ZERO-CHARGE AND SCALE-INVARIANCE PROBLEMS IN THE SOLVABLE FIELD-THEORY MODEL

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Several static models are investigated and the possible behavior of the Green's function and of the vertex parts in the region where perturbation theory is not valid is ascertained with their exact solutions as an example. This analysis shows that the possible solution variants proposed in [2-5] (weak coupling, strong coupling, etc.) are actually realized, and it is possible to trace in explicit form how they can be matched with perturbation theory at high frequencies (energies).

1. INTRODUCTION

Many physical problems (analysis of the scattering amplitude at high energy, investigation of systems consisting of a large number of particles near the phase-transition points, etc.) reduce to a determination of the Green's function of a nonrelativistic quasiparticle (β), capable of decaying into an arbitrary number of different (or like) quasiparticles (α). The spectrum of the quasiparticles α has no gap, i.e., $\epsilon_\alpha(k) \to 0$ as $k^2 \to 0$. For example, when considering the behavior of the scattering amplitude at high energy, the role of the quasiparticles α is played by the vacuum reggeons, and the absence of the gap means that the spin of the Pomeranchuk pole at $k^2 = 0$ is equal to unity.[1]

When no account is taken of the decays we have $\epsilon_{0\beta}(k^2) = \beta' k^2$ and the free Green's function takes the form $G(\epsilon, k) = -1/(\epsilon - \beta' k^2)$. The absence of a gap in the spectrum of the quasiparticles α leads to decay thresholds $\beta \to n\alpha + \beta(\epsilon_{n\beta} = \alpha'\beta' k^2/(\alpha' + n\beta'))$, which are close at small k^2 to the position of the pole corresponding to the quasiparticle β. Owing to these singularities that are close to the pole, the self-energy part Σ, which determines the exact Green's function ($G^{-1}(\epsilon, k^2) = -\epsilon + \beta' k^2 - \Sigma(\epsilon, k^2)$), if calculated by perturbation theory (see, for example, [1]) is large at small ϵ and k^2, i.e., $\Sigma \gg -\epsilon + \beta' k^2$ as $\epsilon, k^2 \to 0$. This means that the interaction should essentially renormalize either the vertex parts of Γ, which determine the interaction force, or the spectrum of the quasiparticles $\epsilon_\beta(k^2)$ and $\epsilon_\alpha(k^2)$. Obviously, perturbation theory cannot be used in this problem, even if the constants that determine the decay amplitudes are small, and therefore it is necessary to use other methods when attempts are made to solve it.

The properties of the possible solutions of these problems were discussed in [2,3], with the self-action of the Pomeranchuk pole ($\beta = \alpha$) as an example. The main physical requirement imposed on the solution in this case was the presence of a pole in the Green's function at least at positive t ($t = -k^2$). (In the region $t > 0$ this pole corresponds at sufficiently large t to a really existing particle lying on the Regge trajectory $j = \beta(t)$). Depending on the character of the interaction, an investigation of problems of this type has revealed the presence of the following possible solution variants:

1. Nonsingular weak coupling. In this solution, Γ is of the order of unity when $k^2 \geq r^2$ (r^2 is the coupling constant of $\beta \to \alpha + \beta$) and Γ tends to zero in a nonsingular manner ($\Gamma \sim k^2$) as $k^2 \to 0$. (An example is $\Gamma = (1 + r^2/k^2)^{-1}$; $\Gamma = 1$ at $k^2 \gg r^2$ and $\Gamma = k^2/r^2$ at $k^2 \ll r^2$.) This solution is realized, in all probability, for the vacuum Regge pole.[2] The spectrum does not change in this solution.

2. Singular weak coupling. Γ tends to zero in a singular manner (for example, $\Gamma = [1 - r^2 \ln(k^2/L)]^{-1}$ is equal to $-1/r^2 \ln(k^2/L)$ as $k^2 \to 0$). This solution corresponds to the so-called "zero charge" and is discussed in detail, for example in [4], for the case of nonvacuum Regge poles. The spectrum likewise remains unchanged in the limit of small k^2, but the corrections to the spectrum in this case are more significant than for solution 1.

3. Strong coupling. Interaction at small k^2 leads to a strong change of the particle spectrum ($\epsilon_\beta \sim (k^2)^{1/\nu}$, and to a renormalization of G and Γ. The form of the self-consistent solution for this case was proposed in [3] ($G = \epsilon^{-\mu} f(k^2/\epsilon^\nu)$, $\Gamma = \epsilon^{-\nu} F(k^2/\epsilon^\nu)$) and was then applied to problems dealing with second-order phase transitions[5] and in nonvacuum Regge poles.[4]

4. Unphysical solution. It may turn out that in some cases at definite values of the interaction constants, the solutions satisfying our physical requirement that a pole be present at least at $t > 0$ do not exist at all. This means that either the pole of the Green's function goes off to infinity, or else the infrared situation arises (for details see [4]).

An essential shortcoming of the approach used in the cited paper,[2-5] however, is that it does not make it possible to obtain the solution in the entire region of the momenta, and it becomes necessary to make self-consistent hypotheses concerning the behavior of the Green's functions in the vertex parts at small momenta. It remains unclear whether this solution is unique, whether it can be matched to the results of perturbation theory at large momenta, and in general whether it can be realized, i.e., whether the solutions satisfy the self-consistent conditions of [3,5].

To clarify this problem, we have investigated a number of statistical models. It is possible to obtain for these problems, on the one hand, an exact expression for the Green's functions, and on the other hand, it is possible to realize in them all the solution types indicated above.

We shall consider, more concretely, a solvable nonrelativistic model with a Hamiltonian¹⁾

$$H = a_k^+ \varepsilon_{\alpha k} a_{-k} + b^+ b[r(a_k^+ + a_k) + 2\lambda a_{k_1}^+ a_{k_2} + \lambda_1 (a_{k_1}^+ a_{k_2}^+ + a_{k_1} a_{k_2})] \quad (1)$$

Here a_k and b are field operators; b is a static field corresponding to a nonrelativistic particle β with infinitely large mass; a_k is the field of the particle α with spectrum $\epsilon_\alpha(k^2) = \Delta + k^2$; r is the amplitude of the transitions $\beta \to \beta + \alpha$ and $\beta + \alpha \to \beta$; λ is the amplitude of the scattering $\alpha + \beta \to \alpha + \beta$; λ_1 is the amplitude of the transitions $\beta \to 2\alpha + \beta$, $\beta + 2\alpha \to \beta$. In (1) we imply integration with respect to k, k_1, and k_2 (we recall that for reggeons the integration with respect to the momenta is carried out in two-dimensional space and that r is pure imaginary. The parameters of the expansion of the perturbation theory for this Hamiltonian are of the order of r^2/Δ, $\lambda \ln \Delta$, $\lambda_1 \ln \Delta$ in the two-dimensional case and $r^2/\Delta^{3/2}$, $\lambda_1/\sqrt{\Delta}$, $\lambda/\sqrt{\Delta}$ in the one-dimensional case; at sufficiently small Δ (and small k^2), perturbation theory is not valid no matter how small the bare constants are.

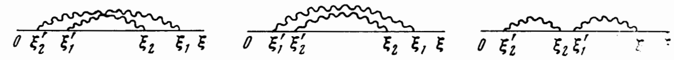
In Sec. 2 of the article we consider the problem with $\lambda_1 = 0$ and $\lambda \neq 0$, $r^2 \neq 0$. This problem has a solution of type 4 in accordance with the classification given above. In this section we discuss the properties of this solution and the limit to which it tends as $\Delta \to 0$.

In the remaining sections we consider the exact solution at $r^2 = 0$, $\lambda_1 \neq 0$, $\lambda \neq 0$ and investigate its properties for two-dimensional and one-dimensional systems. In Sec. 3 (which has a somewhat formal character) we give general formulas, equations, and their solutions, which do not depend on the concrete form of the system. In the following Sec. 4 we discuss in detail the one-dimensional system for which we obtain at $\lambda_1 \neq \lambda$ a singular weak coupling ($\Gamma_2 \sim \sqrt{\omega_1 \omega_2}$ as $\omega_1, \omega_2 \to 0$), and at $\lambda_1 = \lambda$ we obtain a solution of type 3 (i.e., strong coupling). The last case is of greatest interest and makes it possible to trace the matching of the results with those of perturbation theory at large momenta. For the two-dimensional system we obtain a singular weak coupling (see Sec. 5), and our solution makes it possible to verify the parquet equations,[4] with the aid of which the "zero charge" is usually proved.

Of course, the real physical problems that must be solved are quite far from this simplified model, but in our opinion its investigation makes it possible to regard the general analysis in [2-5] with greater confidence; in addition, the problem with $\lambda \neq 0$ and $\lambda_1 \neq 0$ has apparently never been solved before, and this may be of independent interest for certain problems of statistical physics.

