FLUCTUATIONS IN A MAGNETOHYDRODYNAMIC SHOCK WAVE

A. I. AKHIEZER, R. V. POLOVIN, and V. V. ROZHKOV

Physico-technical Institute, Ukrainian Academy of Sciences

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We investigate the influence of fluctuations on a shock wave propagating in a plasma transversely to an external magnetic field. The distribution function of the amplitudes of the magnetic field behind the shock wave is determined. It is shown that the fluctuations can lead to a collapse of the shock wave, and the lifetime of the shock wave is determined. This time turns out to be very short at Mach numbers close to unity.

1. IN the investigation of shock waves it is customary not to take into account the fluctuations of various physical quantities. Therefore, if the states of the medium ahead of the shock wave and the shock-wave intensity are strictly specified, the state of the medium behind the shock wave also is strictly defined. Yet fluctuations can play an important role in shock waves, for they not only give rise to certain distributions of different physical quantities characterizing the shock wave relative to their mean values (which are related by the shock adiabat), but can also lead to a collapse of the shock wave.

The purpose of the present article is to investigate the role of fluctuations in a magnetohydrodynamic shock wave propagating in a plasma transversely to an external magnetic field.

2. Let such a wave, which we assume to be plane (it was first investigated by Sagdeev[1]), propagate in the negative z direction with a velocity u_1. We use a coordinate system moving together with the shock wave. In this system, the state of the plasma does not depend on the time, and in addition, all the quantities do not depend on the coordinates x and y. Ahead of the wave front ($z = -\infty$) the state of the plasma is characterized by the variables

$$n = n_1, \quad u_x = u_y = 0, \quad u_z = u_1,$$
$$H_z = H_1, \quad H_y = H_z = 0, \quad E_x = E_z = 0, \quad E_y = -u_1 H_1 / c.$$

According to [1], the profile of the wave is described by the equations

$$\begin{gathered} m_e \frac{du_y}{d\tau} = \frac{e}{c}(u_x H_1 - u_z H_x) + R_y, \\ m_i \frac{du_z}{d\tau} = \frac{e}{c} u_y H_x, \quad \frac{dH_x}{d\tau} = -\frac{4\pi n_1 u_1 e}{c} u_y, \\ n u_z = n_1 u_1, \quad m_e \frac{du_x}{d\tau} = \frac{e}{c} u_z H_y + R_x, \\ \frac{dH_y}{d\tau} = \frac{4\pi n_1 u_1 e}{c} u_x, \end{gathered} \tag{1}$$

where we have introduced the "effective time" τ, $d\tau = dz/u_z$, $\mathbf{u}$ is the hydrodynamic velocity of the electronic components, n is the electron density, equal to the ion density (the quasineutrality condition is assumed satisfied), e is the electron charge, $m_{e,i}$ are the masses of the electron and of the ion (since $m_e \ll m_i$, the transverse motion of the ion is disregarded), $\mathbf{R} = -m_e \nu \mathbf{u}$, and ν is the frequency of the electron-ion collisions. Finally, the particle velocity spread is assumed to be sufficiently small, $\beta = 8\pi nT/H^2 \ll 1$ (T is the plasma temperature behind the shock wave), so that the hydrodynamic pressure and the viscosity forces are disregarded.

Eliminating the quantities u_y and u_z from the first four equations of (1), we obtain the following equation for the magnetic field H_x:

$$\frac{d^2H_x}{d\tau^2} + \nu\frac{dH_x}{d\tau} + \frac{dV(H_x)}{dH_x} = 0, \tag{2}$$

where

$$V(H_x) = \frac{\omega_h^2 M^2}{2}(H_x - H_1)^2\left[\frac{(H_x + H_1)^2}{16\pi m_i n_1 u_1^2} - 1\right],$$

and ω_h and M are the hybrid frequency and the Mach number ahead of the shock wave,

$$\omega_h = \frac{eH_1}{c\sqrt{m_e m_i}}, \quad M = \frac{u_1\sqrt{4\pi m_i n_1}}{H_1}.$$

From the last two equations of (1) it follows that $u_x = H_y = 0$.

Equation (2) describes damped nonlinear oscillations of a material point (unit mass) in a field with potential energy $V(H_x)$ about the value of the magnetic field H_2:

$$H_2 = -\frac{H_1}{2} + \sqrt{\frac{H_1^2}{4} + 8\pi m_i n_1 u_1^2}$$

(H_2 is the magnetic field behind the shock wave, see Fig. 1).

