ($\beta > 0$) and assume that the regular initial Cauchy data are specified on the line $u = u_0 < 0$ (u_0 = const). It is clear that, at least formally, any initial data can be satisfied by choosing the solution in the form of a superposition of particular solutions (12). On the line $u = 0$ the solution has, generally speaking, a singularity (branch point) representing the "singular" parabola considered in Sec. 1. The field becomes infinite on this line like $u^{-1+\beta}$ (we assume that $0 < \beta < 1$). If the initial data are subjected to limitations that lead to sufficiently good convergence of the indicated superpositions, then the line $u = 0$ is the only singular line of the solution.

In addition, there exists, however, solutions having a singularity on the line $4u = z^2(1 + 4\beta)$ (this line is the second characteristic parabola), and also solutions having an arbitrarily strong singularity on the line $u = 0$. Examples of such solutions may be expressions (10) for $p \neq n$ and $p < 0$. A similar situation occurs also in the case of plane geometry, when $\beta = \alpha = 0$. (When $\alpha = 0$ Eq. (9) describes the case of plane geometry with the concentration gradient perpendicular to the magnetic field.) Namely, by separating variables we obtain particular solutions of the type

$$\varphi_k = e^{ikz}[c_1J_0(2k\sqrt{x}) + c_2N_0(2k\sqrt{x})],$$

which have a singularity at $x = 0$. However, if we take the superposition of such functions, which converge in a non-uniform manner, then we can obtain solutions with singularities elsewhere, for example

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{ikz}J_0(2k\sqrt{x})\,dk = \frac{1}{\sqrt{4x-z^2}},\quad z^2 \leqslant 4x.$$

Such solutions are not considered in connection with problems of absorption and transformation of waves, and are apparently not realized in physical problems of this type.

On the basis of the analogy with the case of plane geometry, and also in view of the considerations advanced in Sec. 1, we shall assume that in the case of a plasma with a two-dimensional inhomogeneity the physical meaning is possessed by solutions represented by a convergent series of the functions (12).

Some additional properties of the solutions in the vicinity of the singular points can be deduced by using the general integral of Eq. (9). It can be verified directly that the general integral of (9) is

$$\varphi(x,z) = u^{\beta}\int_{-1}^{1}\Phi_1[u^{2\beta}(z-2\sqrt{u/(1+4\beta)}\,t)(1-t^2)^{2\beta}](1-t^2)^{-1/2+\beta}\,dt + \int_{-1}^{1}\Phi_2[(z-2\sqrt{u/(1+4\beta)}\,t)(1-t^2)^{-2\beta'}](1-t^2)^{-1/2-\beta'}\,dt, \tag{13}$$

where Φ_1 and Φ_2 are arbitrary functions and $\beta' = \beta/(1 + 4\beta)$. From (13) we can conclude that any solution of Eq. (9) dependent on two arbitrary functions has a singularity—branch point—at positive non-integer β. Indeed, for an arbitrary function Φ_1, such a singularity takes place on the line $u = 0$. It will be missing only in the case when $\Phi_1(\tau) \to 1/\sqrt{\tau}$ as $\tau \to 0$. In this case, however, there arises a singularity on the parabola $4u = z^2(1 + 4\beta)$. The solution will therefore be regular only if $\Phi_1 \equiv 0$; in other words, in the vicinity of the singular saddle point Eq. (9) has a regular solution that depends only on one arbitrary function. Thus, in the presence of a singular saddle point of the characteristics, the regular initial Cauchy data specified on a line lying in the elliptical region lead, generally speaking, to non-unique solutions in the hyperbolic region.

The "physical" solutions referred to above are all solutions of the type (13) with regular functions Φ_1 and Φ_2. At small values of u these solutions take the form

$$\varphi(x,z) = u^{\beta}F_1(u^{2\beta}z) + F_2(z),\qquad E_u \equiv \frac{\partial\varphi}{\partial u} = \frac{1}{u^{1-\beta}}F(u^{2\beta}z), \tag{14}$$

where F_1 and F_2 are arbitrary functions and $F(\tau) = \beta[F_1(\tau) + 2\tau dF_1(\tau)/d\tau]$.