2. THE CASE $\lambda_1 = 0$

When $\lambda_1 = 0$, the problem is solved in the simplest manner. We assume first for simplicity that λ is also equal to zero. We solve the problem in the ξ representation (ξ has the meaning of the imaginary time), in which the bare Green's function $G_0(\xi)$ of the nonrelativistic particle β is equal to $\vartheta(\xi)$ ($\vartheta(\xi > 0) = 1$, $\vartheta(\xi < 0) = 0$). It is easy to note that the total assembly of perturbation-theory diagrams for any static problem is obtained by fixing in each order of perturbation theory the coordinates of the absorption of particles α and by integrating in independent fashion with respect to the coordinates of emission of the particles α. For example, in order r^4 we have the following diagrams:



(2a)

which correspond to different integration regions:

$$\xi > \xi_1 > \xi_2 > \xi_1' > \xi_2' > 0, \quad \xi > \xi_1 > \xi_2 > \xi_2' > \xi_1' > 0,$$
$$\xi > \xi_1 > \xi_1' > \xi_2 > \xi_2' > 0$$

in the integral

$$r^4 \vartheta(\xi) \int^{\xi} d\xi_1 \int^{\xi_1} d\xi_2 \int^{\xi} d\xi_1' \int^{\xi} d\xi_2' D(\xi_1 - \xi_1') D(\xi_2 - \xi_2'), \quad (2)$$

where

$$D(\xi) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} d\omega\, e^{\omega\xi} D(\omega), \quad D(\omega) = \int \frac{d\mathbf{k}}{\omega + \varepsilon_\alpha(\mathbf{k})}.$$

We see thus that to construct the perturbation-theory diagrams it suffices to specify the order of the points of absorption of the particle α and integrate over the emission coordinates in independent fashion. We note that in obtaining the last equation of (2) we have made use of the fact that $D(\xi) = 0$ when $\xi < 0$. This is obvious from (2), for when $\xi < 0$ the contour of the integration with respect to ω can be closed on the right, and there are no singularities in the right-hand half-plane of $D(\omega)$.

Thus, the sum of diagrams of n-th order in r^2 is

$$(r^2)^n \vartheta(\xi) \int_0^{\xi} d\xi_1 \int_0^{\xi_1} d\xi_2 \ldots \int_0^{\xi_{n-1}} d\xi_n \int_0^{\xi} d\xi_1' \int_0^{\xi} d\xi_2' \ldots \quad (3)$$
$$\ldots \int_0^{\xi} d\xi_n' D(\xi_1 - \xi_1') \ldots D(\xi_n - \xi_n').$$

Changing over to the unordered region of integration with respect to the variables $\xi_1, \ldots, \xi_n$, we obtain for (3)

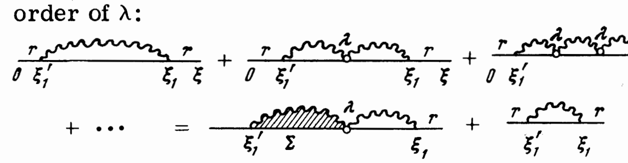
$$\frac{(r^2)^n}{n!} \vartheta(\xi) \left(\int_0^{\xi} d\xi_1 \int_0^{\xi} d\xi_1' D(\xi_1 - \xi_1') \right)^n \quad (4)$$

Consequently the Green's function of the heavy particle is equal to

$$G(\xi) = \vartheta(\xi) \sum_n \frac{(r^2)^n}{n!} \left(\int_0^{\xi} d\xi_1 \int_0^{\xi} d\xi_1' D(\xi_1 - \xi_1') \right)^n \quad (5)$$
$$= \vartheta(\xi) \exp\left\{ r^2 \int_0^{\xi} d\xi_1 \int_0^{\xi} d\xi_1' D(\xi_1 - \xi_1') \right\}.$$

when $\lambda \neq 0$, the solution (5) must be modified only slightly.

Let us consider diagrams of order r^2 but of arbitrary order of λ:



(6)

¹⁾Needless to say, problems with the Hamiltonian (1) can, in principle, be solved by the standard methods [6]; we claim merely a more complete investigation of them.

We can readily see how to sum them over the powers of λ. For the sum of these diagrams we can readily set up the equation

$$\Sigma(\xi_1-\xi_1')=r^2D(\xi_1-\xi_1')-\lambda\int_0^\infty D(\xi_1-\xi_1'')\Sigma(\xi_1''-\xi_1')d\xi_1''. \quad (6')$$

Changing to the ω representation, we obtain

$$\Sigma(\omega)=r^2D(\omega)/[1+\lambda D(\omega)]. \quad (7)$$

Summing over r^2 in exactly the same manner as at λ = 0, we obtain G(ξ) in the form

$$G(\xi)=\vartheta(\xi)\exp\left\{\int_0^\xi\int_0^\xi\Sigma(\xi_1-\xi_1')d\xi_1d\xi_1'\right\}, \quad (8)$$

or, substituting (7) in (8) and integrating explicitly with respect to ξ_1, we obtain

$$G(\xi)=\vartheta(\xi)\exp\left\{r^2\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i\omega^2}e^{\omega\xi}\frac{D(\omega)}{1+\lambda D(\omega)}\right\}. \quad (9)$$

The contour of integration with respect to ω in (9) passes to the right of all the singularities of the integrand (contour C of Fig. 1).

Let us consider now the properties of the solution obtained for the two-dimensional and one-dimensional systems. In the two-dimensional system

$$D(\omega)=\int\frac{d^2k}{\omega+\xi_\alpha(k^2)}=\int\frac{d^2k}{\omega+\Delta+k^2}=\pi\ln\frac{L}{\omega+\Delta}. \quad (10)$$

We shall henceforth omit the coefficient preceding the logarithm, assuming it to be included in r^2 and λ; L is the cutoff radius, which is determined, for example, by the succeeding terms of the expansion in powers of k^2 in the spectrum $\epsilon_\alpha(k^2) = \Delta + k^2 + k^4/L$.

Expression (9) is analytic in the right half-plane of ω (if λ > 0) and has a branch point at ω = − Δ. The picture of the singularities in the ω plane is shown in Fig. 1. When λ < 0 there is, in addition to these singularities, also a pole at some positive value $\omega = \omega_0$, due to the vanishing of the denominator of the integrand $1 + \lambda D(\omega_0) = 0$. Separating the contribution of this pole to (9), we get

$$G(\xi)\propto\vartheta(\xi)\exp\{ae^{\omega_0\xi}\}=\vartheta(\xi)\exp\{aS^{\omega_0}\},\quad \xi\to\infty. \quad (11)$$

Thus, when λ < 0 we have a solution that increases more rapidly than any power of S (ξ = ln S) with increasing S. Such a solution does not satisfy our physical requirements (we recall, for example, that if the particles α and β are taken to be reggeons, then G(ξ) is the scattering amplitude and S is the energy), and consequently the problem with λ < 0 has no solution. This was to be expected, since λ < 0 corresponds to attraction.

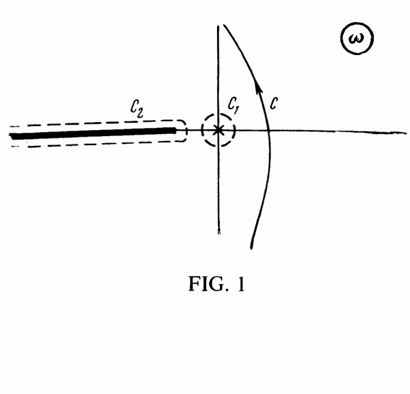


FIG. 1

Let us consider now the asymptotic form of G(ξ) as ξ → ∞ and at a finite gap Δ in the spectrum of the particles α. We close the contour C around the singularities of the integrand in (9) (see Fig. 1, the integration is along C_1 and C_2) and obtain

$$G(\xi)=\vartheta(\xi)\exp\left\{-\frac{r^2}{\Delta}\frac{1}{(1+\lambda l)^2}\right\}\exp\left\{\frac{r^2l}{1+\lambda l}\xi\right. \quad (12)$$

$$\left.+r^2\int_{-\infty}^{-\Delta}e^{\omega\xi}\frac{d\omega}{\omega^2}\left[\left(1+\lambda\ln\frac{L}{\Delta+\omega}\right)^2+\pi^2\lambda^2\right]^{-1}\right\},$$

$$l\equiv\ln(L/\Delta).$$

At fixed Δ and as ξ → ∞, the integral in (12) is of the order of $e^{-\Delta\xi}$, and consequently can be discarded in the asymptotic expression. Then

$$G(\xi)\to\vartheta(\xi)\exp\left\{-\frac{r^2}{\Delta}\frac{1}{(1+\lambda l)^2}+\frac{r^2l}{1+\lambda l}\xi\right\}. \quad (13)$$