Equation (2) with $\nu = 0$ has an energy integral

$$W \equiv \frac{\dot{H}_x^2}{2} + V(H_x) = \text{const}, \quad \dot{H}_x \equiv \frac{dH_x}{d\tau}, \tag{3}$$

corresponding to the integral curves shown in Fig. 2. The shaded region in Fig. 2 corresponds to negative particle density, $n < 0$, and is therefore not realized physically. This region is determined, in accordance with (1), by the inequality

$$H_1^2 + 8\pi m_i n_1 u_1^2 - H_x^2 < 0.$$

As seen from Fig. 2, the integral curves (3) pass through the region $n < 0$ when $W > 0$. The following inequality should therefore be satisfied

$$\tfrac{1}{2}\dot{H}_x^2 + V(H_x) \leqslant 0. \tag{4}$$

It is clear that when the friction forces are taken into account ($\nu > 0$) the phase point on the (H_x, $\dot{H}_x$)

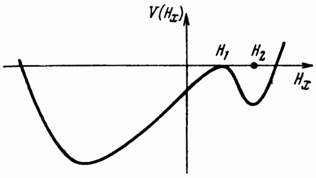

FIG. 1

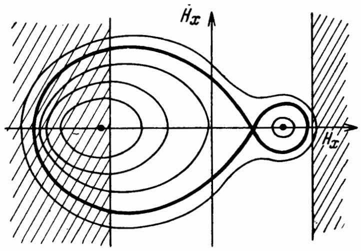

FIG. 2

plane will pass over from one integral curve to the other, corresponding to the smaller value of the "energy" W.

3. To take the fluctuations into account, it is necessary to introduce random forces into the equations of motion. Since the change of the entropy per unit volume of the plasma in a unit time is[2]

$$\frac{dS}{dt} = u_z\frac{\partial S}{\partial z} = -\frac{1}{T}\mathbf{Ru} = -\frac{R_x u_x + R_y u_y}{T}, \tag{5}$$

it follows that, in accordance with fluctuation theory,[3] the quantities R_x and R_y in (1) should be replaced by

$$R_x = -m_e \nu u_x + f_x(\tau), \quad R_y = -m_e \nu u_y + f_y(\tau),$$

where f_x and f_y are random forces satisfying the correlation conditions

$$\langle f_x(\tau)\rangle = \langle f_y(\tau)\rangle = 0,$$
$$\langle f_x(\tau_1) f_x(\tau_2)\rangle = \langle f_y(\tau_1) f_y(\tau_2)\rangle = T m_e \nu\delta(\tau_1 - \tau_2),$$
$$\langle f_x(\tau_1) f_y(\tau_2)\rangle = 0.$$

Repeating the derivation of Eq. (2) in the presence of random forces, we obtain the following equation for H_x:

$$\frac{d^2 H_x}{d\tau^2} + \nu\frac{dH_x}{d\tau} + V'(H_x) = F(\tau), \tag{6}$$

$$F(\tau) = -\frac{4\pi u_1 e n_1}{m_e c} f_x(\tau), \quad V'(H_x) \equiv \frac{dV(H_x)}{dH_x}$$

We see that the equation for H_x contains only the random force f_x, while the random force f_y enters only in the equations for the quantities H_y and u_x, which, as will be explained later, are of no interest.

Under the influence of the random forces, the phase point on the $(H_x, \dot H_x)$ plane will experience "Brownian motion" but it should not fall into the region shown shaded in Fig. 2, where $n < 0$, i.e., the inequality (4) should be satisfied also in the presence of random forces.

4. Equation (6) corresponds to the Fokker-Planck equation[4] for the distribution function $\Phi(H_x, \dot H_x; \tau)$ of the quantities H_x and $\dot H_x = dH_x/d\tau$ at the "instant of time" τ:

$$\frac{\partial\Phi}{\partial\tau} = -\dot H_x\frac{\partial\Phi}{\partial H_x} + \nu\frac{\partial(\Phi\dot H_x)}{\partial\dot H_x} + V'(H_x)\frac{\partial\Phi}{\partial\dot H_x} + 2\pi M^2 n_1 T\nu\omega_h^2\frac{\partial^2\Phi}{\partial\dot H_x^2}. \tag{7}$$

Introducing the amplitude $a(\tau)$ and the phase $\varphi(\tau)$ of the oscillations[5] $H_x = H_2 + a\cos\varphi$, $\dot H_x = -a\omega\sin\varphi$, where $\omega^2 \equiv V''(H_2) = \frac{1}{4}(1 + 12M^2 - 3\sqrt{1 + 8M^2})\omega_h^2$, and averaging the Fokker-Planck equation over the phase φ, we obtain the following equation for the distribution function with respect to the amplitude:

$$\frac{\partial\Phi}{\partial\tau} + \hat L\Phi = 0, \quad \hat L\Phi = -\frac{\partial I_a}{\partial a}, \tag{8}$$
$$I_a = \left(\frac{\nu a}{2} - \frac{2\pi n_1\nu T\xi(M)}{a}\right)\Phi + 2\pi n_1 T\nu\xi(M)\frac{\partial\Phi}{\partial a},$$

where

$$\xi(M) \equiv \frac{M^2\omega_h^2}{2\omega^2} = \frac{2M^2}{1 + 12M^2 - 3\sqrt{1 + 8M^2}}$$