The fact that in the vicinity of the singular saddle point the general solution of (9) has a singularity at least on one of the parabolas $u = 0$ and $4u = (1 + 4\beta)z^2$ is due to the fact that when either of them is approached the region of variation of the argument of the function in the integral (13) tends to zero. On the other hand, in the vicinity of a node, when $\beta < 0$, the situation changes: the behavior of the solution in the vicinity of the parabola $u = 0$ is determined by the form of the function $\Phi_1(\tau)$ as $\tau \to \infty$, and near the parabola $4u = (1 + 4\beta)z^2$ it is determined by its behavior as $\tau \to 0$. There can therefore exist solutions for which the field does not become infinite on any of these parabolas. However, in the saddle point itself, when $u \to 0$ and $z \to 0$ simultaneously, the region of variation of the argument in (13) can be arbitrary, depending on the limit of the ratio u/z^2 as $z \to 0$. Consequently, at the point $x = z = 0$ the field becomes infinite, but the character of the approach to infinity depends on the direction of the approach to the origin. Such a behavior of the solution corresponds to what can be expected at a node on the basis of the considerations of Sec. 1. We shall therefore assume that in physical problems of the type in question there are realized in the vicinity of the node solutions that are finite everywhere with the exception of the singular point itself[2)].

[2)]This is confirmed by the fact that when $\beta < 0$ the singularity on the line $u = 0$ arises only in the case when the field does not decrease when u = const and $z \to \infty$, or else decreases sufficiently slowly (see 13)). A clear explanation of this fact is that Eq. (9) itself describes an infinitely extended layer and the line $u = 0$ is the asymptote of the characteristics that go to infinity. There are obviously no such characteristics in a bounded plasma.

The singularity of the field on the line $u = 0$ (in the case of a saddle point) leads, as in the case of plane geometry, to finite absorption of the energy. This absorption can be described by adding to ϵ_{xx} a small imaginary increment, which is equivalent to replacing u by $u + i\delta$. The energy absorbed in the vicinity of the line $u = 0$ is proportional, when $\delta \to 0$, to the integral

$$\delta \int |E_u|^2 \, du \, dz.$$

Substituting here expression (14) for E_u and introducing new integration variables $t = u/\delta$ and $\tau = z\delta^{2\beta}$, we see that this integral does not depend on δ.

So far we have considered the particular case of Eq. (2) at $\epsilon_{xz} = \epsilon_{zx} = 0$ and $k_y = 0$. In the general case and for solutions of the type $\varphi = \varphi(X, Z) \exp(ik_y y)$ we confine ourselves to finding a solution in the approximation (14), by using exactly the same expansion for the components of the tensor ϵ_{ik} as in the derivation of (6). Changing over in (2) to new variables

$$u = x + {}^1/_2\beta_1 z^2, \quad \beta_1 = \beta - \alpha_0$$

and retaining only the term with the derivatives of the potential φ with respect to u, we obtain a first-order equation for $E_u \equiv \partial\varphi/\partial u$ (we are considering the case when $z^2 \sim |x| \ll 1$:

$$u\frac{\partial E_u}{\partial u} - 2\beta z\frac{\partial E_u}{\partial z} + (e_{10} - \beta_1 + e_{20}\beta_1 z)E_u = 0,$$

where

$$e_{10} = e_1 a|_{x=0,\, z=0}, \qquad e_{20} = \frac{e_2 a}{\sqrt{\eta_0}}\Big|_{x=0,\, z=0}.$$

The general integral of this equation is

$$E_u = u^{-e_{10}+\beta_1}\exp\left[\frac{e_{20}\beta_1 z}{2\beta}\right]F(u^{2\beta}z). \tag{15}$$

Using the explicit form of the coefficients e_{10} and β_1, we can readily show that $\beta_1 - e_{10} = \beta - 1 + i\nu$, where

$$\nu = a\left[k_y(\varepsilon_{xy} + \varepsilon_{yx}) + \frac{i}{2}\frac{\partial(\varepsilon_{xz} - \varepsilon_{zx})}{\partial Z}\right]_{x=0,\, z=0}$$

is a real constant, so that $|E_u|$ tends to infinity as $u \to 0$ in accordance with the same law as considered in the particular case above.