Expression (13) corresponds to a pole of G(ω) at

$$\omega=\beta_0=\frac{r^2l}{1+\lambda l}. \quad (14)$$

Expanding (12) in powers of the integral

$$\int_{-\infty}^{-\Delta}(\ldots)\sim e^{-\Delta\xi},$$

we obtain the contributions made to G(ξ) by the branch points in the ω plane:

$$\omega_n=\beta_0-n\Delta. \quad (15)$$

What happens when Δ → 0? Obviously, all the thresholds come closer together and give rise at Δ = 0 to a complicated singularity at the point $\omega = r^2/\lambda$, leading to the following asymptotic form as ξ → ∞:

$$G(\xi)\to\exp\left\{\frac{r^2\xi\ln L\xi}{1+\lambda\ln L\xi}\right\}\to\exp\left\{\frac{r^2}{\lambda}\xi-\frac{r^2\xi}{\lambda\ln L\xi}\right\}. \quad (16)$$

In a one-dimensional system

$$D(\omega)=\int\frac{dk}{\omega+\Delta+k^2}=\frac{1}{\sqrt{\omega+\Delta}}$$

The analytic properties are the same as in the two-dimensional system, but the concrete formulas are modified in trivial fashion. In particular, $\beta_0 = r^2/(\sqrt{\Delta} + \lambda) \to r^2/\lambda$ as Δ → 0, and when ξ → ∞ we have

$$G(\xi)=\exp\left\{\frac{r^2}{\lambda}\xi+\text{const}\cdot\sqrt{\xi}\right\}. \quad (17)$$

Thus we see that we obtain for our problem a solution having no pole if Δ = 0. The Green's function has a finite limit here as Δ → 0, but the residue Z_β at the pole tends to zero (at $r^2 > 0$). In exactly the same manner, the admixture of states with a finite number of particles α is small, and the main contribution to the Green's function is made by states with n ~ Δ^{-1}, i.e., a situation analogous to the infrared catastrophe in electrodynamics sets in. This is most clearly manifest in the wave function of the β particles. It can be obtained in standard fashion[4] with the aid of a canonical transformation that annihilates the terms linear in a^+a in the Hamiltonian (1),

$$|\beta\rangle=\exp\left\{\int f(k)(a_k^+-a_k)dk\right\}b^+|0\rangle, \quad (18)$$

where

$$f(k)=\frac{r}{\varepsilon_\alpha(k)}\left(1+\lambda\int\frac{dk}{\varepsilon_\alpha(k)}\right)^{-1}. \tag{19}$$

The admixture of the n-particle state is described by the Poisson formula

$$|\langle\beta|a_1^+a_2^+\dots a_n^+b^+|0\rangle|\sim\exp\left\{-\int f(k)\,dk\right\}\frac{f_1^2\dots f_n^2}{n!}. \tag{20}$$

Inasmuch as at small momenta we have

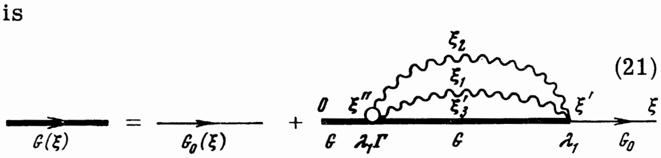
$$f\sim\frac{r}{\Delta}\frac{1}{1+\lambda l},$$

the average number of particles with zero momentum is of the order of $r/\Delta\lambda l$.

Thus, the static problem with $\lambda_1 = 0$ has no solution satisfying the physical requirement that the Green's function have a pole. We note that at $\lambda = 0$ the pole is located at $\omega = \beta_0 = -r^2 l$ and goes off to $-\infty$ as $\Delta \to 0$. Then, $G(\xi)$ is equal to $\exp\{r^2\xi \ln \xi\}$, i.e., it increases more rapidly than any power of S. This solution, as before, does not satisfy the physical requirement, although we do not encounter the infrared situation here (the pole simply goes off to $-\infty$). It can be shown with the aid of the variational principle that for the two-dimensional and one-dimensional problems with a Hermitian Hamiltonian with $\lambda_1 = \lambda = 0$, the fact that the pole goes off to infinity remains in force also when recoil is taken into account, i.e., also when $\beta' \neq 0$.

3. SOLUTION OF PROBLEM AT r = 0 (GENERAL ANALYSIS)

In this section we consider the interaction of reggeons at r = 0 (see (1)). Since allowance for the cross scattering of the reggeons entails no difficulty (we have seen that in the preceding section), we assume for the time being, for simplicity, that $\lambda = 0$ (see (1)). The Dyson equation for the Green's function of this problem is



(21)

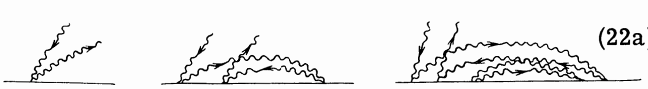
or

$$G(\xi)=G_0(\xi)+\lambda_1^2\int G(\xi'')\Gamma_2(\xi'',\xi_1,\xi_2,\xi_3')G(\xi'-\xi_3')D(\xi'-\xi_1)\times D(\xi'-\xi_2)G_0(\xi-\xi')\,d\xi_1\,d\xi_2\,d\xi_3'\,d\xi'\,d\xi''. \tag{21'}$$

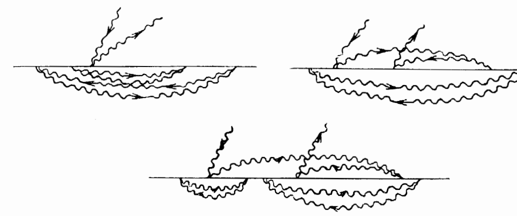
Instead of the vertex Γ_2 it turns out to be more convenient to consider the function F_2 defined by the equation

$$\int G(\xi'')\Gamma_2(\xi'',\xi_1,\xi_2,\xi_3')G(\xi-\xi_3')\,d\xi''\,d\xi_3'=F_2(\xi_1,\xi_2,\xi)G(\xi). \tag{22}$$

The function $F_2(\xi_1, \xi_2, \xi)$ is convenient because out of the entire assembly of diagrams for $G\Gamma G$, owing to the independent integration with respect to the coordinates of the α-Reggeon emission (see Sec. 2), it receives contributions only from diagrams that can be represented with the aid of a single directed α-Reggeon line, for example


(22a)

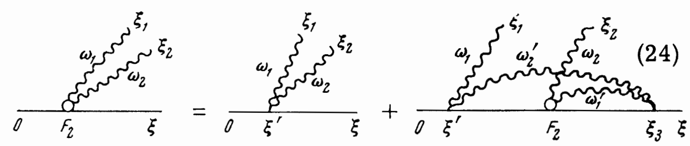
On the other hand, all the closed lines, for example


(22b)

are separated in the form of a factor $G(\xi)$ in (22). Substituting (22) in (21'), we get

$$G(\xi)=\vartheta(\xi)\exp\left\{\lambda_1^2\int_0^\xi d\xi_1\,d\xi_2\,d\xi' F_2(\xi_1,\xi_2,\xi')D(\xi'-\xi_1)D(\xi'-\xi_2)\right\}. \tag{23}$$

For $F_2(\xi_1, \xi_2, \xi)$ we can write the simple equation



(24)

or

$$\int_0^\xi F_2(\xi_1',\xi_2',\xi)D(\xi_1-\xi_1')D(\xi_2-\xi_2')\,d\xi_1'\,d\xi_2' \tag{24'}$$

$$=\int_0 D(\xi_1-\xi')D(\xi_2-\xi')\,d\xi'+\lambda_1^2\int_0^\xi D(\xi_1-\xi')D(\xi_3-\xi')\times D(\xi_2-\xi_2')D(\xi_3-\xi_1')F_2(\xi_1',\xi_2',\xi_3)\,d\xi_1'\,d\xi_2'\,d\xi_3\,d\xi'.$$

If we consider values of ξ_1 and ξ_2 smaller than ξ, then the upper limit of integration with respect to ξ_1', ξ_2' and ξ' in (24) must be set equal to infinity, since $D(\xi_i - \xi_i') = 0$ when $\xi_i' > \xi_i$. We then obtain an equation containing F_2 only in the region $\xi_1, \xi_2 < \xi$. It is convenient to change over in this equation to a mixed representation, in which

$$F_2(\xi_1,\xi_2,\xi)=\frac{1}{(2\pi i)^2}\int_C e^{\omega_1\xi_1+\omega_2\xi_2}F_2(\omega_1,\omega_2,\xi)\,d\omega_1\,d\omega_2,$$

with $F_2(\xi_1, \xi_2, \xi)$ defined for $\xi_1, \xi_2 > \xi$ in accordance with (24') with an infinite upper limit in the integration with respect to ξ_1', ξ_2' and ξ' (the contour C is to the right of all the singularities of the integrand).