The quantity I_a represents the "particle" flux density in the space a. We call attention to the fact that the expression for I_a does not contain $V(H_x)$ explicitly and the form of the "potential energy" $V(H_x)$ determines only the function $\xi(M)$.

Since the phase point should not fall in the region shown shaded in Fig. 2, the function $\Phi(a; \tau)$ should satisfy the condition

$$\Phi(a_0; \tau) = 0, \tag{9}$$

where the boundary value of the amplitude a_0 is determined by the equation $W = 0$ averaged over the phase φ:

$$\overline{{}^1/_2\dot H_x^2} + \overline{V(H_x)} = 0. \tag{10}$$

To obtain a_0 from this, we note that the distribution function $\Phi(a_0; \tau)$ is small in the region of large amplitudes a, and therefore it is possible to modify somewhat the potential $V(a)$ in the region of large a and to assume it to be throughout a quadratic function of H_x:

$$V(H_x) = {}^1/_2\omega^2(H_x - H_2)^2 + V(H_2).$$

Equation (10) then takes the form

$${}^1/_2 a_0^2\omega^2 = -V(H_2),$$

whence

$$a_0^2 = \frac{(\sqrt{1 + 8M^2} - 3)^2(4M^2 - 1 - \sqrt{1 + 8M^2})}{8(1 + 12M^2 - 3\sqrt{1 + 8M^2})}H_1^2. \tag{11}$$

Since the operator $\hat L$ has a singularity at $a = 0$, the solution $\Phi(a; \tau)$ should vanish when $a = 0$

$$\Phi(0; \tau) = 0. \tag{12}$$

5. It is easily seen that Eq. (8) admits of a stationary solution $\Phi_0(a)$:

$$\Phi_0(a) = Ca\exp\left[-\frac{a^2}{8\pi n_1 T\xi(M)}\right], \tag{13}$$

which satisfies the boundary condition (12), but not the boundary condition (9). Using this solution, we can rewrite (8) in the form

$$\frac{\partial\Phi}{\partial\tau} = 2\pi n_1\nu T\xi(M)\frac{\partial}{\partial a}\left[\Phi_0\frac{\partial}{\partial a}\left(\frac{\Phi}{\Phi_0}\right)\right] \equiv -\hat L\Phi.$$

It is easy to verify that the operator $\hat L$ which enters in this equation

$$\hat L\Phi = -2\pi n_1\nu T\xi(M)\frac{\partial}{\partial a}\left[\Phi_0\frac{\partial}{\partial a}\left(\frac{\Phi}{\Phi_0}\right)\right], \tag{14}$$

is self-adjoint[6,7] on the class of functions satisfying the boundary conditions (9) and (12), for the following definition of the scalar product:

$$(\Phi_1, \Phi_2) = \int_0^{a_0}\frac{\Phi_1(a)\Phi_2(a)}{\Phi_0(a)}da. \tag{15}$$

Therefore the solution of Eq. (8) can be sought in the form of the series

$$\Phi(a;\tau)=\sum_n c_n e^{-\lambda_n\tau}\Phi_n(a), \tag{16}$$

where $\Phi_n(a)$ is the eigenfunction of the operator $\hat{L}$, corresponding to the eigenvalue λ_m, $\hat{L}\Phi_n = \lambda_n\Phi_n$, and satisfying the boundary conditions (9) and (12).

Since the operator L is positive-definite, all its eigenfunctions are non-negative, $\lambda_n \geq 0$. It is easily seen that the minimum eigenvalue of this operator is positive. Indeed,[8]

$$\lambda_1=\min(\Phi,\hat{L}\Phi)=2\pi n_1 T\nu\xi(M)\min\int_0^{a_0}\Phi_0\left[\frac{d}{da}\left(\frac{\Phi}{\Phi_0}\right)\right]^2 da,$$

where Φ is an arbitrary function satisfying the conditions (9) and (12) and the normalization condition $(\Phi, \Phi) = 1$. Were it not necessary to satisfy the condition (9), then we would obtain $\lambda_1 = 0$ for $\Phi = \Phi_0$. If condition (9) is satisfied, on the other hand, $\Phi \neq \Phi_0$ and therefore $\lambda_1 > 0$.