3. SOLUTION FAR FROM THE SINGULAR POINTS

Expressions (14) and (15) are suitable only in the vicinity of a singular point, i.e., at small x and z. Yet as indicated in Sec. 1, the solutions should have a singularity along the entire line representing the corresponding branches of the characteristics $u_1 = 0$ (when $z < 0$) and $u_2 = 0$ (when $z > 0$). Let us consider the solutions of Eqs. (2) in the vicinity of such a line, assuming for concreteness that $z < 0$. We change over in (2) from the variables X and Z to new variables $V_1(x, z)$ and z, where $V_1(x, z)$ is the solution of (4) that goes over into the function u_1 from (7) as $z \to 0$ and $x \to 0$. Then, recognizing that $\partial V_1/\partial x$ become infinite when $V_1 = 0$ and retaining in the region $|V_1| \ll 1$ only the terms proportional to $\partial V_1/\partial x$, we obtain

$$(\sigma + \mu_1\varepsilon_{zz})\frac{\partial^2\varphi}{\partial V_1\,\partial z} + d\frac{\partial\varphi}{\partial V_1} = 0,$$

where

$$d = \frac{a\sqrt{\eta_0}}{2}\left[e_1 + \mu_1 e_2 + 2\sigma\frac{\partial\mu_1}{\partial X} + \varepsilon_{zz}\left(\mu_1\frac{\partial\mu_1}{\partial X} + \frac{\partial\mu_1}{\partial Z}\right)\right]_{V_1=0},$$

and it can be assumed that the coefficients in this equation depend only on z. Then we obtain for the field $E_V \equiv \partial\varphi/\partial V_1$

$$E_V = \Phi(V_1)\exp\left\{-\int\frac{d}{\sigma + \mu_1\varepsilon_{zz}}dz\right\}, \tag{16}$$

where Φ is an arbitrary function.

We shall show now that expression (16) goes over into (15) at small z and at a suitably chosen function Φ. To this end we note, first, that by introducing in (7) the quantity $u = x + (1/2)\beta_1 z^2$ in place of x and assuming $|u| \ll z^2$, we obtain

$$V_1 = u_1 = u^\beta(-z)^{1/2}c, \quad c = ({}^1/_2 + 2\beta)^{1/2+\beta}/(2\beta)^\beta.$$

Introducing in similar fashion the quantity $u = (x - x_0)$, where $x_0(z)$ is the equation of the "singular" characteristic far from the singular point, and solving the characteristic equation (4) linearized with respect to u, we find that the general integral of this equation V_1, which goes over into u_1 when $z \to 0$, is given by

$$V_1 = cu^\beta\sqrt{-z}\exp\left\{\beta\int_0^z\left[\sqrt{\eta_0}\frac{\partial\mu_1}{\partial x}\Big|_{x=x_0} - \frac{1}{2\beta z'}\right]dz'\right\}.$$

Substituting this expression in (16) and putting

$$\Phi(\tau) = \tau^\gamma F(-\tau^2/c^2), \quad \gamma = (-e_{10} + \beta_1 + 1)/\beta - 1,$$

where F is the arbitrary function contained in (15), we can verify that when $z \to 0$ expressions (15) and (16) coincide, apart from an inessential constant multiplier.

Thus, the electric field $E = E_V \partial V_1/\partial u$ far from the singular point can be represented in the form

$$E_u = u^{-e_{10}+\beta_1}F(u^{2\beta}ze^{2f_1})\exp\left[-f_2 + (\gamma+1)f_1\right], \tag{17}$$

where

$$f_1 = \beta\int_0\left(\sqrt{\eta_0}\frac{\partial\mu_1}{\partial x}\Big|_{u=0} - \frac{1}{2\beta z'}\right)dz',$$

$$f_2 = \int_0\left(\frac{d}{\sigma + \mu_1\varepsilon_{zz}} + \frac{e_{10} - \beta_1 - 1}{2\beta z'}\right)dz'.$$

It follows from (17) that the type of singularity of the field is determined by the parameters of the plasma in the vicinity of the singular point.

Of definite interest is also the structure of the "singular" curve at the place where it encounters the parabolic line. If such an encounter point is not a "singular" point (node), then the curvature of the boundary does not play any role and we can use the model equation

$$(\sigma^2 - z)\frac{\partial^2\varphi}{\partial x^2} + 2\sigma\frac{\partial^2\varphi}{\partial x\,\partial z} + \frac{\partial^2\varphi}{\partial z^2} = 0, \quad \sigma = \text{const}, \tag{18}$$

for which the structure of the characteristics

$$u_{1,2} = x - \sigma z \pm {}^2/_3 z^{3/2}$$

is analogous in the vicinity of the parabolic line $z = 0$ to the structure of the characteristics of Eq. (2).