Then

$$F_2(\omega_1,\omega_2,\xi)=\frac{1}{\omega_1+\omega_2}+\lambda_1^2\int\frac{d\omega_1'\,d\omega_2'}{(2\pi i)^2}\frac{D(\omega_1')D(\omega_2')\exp\{(\omega_1'+\omega_2')\xi\}F_2(\omega_1',\omega_2,\xi)}{(\omega_1+\omega_2')(\omega_1'+\omega_2')}. \tag{25}$$

It is easy to verify that

$$F_2(\omega_1,\omega_2,\xi)=\frac{1}{\omega_1+\omega_2}[\gamma_1(\omega_1,\xi)\gamma_1(\omega_2,\xi)-\lambda_1^2\gamma_2(\omega_1,\xi)\gamma_2(\omega_2,\xi)], \tag{26}$$

where

$$\gamma_2(\omega_1,\xi)=\frac{1}{2\pi i}\int\frac{\exp\{\omega_1'\xi\}}{\omega_1+\omega_1'}D(\omega_1')\gamma_1(\omega_1',\xi)\,d\omega_1', \tag{27}$$

and $\gamma_1(\omega, \xi)$ satisfies the equation

$$\gamma_1(\omega_1,\xi)=1+\lambda_1^2\int\frac{d\omega_1'\,d\omega_2'}{(2\pi i)^2}\frac{D(\omega_1')D(\omega_2')\exp\{(\omega_1'+\omega_2')\xi\}\gamma_1(\omega_1',\xi)}{(\omega_1+\omega_2')(\omega_1'+\omega_2')} \tag{28}$$

We now put $\lambda \neq 0$. Since we have considered in (24) and (23) $\xi_1, \xi_2 < \xi$, it follows that the scattering of the vacuum Reggeons (α) by the β Reggeon occurs as if the α Reggeon were to move in an external field. This makes it necessary to replace the vacuum-Reggeon Green's function

$$D_0(\omega) = \int \frac{dk}{\omega + k^2 + \Delta} \tag{29}$$

everywhere by

$$D(\omega) = \frac{D_0(\omega)}{1 + \lambda D_0(\omega)}, \tag{30}$$

just as in the preceding section.

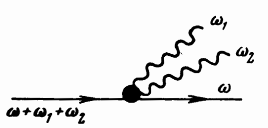
In finding $\Gamma_2(\omega_1, \omega_2, \omega)$ (Γ_2 is the vertex of the decay of the β Reggeon into two α and β Reggeons) it is necessary to know the correct value of $F_2^\lambda(\xi_1, \xi_2, \xi)$ for $\xi_1, \xi_2 > \xi$; this can be readily obtained in the mixed representation:

$$F_2^\lambda(\omega_1, \omega_2, \xi) = D_0(\omega_1) D_0(\omega_2) \int \frac{d\omega_1'\, d\omega_2'}{(2\pi i)^2} \times \frac{D(\omega_1') D(\omega_2') \exp\{(\omega_1' + \omega_2')\xi\} F_2(\omega_1', \omega_2', \xi)}{D_0(\omega_1') D_0(\omega_2') (\omega_1' - \omega_1)(\omega_2' - \omega_2)}; \tag{31}$$

where D and D_0 are defined in accordance with (29) and (30), and $F_2(\omega_1, \omega_2, \xi)$ satisfies Eq. (25), in which $D(\omega_i')$ is taken to mean expression (30). According to (31) we have (we have used (26) in this case)

$$G(\omega + \omega_1 + \omega_2)\Gamma_2(\omega_1, \omega_2, \omega) G(\omega) = \int \frac{d\omega_1'\, d\omega_2'}{(2\pi i)^2} d\xi \times \exp\{-(\omega + \omega_1 + \omega_2 - \omega_1' - \omega_2')\xi\} G(\xi) \frac{D(\omega_1') D(\omega_2')}{D_0(\omega_1') D_0(\omega_2')} \times \frac{\gamma_1(\omega_1', \xi)\gamma_1(\omega_2', \xi) - \lambda_1^2 \gamma_2(\omega_1', \xi)\gamma_2(\omega_2', \xi)}{(\omega_1' - \omega_1)(\omega_2' - \omega_2)(\omega_1' + \omega_2')} \equiv \int e^{-\omega\xi} G(\xi) f_2(\omega_1, \omega_2, \xi)\, d\xi. \tag{32}$$

The notation in (32) is clear from the following diagram:



(32′)

The function $G(\xi)$ can also be expressed in terms of the solution of (28), namely

$$G(\xi) = \vartheta(\xi) \exp\left\{ \int_0^\xi d\xi' \int \frac{d\omega_1\, d\omega_2}{(2\pi i)^2} \frac{D(\omega_1) D(\omega_2)}{\omega_1 + \omega_2} \times \exp\{(\omega_1 + \omega_2)\xi'\} \gamma_1(\omega_1, \xi') \right\} \tag{33}$$

$$= \vartheta(\xi) \exp\left\{ \int_0 d\xi' \int \frac{d\omega}{2\pi i} [\gamma_1(\omega, \xi') - 1] \right\}$$

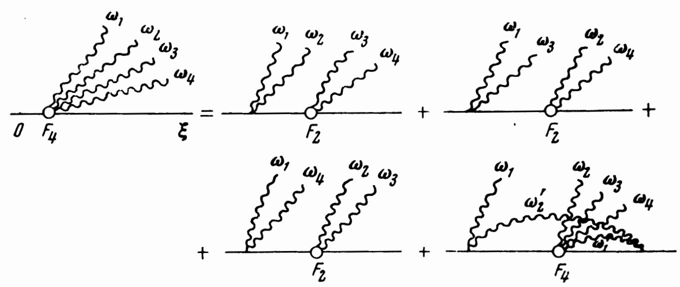
We note that as $\xi \to \infty$, expression (33) contains the factor $\exp(\delta E_\beta \xi)$, which describes the shift of the energy of the single-particle state

$$\delta E_\beta = \int \frac{d\omega}{2\pi i} [\gamma_1(\omega, \infty) - 1]. \tag{34}$$

The energy shift is immaterial for our purposes, and we shall henceforth omit this factor, with the understanding that the frequencies ω are reckoned from the exact energy of the single-particle state.

Thus, the problem will be completely solved if Eq. (28) is solved.

Let us stop to discuss one interesting feature of the static problems. In these problems, the amplitudes of the transition of the Reggeon into n particles α are expressed in simple fashion in terms of the amplitude $\beta \to 2\alpha + \beta$. Indeed, if we introduce the functions $F_n(\xi_1, \xi_2, \ldots, \xi_n, \xi)$ (n is the number of vacuum Reggeons) in the same manner as $F_2(\xi_1, \xi_2, \xi)$ (see (22)), then we obtain simple equations for these functions. For concreteness, we write them for $F_4(\xi_1, \xi_2, \xi_3, \xi_4, \xi)$: and directly in the mixed representation:



or

$$F_4(\omega_1, \omega_2, \omega_3, \omega_4, \xi) = \frac{1}{\omega_1 + \omega_2} F_2(\omega_3, \omega_4, \xi) + \frac{1}{\omega_1 + \omega_3} F_2(\omega_2, \omega_4, \xi) + \frac{1}{\omega_1 + \omega_4} F_2(\omega_2, \omega_3, \xi) + \frac{\lambda_1^2}{(2\pi i)^2} \int d\omega_1'\, d\omega_2' \frac{D(\omega_1') D(\omega_2') \exp\{(\omega_1' + \omega_2')\xi\}}{(\omega_1 + \omega_2')(\omega_1' + \omega_2')} F_4(\omega_1', \omega_2, \omega_3, \omega_4, \xi). \tag{35}$$

It is easy to see that the solution of (35) is

$$F_4(\omega_1, \omega_2, \omega_3, \omega_4, \xi) = F_2(\omega_1, \omega_2, \xi) F_2(\omega_3, \omega_4, \xi) + F_2(\omega_1, \omega_3, \xi) F_2(\omega_2, \omega_4, \xi) + F_2(\omega_1, \omega_4, \xi) F_2(\omega_2, \omega_3, \xi). \tag{36}$$

Analogously

$$F_n = \sum \prod_{ik} F_2(\omega_i, \omega_k, \xi). \tag{37}$$

The summation is over all possible combinations of particles by pairs. Such a form of F_n leads to a simple expression for Γ_n, namely

$$G\left(\omega + \sum \omega_i\right) \Gamma_n(\omega_1, \ldots, \omega_n, \omega) G(\omega) = \sum \int e^{-\omega\xi} G(\xi) \prod_{ik} f_2(\omega_i, \omega_k, \xi)\, d\xi. \tag{38}$$