At large values of τ, all the terms of (16) "die out" except for the first,

$$\Phi(a;\tau)\approx e^{-\lambda_1\tau}\Phi_1(a).$$

We see that the quantity $\tau_0 = 1/\lambda_1$ determines the "lifetime" of the distribution function $\Phi(a; \tau)$, i.e., the lifetime of the shock wave due to the influence of the fluctuations.

Since this time is large, the function $\Phi_1(a)$ differs little from $\Phi_0(a)$. Using this circumstance, we can easily show[9] that

$$\Phi_1(a)=Cae^{-x}\left[1+\frac{\Psi(x)}{\Psi(x_0)}\right], \tag{17}$$

$$\tau_0=\frac{1}{\nu}\Psi(x_0), \tag{18}$$

where $\Psi(x) = \overline{\mathrm{Ei}}(x) - \ln x - \gamma$,

$$x=\frac{a^2}{8\pi n_1 T\xi(M)},\qquad x_0=\frac{a_0^2}{8\pi n_1 T\xi(M)},$$

$\gamma = 0.577\ldots$ is the Euler function, and $\overline{\mathrm{Ei}}(x)$ is the modified integral exponential function[10, 11],

$$\overline{\mathrm{Ei}}(x)={}^1/_2[\mathrm{Ei}(x+i0)+\mathrm{Ei}(x-i0)],\qquad \mathrm{Ei}(z)=\int_{-\infty}^{z}\frac{e^t}{t}dt.$$

6. Let us see how the lifetime of the shock wave depends on the Mach number M, which, according to [1], lies in the interval $1 < M < 2$.

It follows from (18) that at Mach numbers close to unity $0 < M - 1 \ll \beta^{1/2}$, $\beta = 8\pi n_1 T/H_1^2$, the lifetime of the shock wave is

$$\tau_0={}^{64}/_{27}(M-1)^3/\beta\nu. \tag{19}$$

This quantity does not exceed the free path time of the electron $1/\nu$. Thus, at values of M close to unity the shock wave actually does not exist. With increasing M, τ_0 increases and when $\beta^{1/3} \ll M - 1 < 1$ it is determined by the formula

$$\tau_0=e^{x_0}/\nu x_0=\beta e^{1/\beta}/\nu. \tag{20}$$

This is much larger than the free-path time $1/\nu$.

Thus, the shock waves considered by us exist at Mach numbers exceeding $1 + \beta^{1/3}$.[1)]

7. Let us stop to discuss now the role of the random force f_y. We have assumed that $H_y = \dot{H}_y = 0$ for $f_y = 0$. At nonzero values of f_y, a nonzero field H_y is produced but it is much smaller than H_x if $\beta \ll 1$. The state of the shock wave should then be represented by the phase point on the two-dimensional space $(H_x, \dot{H}_x)$ in the four-dimensional space $(H_x, H_y, \dot{H}_x, \dot{H}_y)$. In this case the potential energy will depend not only on H_x but also on H_y:

$$V(H_x,H_y)=\frac{\omega_h^2M^2}{2}\left[-(H_x-H_1)^2-H_y^2+\frac{(H_x^2-H_1^2+H_y^2)^2}{16\pi m_i n_1 u_1^2}\right] \tag{21}$$

The state of the plasma behind the shock wave corresponds to the point $H_x = H_2$, $H_y = \dot{H}_x = \dot{H}_y = 0$, which obviously is a saddle point of the surface (21). In spite of the fact that the point represented in the state of the plasma behind the shock wave is a saddle and not a stable node (the surface (21) has two nodal points—an unstable node corresponding to the state ahead of the shock wave $H_x = H_1$, $H_y = \dot{H}_x = \dot{H}_y = 0$, and a stable node located in the inadmissible region $n < 0$), the profile of the shock wave is nonetheless stable. This is connected with the character of the permissible perturbations, namely, the values of H_x, H_y, $\dot{H}_x$, and $\dot{H}_y$ should be specified in the problem of the shock-wave structure not only at $\tau = -\infty$ (i.e., at a certain initial instant of time, as in the problem of motion of a material point), but also at $\tau = +\infty$, and furthermore in such a way that in the final state the phase point must be located in the saddle point. Indeed, the final state should correspond to a singular point, but this point can be only a saddle and not a stable node, which is not admissible since $n < 0$ for it. It can therefore be concluded that the fluctuations should be such as not to violate this condition; in other words, even in the presence of fluctuations the phase point representing the shock wave at $\tau \to +\infty$ should correspond to $H_y = \dot{H}_y = 0$.

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1) It is interesting to note that in a solitary wave (soliton), under this condition, there arises a current instability that leads to collisionless dissipation [12].

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