In the "characteristic" variables u_1 and u_2, Eq. (18) is an Euler-Poisson equation[9], the general integral of which is known. Using this circumstance, we can write

$$\frac{\partial\varphi}{\partial x} = \int_{x_1}^{x_2}\Phi_1(t)\frac{dt}{[(x_2 - t)(t - x_1)]^{1/6}}$$

$$+(x_1-x_2)^{2/3}\int_{x_1}^{x_2}\Phi_2(t)\,\frac{dt}{[(x_2-t)(t-x_1)]^{5/6}},\qquad(19)$$

where x_1 and x_2 are the coordinates of the points where the characteristic passing through the considered point x, z encounter the parabolic line z = 0, and Φ_1 and Φ_2 are arbitrary functions, with $\Phi_2(x) \sim \partial\varphi/\partial x|_{z=0}$ and $\Phi_1(x) \sim \partial^2\varphi/\partial x\partial z|_{z=0}$. We assume now that at a certain point x = 0 of the parabolic line there arises a "singular" characteristic. Then at least one of the functions $\Phi_1(x)$ or $\Phi_2(x)$ should have a singularity at x = 0, for only in this case will the field have a singularity along this characteristic. But then it follows from (19) that a similar singularity will be possessed by the field also on the second characteristic that emerges from the point x = 0, i.e., a "reflection" of the singular point from the parabolic line takes place.

In conclusion, let us consider the solution of the wave equation in a plane-layered medium in the vicinity of the asymptote of the characteristics, without assuming that the dependence on the coordinate z has the form of a plane wave (the inhomogeneity is directed along the x axis). Such solutions are interesting in connection with the fact that they are analogous to the solutions in the vicinity of the line of hybrid resonance in a bounded plasma. Using variables analogous to those employed in Sec. 1, the wave equation (2) in the vicinity of the asymptotes of the characteristic x = 0 can be represented in the form

$$x\frac{\partial^2\varphi}{\partial x^2}+2\sigma\frac{\partial^2\varphi}{\partial x\,\partial z}-\frac{\partial^2\varphi}{\partial z^2}+\frac{\partial\varphi}{\partial x}=0.$$

The general integral of this equation is of the form

$$\varphi=\int_{-1}^{1}\Phi_1\left[\exp\left(-\frac{z+2\sqrt{x}\,t}{2\sigma}\right)(1-t^2)x\right]\frac{dt}{\sqrt{1-t^2}}$$

$$+\int_{-1}^{1}\Phi_2\left[\exp\left(\frac{z+2\sqrt{x}\,t}{2\sigma}\right)(1-t^2)\right]\frac{dt}{\sqrt{1-t^2}},$$

where Φ_1 and Φ_2 are arbitrary functions. If they are regular functions of their arguments, which corresponds to specifying regular initial data on the line intersecting the line x = 0, then the field has no singularities at finite z. However, with increasing z, the dependence of φ on x becomes increasingly stronger and, in particular, the field component E_x increases at small x exponentially. Such a behavior of the solution corresponds fully to the qualitative picture based on the concept of condensation of equipotentials.

[1] V. V. Zheleznyakov and E. Ya. Zlotnik, Izv. Vuzov Radiofizika 5, 644 (1962).

[2] V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in a Plasma), Nauka, 1967.

[3] S. S. Moiseev, Paper at 7th Internat. Conference on Phenomena in Ionized Gases, Belgrade, 1965.

[4] T. H. Stix, Phys. Rev. Lett., 15, 878 (1965).

[5] A. D. Piliya and V. I. Fedorov, Zh. Eksp. Teor. Fiz. 57, 1198 (1969) [Sov. Phys.-JETP 30, 653 (1970)].

[6] A. D. Piliya, Zh. Tekh. Fiz. 36, 2103 (1966) [Sov. Phys.-Tech. Phys. 11, 1567 (1967)].

[7] V. V. Dolgopolov, ibid. 36, 273 (1966) [11, 198 (1966)].

[8] V. V. Nemytskiĭ and V. V. Stepanov, Kachestvennaya teoriya differentsial'nykh uravneniĭ (Qualitative Theory of Differential Equations), Gostekhizdat, 1949.

[9] F. Tricomi, Lectures on Partial Differential Equations, (Russian Translation), IIL, 1957.

[10] M. M. Smirnov, Vyrozhdayushchiesya éllipticheskie i giperbolicheskie uravneniya (Degenerate Elliptic and Hyperbolic Equations), Nauka, 1966.

Translated by J. G. Adashko
44