A particularly simple form is possessed by the $G\Gamma_n$ in the limiting case as $\omega \to 0$:

$$G\Gamma_n = \sum \prod (G\Gamma_2), \tag{39}$$

with

$$[G(\omega + \omega_1 + \omega_2)\Gamma_2(\omega_1, \omega_2, \omega)]_{\omega=0} = \gamma_1^0(\omega_1)\gamma_1^0(\omega_2) / (\omega_1 + \omega_2) \cdot \times D(\omega_1) D(\omega_2) / D_0(\omega_1) D_0(\omega_2), \tag{40}$$

where

$$\gamma_1^0(\omega) = \gamma_1(\omega_1, \xi)|_{\xi \to \infty}.$$

We note that as $\omega \to 0$ the quantities $G\Gamma_n$ enter in the unitarity condition for $G(\omega)$ and their determination solves the problem in a certain sense:

$$\operatorname{Im} G(\omega) = \sum \frac{1}{n!} \int |(G\Gamma_n)|^2_{\omega=0,\, \omega_i = -k_i^2} \delta\left(\omega + \sum k_i^2\right) \prod dk_i. \tag{41}$$

Thus, we gain much knowledge of the Green's function of the β Reggeon if we obtain the solution of (28) as $\xi \to \infty$. It is easy to show that the solution of the complete static problem with Hamiltonian (1) can be obtained in terms of the solution of (28) (see Appendix I) when $r \neq 0$, $\lambda_1 \neq 0$, $\lambda \neq 0$.

Let us now find $\gamma_1(\omega, \infty) \equiv \gamma_1^0(\omega)$. As $\xi \to \infty$ it suffices for us to confine ourselves in the integration with

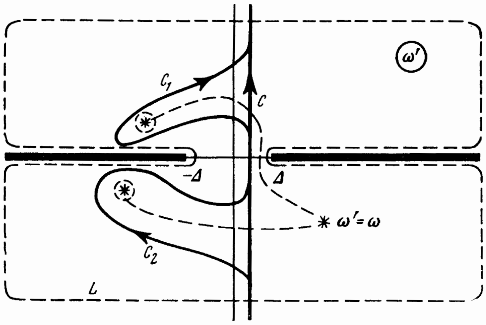


FIG. 2

respect to ω_2' in (28) to the contribution of the pole at $\omega_2' = -\omega_1'$ (the remainder constitutes at most a power-law small quantity in terms of $1/\xi$), and therefore $\gamma_1^0(\omega)$ satisfies the equation

$$\gamma_1^0(\omega) = 1 + I_c, \tag{42}$$

where

$$I_c \equiv \frac{\lambda_1^2}{2\pi i}\int_c d\omega' \frac{D(\omega')D(-\omega')\gamma_1^0(\omega')}{\omega-\omega'}, \tag{42'}$$

and the integration contour C is shown in Fig. 2.

The singularities of γ_1^0 arise from the fact that the pole $\omega' = \omega$ and the cut $D(\omega')$ pinch the integration contour. We see therefore that $\gamma_1^0(\omega)$ is analytic on the plane with a cut along the real axis from $\omega = -\Delta$ to $\omega = -\infty$. The value of the function $\gamma_1^0(\omega)$ on the upper edge of this cut, $\gamma_1^{0+}(\omega)$, is (see Fig. 2)

$$\gamma_1^{0+}(\omega) = 1 + I_{c_1} = \lambda_1^2\gamma_0^+(\omega)D^+(\omega)D(-\omega) + 1 + I_c \tag{43}$$

and on the lower edge

$$\gamma_1^{0-}(\omega) = 1 + I_{c_2} = \lambda_1^2\gamma_0^-(\omega)D^-(\omega)D(-\omega) + 1 + I_c. \tag{44}$$

Comparing (43) and (44) we get

$$\frac{\gamma_1^{0+}(\omega)}{\gamma_1^{0-}(\omega)} = \frac{1-\lambda_1^2 D^-(\omega)D(-\omega)}{1-\lambda_1^2 D^+(\omega)D(-\omega)}. \tag{45}$$

The solution of the boundary-value problem (45) is

$$\gamma_1^0(\omega) = c\exp\left\{\frac{1}{2\pi}\int_{-\Delta}^{-\infty}\frac{d\omega'}{\omega'-\omega}\,\mathrm{Im}\ln\frac{1-\lambda_1^2 D^-(\omega')D(-\omega')}{1-\lambda_1^2 D^+(\omega')D(-\omega')}\right\}. \tag{46}$$

Since perturbation theory is valid as $\omega \to \infty$, we get $c = \gamma_1^0(\infty) = 1$. We note that for the two-dimensional and one-dimensional cases, the value of $1 - \lambda_1^2 D^+(\omega)D(-\omega)$ as $\omega \to 0$ is $1 - \lambda_1^2/\lambda^2$, and as $\omega \to \infty$ its value is 1, and consequently when $\lambda_1^2 > \lambda^2$ the logarithm $\ln[1 - \lambda_1^2 D(\omega) \times D(-\omega)]$ acquires a phase 2π as ω varies along the contour L (see Fig. 2). Since in this case $\gamma_1^0(\omega) = 1$ at large ω (on a circle of large radius), the function $\gamma_1^0(\omega)$ has in the case of $\lambda_1^2 > \lambda^2$ a pole on the real axis to the right of the origin. Just as in the preceding section (see (11)), this leads to a solution that does not agree with the physical requirements after $\gamma_1^0(\omega)$ is substituted in (33). Thus, a solution exists only when $\lambda_1^2 \le \lambda^2$ and $\lambda > 0$.

4. ONE-DIMENSIONAL MODEL. SCALE INVARIANCE

Let us now determine the properties of the Green's function and of the vertex parts in the one-dimensional model. We recall that in the one-dimensional model the Green's function of the α Reggeon is

$$D_0(\omega) = (\omega+\Delta)^{-1/2}, \quad D(\omega) = D_0/(1+\lambda D_0) = 1/[\lambda + (\omega+\Delta)^{1/2}].$$

The constants λ_1 and λ satisfy the stability condition $\lambda \ge |\lambda_1|$.

Let us consider first the case when the amplitude of the decays $|\lambda_1|$ is smaller than the maximum value of λ. In this case it turns out that the solution corresponds to the singular weak coupling in accordance with the classification proposed in the Introduction. It is seen first of all from (45) and (46) that when $\omega \ll \lambda_1^2, \lambda^2$ the function $\gamma_1(\omega, \xi = \infty)$ tends to a finite limit

$$\gamma_1^0(\omega) \to A(\lambda_1/\lambda), \quad \omega \ll \lambda^2. \tag{47}$$

It then follows from (39) and (40) that the quantities $G\Gamma_n|_{\omega=0}$ that enter in the unitarity condition (41) for the Green's function are homogeneous functions, of degree zero, of the frequencies ω_i and of the gap Δ. For example

$$G(\omega_1+\omega_2)F_2(\omega_1,\omega_2,0) = \frac{\lambda_1 A^2(\lambda_1/\lambda)}{\lambda^2}\frac{\sqrt{\omega_1+\Delta}\sqrt{\omega_2+\Delta}}{\omega_1+\omega_2}. \tag{48}$$

Substituting these expressions in the unitarity condition (41), we see that the individual terms in (41) are homogeneous functions of ω and Δ:

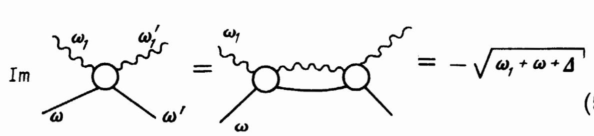
$$\mathrm{Im}\,G(\omega)|_n = \int (G\Gamma_n)^2\,\delta\left(\omega + n\Delta + \sum k_i^2\right)d\mathbf{k}_1\ldots d\mathbf{k}_n \tag{49}$$

$$= \frac{1}{\omega}f_n\left(\frac{\omega}{\Delta}\right)\left(\frac{\lambda_1 A^2\sqrt{\omega}}{\lambda^2}\right)^n.$$

In the region $\omega \sim \Delta \ll \lambda^2$ of interest to us, these terms are small compared with the pole term

$$G(\omega) \to \frac{Z(\lambda_1/\lambda)}{\omega}. \tag{50}$$

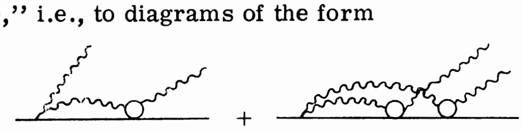
Thus, when $\Delta, \omega \ll \lambda^2, \lambda_1^2$ the Green's function coincides with the free one, apart from the renormalized factor $Z(\lambda_1/\lambda)$ and the shift of the pole position (we recall that the frequencies ω of the particle β are reckoned in all the formulas from the position of the renormalized pole). The vertex parts $\Gamma_n(\omega_1, \ldots, \omega_n, \omega)$ of the decays $\beta \to \beta + n\alpha$ are homogeneous functions of Δ, ω_i, and ω of order ω. For example, $\Gamma_2(\omega_1\omega_2, \omega = 0) = \lambda_1 A^2\sqrt{\omega_1+\Delta}\sqrt{\omega_2+\Delta}/\lambda^2 Z$. The amplitudes of the scattering $n\alpha + \beta \to m\alpha + \beta$ have a larger order of magnitude; for example, the scattering amplitude Λ is determined by the two-particle states and its value at $\omega, \Delta \ll \lambda$ is



(51)

The corrections to this expression, due to the many-particle states, contain the small parameter ω/λ_1^2, in analogy with (49).

The relatively large order of magnitude of the scattering amplitude Λ explains why the decay amplitude Γ_2 is different from the nonrenormalized value $\Gamma_2 = \lambda_1$, namely, this is due to the "interaction in the final state," i.e., to diagrams of the form



These diagrams are sign-alternating ($\lambda < 0$) and cancel out the nonrenormalized coupling constant λ_1.

The solution considered here is an example of the singular weak coupling in accordance with the classification proposed in the introduction, and has scale invariance, i.e., it maintains its structure when the frequencies (time) and Δ are subjected to a scale transformation.)We note that the solution in Sec. 2 of the problem with $r \neq 0$, $\lambda \neq 0$, $\lambda_1 = 0$ had no scale invariance.)

Let us now consider the limiting case when the decay amplitude λ_1 reaches its maximum possible value

$$|\lambda_1| = \lambda. \tag{52}$$

This limiting case is of interest because the solution under condition (52) corresponds to strong coupling. Indeed, when $|\lambda_1| = \lambda$, the coefficient $A(\lambda_1/\lambda)$ in (48) becomes infinite, $A \sim \lambda/\sqrt{\lambda^2-\lambda_1^2}$. When $|\lambda_1| = \lambda \gg \omega, \Delta$ the function $\gamma_1^0(\omega)$ equals, in accordance with (46),

$$\gamma_1{}^0(\omega) \to \frac{\text{const}\cdot\lambda^{1/2}}{(\omega+\Delta)^{1/4}}\exp\left\{\frac{1}{\pi}\int_1^\infty \frac{dz}{z+\omega/\Delta}\left[\operatorname{arctg}\sqrt{\frac{z-1}{z+1}}-\frac{\pi}{4}\right]\right\} = \frac{\lambda^{1/2}}{(\omega+\Delta)^{1/4}} f\left(\frac{\omega}{\Delta}\right). \tag{53}$$

If we now find the value of $G\Gamma_2(\omega_1, \omega_2, 0)$ in accordance with the formula (40), then it turns out that $G\Gamma_2$ is independent of the coupling constant:

$$G(\omega_1+\omega_2)\Gamma_2(\omega_1,\omega_2,0)\,|_{\omega\to 0} \to \frac{[(\omega_1+\Delta)(\omega_2+\Delta)]^{1/4}}{\omega_1+\omega_2} f\left(\frac{\omega_1}{\Delta}\right) f\left(\frac{\omega_2}{\Delta}\right). \tag{54}$$

In order to find the Green's function $G(\omega)$ when $\omega \ll \lambda^2$, it is no longer convenient, as before, to use the unitarity condition (or the Lehmann expansion) (41) since this gives rise to the series

$$G(\omega) \sim \frac{1}{\omega}\sum_n \varphi_n\left(\frac{\omega}{\Delta}\right)\left(\ln\frac{\lambda^2}{\omega}\right)^n, \tag{55}$$

which contains powers of the large parameter $\ln(\lambda^2/\omega)$. Instead of summing this series, we make use of formula (33), which expresses $G(\xi)$ in terms of $\gamma_1(\omega, \xi)$. Since $\gamma_1(\omega, \xi)$ has no singularities in ω at finite values of ξ, the integral in (33) reduces to the residue at $\omega = \infty$. It is shown in Appendix II that the asymptotic form of $\gamma_1(\omega, \xi)$ when $\xi \gg \lambda^{-2}$ and $\omega\xi \to \infty$ is

$$\gamma_1(\omega, \xi) = 1 - 1/16\omega\xi. \tag{56}$$

Then (33) yields

$$G(\xi) = (\xi_0/\xi)^{1/16}, \quad G(\omega) = (\omega\xi_0)^{1/16}/\omega. \tag{57}$$

Here $\xi_0 \sim \lambda^{-2}$ is the lower limit of the logarithmic integral

$$\int_{\xi_0}^{\xi} d\xi'/\xi'.$$

Using the solution obtained in Appendix II for $\gamma_1(\omega, \xi)$ at $\omega \ll \lambda^2$, $\xi \gg \lambda^{-2}$:

$$\gamma_1(\omega,\xi) = \frac{c\xi^{-1/4}}{\sqrt{\omega}}\int\frac{dp}{2\pi i}p^{-5/4}(p+1)^{-1/4}e^{p\omega\xi}, \tag{58}$$

We can find also the vertex parts Γ_n in accordance with formulas (32), (37), (38), and (40).

Thus, the solution has a power-law asymptotic form at $\Delta = 0$, i.e., it corresponds to strong coupling (see the Introduction). At finite $\Delta \ll \lambda^2$, the Green's function and the vertex parts are homogeneous functions of the type

$$G(\omega) = \frac{1}{\omega}\left(\frac{\omega}{\lambda^2}\right)^{1/16} g\left(\frac{\omega}{\Delta}\right), \tag{59}$$

$$G(\omega)\Gamma_n(\omega,\omega_i,k_i) = \omega^{-n/4} g_n\left(\frac{\omega_i}{\omega},\frac{\omega}{\Delta},\frac{k_i^2}{\omega}\right). \tag{60}$$

The ratio of the powers of G and Γ_n confirms the similarity hypothesis of [3], whereby all diagrams of any quantity ($G, \Gamma, \ldots$) containing in their interior Green's functions and vertex parts, have the same order of magnitude. In our case the similarity law means that $(G\Gamma_n)^2\Delta^{n/2} \sim 1$.

5. TWO-DIMENSIONAL MODEL. THE ZERO-CHARGE PROBLEM

Let us now consider the properties of the solution in the two-dimensional model. If we confine ourselves only to quadratic terms in the spectrum $\epsilon_\alpha(k) = \Delta + k^2$, then theory gives rise to logarithmic divergences at large momenta. The asymptotic form of the solution when the momenta and Δ are much smaller than the cutoff radius L can be investigated by the usual methods, by summing the principal diagrams; this leads, as in the relativistic theory,[7] to zero charge (see [4]).

$$\Gamma_2 \to \left[1+(\lambda-\lambda_1)\ln\frac{L}{\omega}\right]^{-1}\left[1+(\lambda+\lambda_1)\ln\frac{L}{\omega}\right]^{-1} \to 0. \tag{61}$$

This formula was obtained in [4] for the general case when the particles β (non-vacuum Reggeons) have $\beta' \neq 0$ (moving Regge pole). Just as in the relativistic theory, the "parquet" solution (61) can be verified only when $\lambda, \lambda_1 \ll 1$, and when $\lambda_1, \lambda \sim 1$ it is possible to have formally a solution corresponding to strong coupling, i.e., to a finite charge.

Let us now find the asymptotic form of Γ_2 in the static model. It turns out that formula (61) is valid for all λ_1 and λ satisfying the stability condition ($|\lambda_1| \le \lambda$), if $\ln(L/\omega_1) \to \ln(L/\omega_2) \gg 1$. Let us rewrite formula (40) for $G\Gamma_2\,|_{\omega=0}$ in a more convenient form:

$$\Gamma_2 = \frac{G^{-1}(\omega_1+\omega_2)}{\omega_1+\omega_2}\chi(\omega_1)\chi(\omega_2), \tag{62}$$

where

$$\chi(\omega) = \exp\left\{\frac{1}{\pi}\int_\Delta^L \frac{d\varepsilon}{\varepsilon+\omega}\operatorname{arctg}\pi\Lambda(\varepsilon)\right\}, \tag{63}$$

$$\Lambda(\varepsilon) = \lambda\frac{1+(\lambda^2-\lambda_1^2)l_+}{(1+\lambda l_+)(1+\lambda l_-)-\lambda_1^2 l_+ l_-}, \quad l_\pm = \ln\frac{L}{\varepsilon\pm\Delta}. \tag{64}$$

(We have substituted $D_0 = l_+$ in the general expression (46) for $\gamma_1^0(\omega) = (D_0/D)\chi(\omega)$. We see now that for all λ and λ_1 satisfying the condition $|\lambda_1| \le \lambda$, the value of $\Lambda(\epsilon)$ tends to zero as $\ln(L/\epsilon) \to \infty$, and we can put $\tan^{-1}\pi\Lambda = \pi\Lambda$. Then we have, with logarithmic accuracy,

$$\chi(\omega) = \left[\left(1+\lambda\ln\frac{L}{\Delta+\omega}\right)^2 - \lambda_1^2\ln^2\frac{L}{\Delta+\omega}\right]^{-1/2}. \tag{65}$$

Thus, $G\Gamma_2 \sim 1/\omega\ln^2(L/\omega)$ when $\lambda_1 \neq \lambda$ and $G\Gamma_2 \sim [\omega\ln(L/\omega)]^{-1}$ when $|\lambda_1| = \lambda$. According to the unitarity condition (41) and the recurrence formula (39) for $G\Gamma_n$, the quantity $\omega G\Gamma_2$ is the effective renormalized charge, i.e., the perturbation theory expansion parameter. The effective charge is logarithmically small, so that the Green's function is determined by the pole term

$G \to Z/\omega$, and the vertex Γ_2 equals, in accordance with (62) and (65),

$$\Gamma_2 \to \frac{1}{Z}\left[(1+\lambda l_1)^2 - \lambda_1^2 l_1^2\right]^{-1/2}\left[(1+\lambda l_2)^2 - \lambda_1^2 l_2^2\right]^{-1/2}, \quad l_{1,2} = \ln\frac{L}{\omega_{1,2}+\Delta}. \tag{66}$$

If the frequencies ω_1 and ω_2 are of the same order, i.e., $l_1 = l_2$, then this formula goes over into the "parquet" solution (61). We note that the quantity $\Lambda(\epsilon)$ in (64) is the amplitude of the $\alpha + \beta \to \alpha + \beta$ scattering with energy ϵ, as can be readily verified from the "parquet" equations.[4]

Thus, in the two-dimensional static model, the renormalized charges λ_{1c} and λ_c vanish with increasing cutoff radius for all nonrenormalized constants λ_1 and λ. If we vary the nonrenormalized constants $\lambda_1(L)$ and $\lambda(L)$ together with the radius L, leaving the renormalized λ_{1c} and λ_c fixed, then we encounter the same paradox as in quantum electrodynamics,[7] namely, the renormalized amplitudes, for example

$$\Lambda_c(\varepsilon) = \lambda_c \frac{1 + (\lambda_c^2 - \lambda_{1c}^2) l_+}{(1+\lambda_c l_+)^2 - \lambda_{1c}^2 l_-^2}, \tag{67}$$

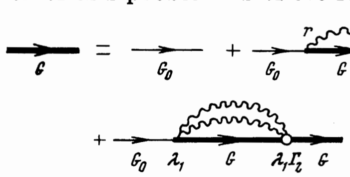
have logarithmic poles at $\epsilon \sim \Delta \exp\{-1/(\lambda_c \pm \lambda_{1c})\}$.

From the point of view of applications to the Reggeon problem,[4] where L is fixed and $\omega, \Delta \to 0$, the solution (61) contains no paradoxes and simply denotes the screening of the interaction at low frequencies and momenta.

In conclusion we wish to thank A. I. Larkin, who called our attention to the one-dimensional model.

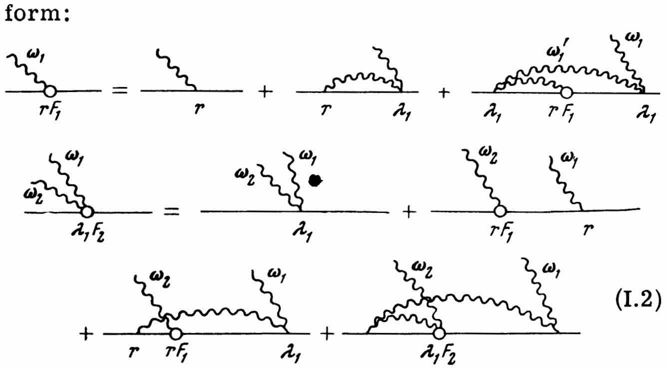
APPENDIX I

We consider here the solution of the complete problem, when all the constants r, λ, and λ_1 are not equal to zero. We shall not analyze this solution in detail, since its properties are similar to those of the case $r \neq 0$, $\lambda \neq 0$, $\lambda_1 \neq 0$ (see Sec. 2), i.e., this problem has no solution in the sense indicated in the introduction. The Dyson equation of this problem is of the form



(I.1)

If we introduce in lieu of Γ_1 and Γ_2 the functions F_1 and F_2 in analogy with the procedure used in Sec. 3, then we can easily write for them equations in graphic form:



(I.2)

We assume that $\lambda = 0$; λ is accounted for by the method already described above (see (29) and (30)).

In the mixed representation, the equation for F_1 is

$$F_1(\omega,\xi) = \frac{1}{\omega} + \frac{\lambda_1}{\omega}\left(D(\omega,\xi) - D(0,\xi)\right) + \frac{\lambda_1^2}{(2\pi i)^2}\int d\omega_1' d\omega_2' \frac{D(\omega_1')D(\omega_2')\exp\{(\omega_1'+\omega_2')\xi\}F_1(\omega_2',\xi)}{(\omega+\omega_1')(\omega_1'+\omega_2')}. \tag{I.3}$$

It is easy to verify that a solution of this equation is

$$F_1(\omega,\xi) = \omega^{-1}[\gamma_1(\omega,\xi) - \lambda_1\gamma_2(\omega,\xi)], \tag{I.4}$$

where γ_1 and γ_2 are defined by (27) and (28).

The following obvious statement can be made with respect to $F_2(\omega_1, \omega_2, \xi)$:

$$F_2(\omega_1,\omega_2,\xi) = \frac{r^2}{\lambda_1}F_1(\omega_1,\xi)F_1(\omega_2,\xi) + F_2^{(r=0)}(\omega_1,\omega_2,\xi); \tag{I.5}$$

$F_2^{(r=0)}(\omega_1, \omega_2, \xi)$ is the solution of the problem at $r = 0$, i.e., it satisfies Eq. (25).

Since the Green's function is equal to

FORMULAS
$$G(\xi) = \vartheta_0(\xi)\exp\left\{\int_0^\xi \beta(\xi')d\xi'\right\}, \tag{I.6}$$

$$\begin{aligned}\beta(\xi) &= r^2\int D(\omega)F_1(\omega,\xi)e^{\omega\xi}d\omega + \lambda_1^2\int F_2(\omega_1,\omega_2,\xi)D(\omega_1)D(\omega_2)e^{(\omega_1+\omega_2)\xi}d\omega_1 d\omega_2 \\ &= r^2\left[\int D(\omega)F_1(\omega,\xi)e^{\omega\xi}d\omega + \lambda_1\left(\int D(\omega)e^{\omega\xi}F_1(\omega,\xi)d\omega\right)^2\right] \\ &\quad + \lambda_1^2\int F_2^{(r=0)}(\omega_1,\omega_2,\xi)e^{(\omega_1+\omega_2)\xi}D(\omega_1)D(\omega_2)d\omega_1 d\omega_2\end{aligned} \tag{I.7}$$

($d\omega = d\omega/2\pi i$), we obtain by using the equation for $\gamma_1(\omega, \xi)$

$$G(\xi) = \vartheta_0(\xi)G^{(r=0)}(\xi)\exp\left\{r^2\int_0^\xi \bar\beta(\xi')d\xi'\right\}, \tag{I.8}$$

where

$$\bar\beta(\xi) = [\gamma_2(0,\xi) - \lambda_1^{-1}\gamma_1(0,\xi) + \lambda_1^{-1}] + \lambda_1[\gamma_2(0,\xi) - \lambda_1^{-1}\gamma_1(0,\xi) + \lambda_1^{-1}]^2. \tag{I.9}$$

The new position of the pole E_β is determined from (I.9):

$$E_\beta = E_\beta(r^2=0) + r^2[(\lambda^{-1} - \lambda_1^{-1})\gamma_1^0(0) + \lambda_1^{-1}][2 + (\lambda_1\lambda^{-1} - 1)\gamma_1^0(0)].$$

It is seen, in particular, that at $\lambda = \lambda_1$ the pole shifts by $2r^2/\lambda$, and when $\lambda = -\lambda_1$ it goes off to infinity ($\gamma_1^0 \sim 1/\sqrt{\lambda^2 - \lambda_1^2}$). The Green's function, for example, for the most interesting case of the one-dimensional problem, is of the order of $\exp(\xi^{1/2})$ when $\lambda_1^2 \neq \lambda^2$ and of the order of $\exp(\xi^{3/4})$ when $\lambda^2 = \lambda_1^2$, i.e., the interaction due to the presence of λ_1 and λ in the Hamiltonian indeed does not alter the main conclusion that when $r^2 \neq 0$ we encounter the infrared situation in this problem.

APPENDIX II

We obtain here the solution of (28) for $\lambda_1^2 = \lambda^2$ in the region $\omega \ll \lambda^2$, $\xi \gg 1/\lambda^2$. In this region we can rewrite (28), using the expansion of D for $\omega \ll \lambda^2$, in the form

$$\gamma_1(\omega,\xi) = 1 + \frac{1}{(2\pi i)^2}\int\frac{\exp\{(\omega_1'+\omega_2')\xi\}\gamma_1(\omega_1',\xi)}{(\omega+\omega_2')(\omega_1'+\omega_2')}d\omega_1' d\omega_2' - \frac{1}{(2\pi i)^2\lambda}\int\frac{(\sqrt{\omega_1'}+\sqrt{\omega_2'})\exp\{(\omega_1'+\omega_2')\xi\}\gamma_1(\omega_1',\xi)}{(\omega+\omega_2')(\omega_1'+\omega_2')}d\omega_1' d\omega_2' + C_0(\xi); \tag{II.1}$$

$C_0(\xi)$ is a certain function of ξ, resulting from the integration with respect to $\omega_1'\omega_2' \gtrsim \lambda^2$ in the right-hand side

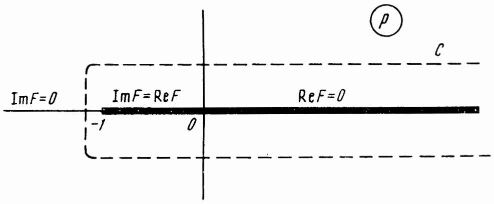


FIG. 3

of (28). Recognizing that $\gamma_1(\omega, \xi)$ has no singularities in ω, we can easily rewrite (II.1) in the form

$$0 = G(\xi) - \frac{1}{(2\pi i)^2} \int \frac{(\sqrt{x_1} + \sqrt{x_2})\, e^{(x_1+x_2)} \gamma_1(x_1, \xi)}{(x + x_2)(x_1 + x_2)} dx_1\, dx_2, \qquad \text{(II.2)}$$

where $x = \omega\xi$, $G(\xi) = \sqrt{\xi}(1 + C_0(\xi))$. Alternately, integrating with respect to x_2, we have

$$\sqrt{x}\, \gamma_1(x, \xi) = G(\xi) + \frac{1}{2\pi i} \int \frac{(-\sqrt{x_1}\, e^{-x+x_1} + \sqrt{-x_1})\, \gamma_1(x_1, \xi)}{x_1 - x} dx_1. \qquad \text{(II.3)}$$

It is convenient to introduce the function $\chi(x, \xi) = e^x \sqrt{x}\, \gamma(x, \xi)$. For this function, Eq. (II.3) takes the form

$$\chi(x) = G(\xi)\, e^x - \frac{1}{2\pi i} \int \frac{(1 - i e^{x-x_1} \operatorname{sign} x_1)\, \chi(x_1)}{x_1 - x} dx_1. \qquad \text{(II.4)}$$

Changing over to the Laplace transform

$$\chi(p) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} \chi(x)\, dx,$$

we obtain for $\chi(p)$ the equation

$$\begin{aligned} \chi(p) &= G(\xi)\, \delta(p+1) + \vartheta(p)\chi(p) - \vartheta(p+1) \frac{i}{2\pi} \int e^{px_1} \chi(x_1) \operatorname{sign} x_1\, dx_1 \\ &= G(\xi)\, \delta(p+1) + \vartheta(p)\chi(p) - \vartheta(p+1) \frac{1}{\pi} \mathrm{P} \int \frac{\chi(p')\, dp'}{p - p'}. \end{aligned} \qquad \text{(II.5)}$$

It is easy to see that the homogeneous equation (at $G(\xi) = 0$) reduces to the need for reconstructing the analytic function

$$F(p) = \frac{1}{\pi} \int_{-1}^{\infty} \frac{\chi(p')\, dp'}{p - p'},$$

where Re F and Im F satisfy the condition

$$\operatorname{Im} F = \vartheta(p) \operatorname{Im} F + \vartheta(p+1) \operatorname{Re} F, \qquad \text{(II.6)}$$

as is shown in Fig. 3.

Such a function can be found readily; it is equal to

$$F(p) = [(-p-1)(-p)]^{-1/4}. \qquad \text{(II.7)}$$

The solution of the inhomogeneous equation is

$$F(p) = G(\xi) / (-p-1)^{5/4} (-p)^{1/4}. \qquad \text{(II.8)}$$

Returning to the function $\gamma_1(\omega, \xi)$, we have for it

$$\gamma_1(\omega, \xi) = \frac{G(\xi)}{\sqrt{x}} \int_c \frac{e^{-(p+1)x}}{(-p-1)^{5/4} (-p)^{1/4}} dp \qquad \text{(II.9)}$$

(The contour C is shown in Fig. 3), which is better rewritten in the form

$$\gamma_1(\omega, \xi) = G(\xi) \int_c dq\, e^{-q} \frac{1}{(-q)^{5/4} (-q + x)^{1/4}}. \qquad \text{(II.10)}$$

We reconstruct the function $G(\xi)$ by examining the behavior of $\gamma_1(\omega, \xi)$ as $x \to +\infty$:

$$\gamma_1(\omega, \xi) \to \frac{G(\xi)}{x^{1/4}} \alpha, \quad \alpha = \int_c dq \frac{e^{-q}}{(-q)^{5/4}}. \qquad \text{(II.11)}$$

On the other hand, we know a solution for $\xi \to \infty$ and for ω fixed but smaller than λ^2 (see Sec. 3, formula (46)), namely

$$b / \omega^{1/4}, \qquad \text{(II.12)}$$

where b is a certain constant. Comparing (II.11) and (II.12) we obtain

$$G(\xi) = b\xi^{1/4} / \alpha. \qquad \text{(II.13)}$$

We now find the next term of the expansion in $1/x$ as $x \to \infty$. It is equal to $G(\xi)\gamma/x^{5/4}$, where

$$\gamma = \int \frac{dq}{(-q)^{1/4}} e^{-q}. \qquad \text{(II.14)}$$

Thus, comparing (II.14), (II.13), and (II.11) we get for γ_1 the following behavior as $x \to \infty$:

$$\gamma_1(x, \xi) = \left(1 - \frac{\gamma}{4x\alpha}\right) \frac{b}{\omega^{1/4}}, \qquad \text{(II.15)}$$

or, since $\alpha = 4\gamma$,

$$\gamma_1(x, \xi) = \left(1 - \frac{1}{16\omega\xi}\right) \gamma_0(\omega). \qquad \text{(II.16)}$$

We recall that $\omega \ll \lambda^2$ and $\xi \gg 1/\lambda^2$. It is easily seen that formula (II.16) is valid also when $\omega \geq \lambda^2$. This is connected with the fact that the equation for the correction to $\gamma_1^0(\gamma_1')$ has for $\xi \to \infty$ the form

$$\begin{aligned} \gamma_1' = {} & \frac{1}{(2\pi i)^2} \int_c \frac{D(\omega_1') D(\omega_2')}{(\omega + \omega_2')(\omega_1' + \omega_2')} \exp\{(\omega_1' + \omega_2')\xi\} \\ & \times \gamma_1^0(\omega_1', \xi)\, d\omega_1'\, d\omega_2' + \frac{1}{2\pi i} \int \frac{D(\omega_1') D(-\omega_1') \gamma_1'(\omega_1', \xi)}{\omega - \omega_1'} d\omega_1'. \end{aligned} \qquad \text{(II.17)}$$

In the first integral, the pole $\omega_1' + \omega_2' = 0$ lies outside the integration contour, and therefore the values $\omega\xi \sim 1$ play an important role in it, and it is necessary to take the expression (II.10) for γ_1^0. Substituting it in (II.17), we find that the first integral behaves like $\xi^{-5/4}$ as $\xi \to \infty$, and therefore the terms of order $1/\xi$ should be determined as $\xi \to \infty$ from the solution of the homogeneous equation. It is easy to verify that (II.16) is indeed a solution of this homogeneous equation, if the pole at $\omega' = 0$ lies on the left side of the contour of integration with respect to ω'.

In conclusion we wish to note that the phenomena occurring in (II.1) are precisely those characteristic of strong coupling,[6] namely, the contributions from the region of large ω ($\omega \gtrsim \lambda^2$) cause cancellation of the nonrenormalized constant (which is equal to unity in (II.4)), and therefore the solution is obtained accurate to a certain function of $\xi(G(\xi))$, which, in turn, is determined from the matching of the results with those in the region of high frequencies.

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