ACCELERATION OF CHARGED PARTICLES IN AN ALTERNATING MAGNETIC FIELD

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It is shown that acceleration of particles in a homogeneous magnetic field that varies periodically with time (Alfven magnetic pumping) reduces to a diffusion of the particles in momentum space; a connection is established between the diffusion coefficient and the turbulence spectrum dh^2/dk (k--wave number). The self-consistent problem of the wave and particle spectrum is formulated and solved, and it is found that $dh^2/dk \sim k^{-2}$. It follows from the obtained solutions that when account is taken of the synchrotron radiation, the diffusion coefficient D_0 of ultrarelativistic electrons does not differ from the Fermi value. An expression is obtained for the minimum concentration of the accelerated particles, at which the cyclotron instability ensures the scattering necessary for effective acceleration. It is shown that this effect can ensure generation of electrons that yield the observed x-ray synchrotron radiation of the Crab nebula and of the pulsar located at the center of the nebula, and numerical estimates are given of the values of W_0 and of the x-ray electron concentration.

1. INTRODUCTION

If a slow periodic time variation of a uniform magnetic field B in a turbulent plasma is accompanied by conservation of the adiabatic invariants p_z = const, $p_\perp^2/B$ = const ($p_\perp$ and p_z are the particle-momentum components perpendicular and parallel to the field), then the total momentum p of the charged particles can increase exponentially relative to the time t as a result of betatron acceleration and non-adiabatic scattering by the hydromagnetic turbulence (the Alfven magnetic pumping)[1-8]. In a denser plasma, the role of the scattering can be assumed by particle collisions[6,7].

At the present time, in connection with the discovery of pulsars and of the alternating magnetic fields corresponding to them, interest has been renewed in magnetic pumping as one of the possible mechanisms for accelerating charged particles in the atmosphere of a pulsar[9].

As applied to a single particle, Alfven obtained the following equation for the rate of growth of its energy in time:

$$dp/p = dt/\tau, \tau = T/\ln\delta, \delta = p_1/p_0,$$

where p_0 and p_1 are the values of the total momentum before and after one cycle T of the variation of B[2,5,6]. Schluter[7] developed the Alfven analysis further, proving that the character of the periodic variations of the field is of no principal significance. He has shown in this paper that in the particular case of harmonic variation of the field

$$B = B_0(1 + \beta\cos Wt), \beta < 1 \qquad (1)$$

the effect of acceleration for an ensemble of particles having the same energy is maximal at κ = W, where κ is the particle collision frequency.

However, in none of the cited papers was a kinetic equation introduced for the particle distribution function f(p, t) under magnetic-pumping conditions. Considerations to the effect that Alfven acceleration can explain the observed spectrum of the cosmic rays $\sim p^{-2,6}$ are advanced in[4,5], but these considerations are not based on the kinetic equation and are more readily qualitative in character.

In none of the cited papers is there a discussion of the source of the turbulent pulsations that ensure the turbulence intensity needed for the acceleration.

We show in the present paper that the kinetic equation for the distribution function $\bar{f}(p, t)$ averaged over the fast turbulent pulsations can be reduced in the quasilinear approximation, in the presence of a force $F = \frac{1}{2}(p_\perp/B)dB/dt$, to the diffusion equation in momentum space, with a diffusion coefficient D(p) that depends on the parameters of the alternating magnetic field and on the turbulence spectrum $\Phi(k) = dh^2/dk$ (h^2 is the turbulence intensity and k is the wave number (see Sec. 2)). As is well known[10], in the diffusion approximation the turbulence spectrum determines the rate of growth (or damping) of the plasma oscillations, which in turn depends on $\bar{f}(p, t)$, so that the problem of finding the spectrum of the waves and of the particles should be formulated in a self-consistent manner. In the present paper (see Sec. 3) we propose to obtain a self-consistent system of equations for $\bar{f}(p, t)$ and $\Phi(k)$ by using the equation of the boundary of the cyclotron instability $\gamma = 0$ (γ is the increment), which arises in an alternating field as a result of the anisotropy of the angular distribution with respect to the velocities. As is well known, an instability of this type is due to resonance between the waves and the particles at the Larmor frequency, with account taken of the Doppler effect[11].

In Secs. 4 and 5 we obtain, by successive approximations, a solution of the self-consistent problem without using numerical methods, for nonrelativistic and ultrarelativistic stationary and nonstationary accelerations, and also for ultrarelativistic stationary acceleration of electrons with allowance for the synchrotron radiation. It is shown that at the plasma instability boundary the turbulence spectrum has a universal form $\Phi(k) = \Phi_0/k^2$. We compare the obtained diffusion coefficient with the Fermi coefficient, and also with the diffusion coefficient for turbulent acceleration, as calculated in[12,13]. We show that the Alfven acceleration leads in the presence of synchrotron radiation to the same particle spectrum, which is formed according to[12] under Fermi acceleration. In Sec. 5 we investigate the limit of applicability of the statistical Alfven acceleration with allowance for the synchrotron radiation.

All the results are obtained for a collisionless plasma under the following assumptions: 1) the variation of the magnetic field is given by Eq. (1); 2) the plasma is assumed to be cold and the fraction of accelerated particles small; 3) the velocities of the fast particles exceed the Alfven velocity c_A; 4) the main scale of turbulence L is much larger than the Larmor radii of the fast particles (in this case L does not exceed $L \sim Tc_A$, where $T = 2\pi/W$); 5) the angular distribution of the wave vectors of the pulsations is isotropic or has a maximum at small angles to the field, so that the main contribution to the scattering is made by waves traveling along B.

In Sec. 6, devoted to the Crab nebula, it is shown that the considered type of acceleration of electrons with allowance for the synchrotron radiation can account for the power of the synchrotron x-radiation of the nebula. On the basis of the obtained solutions and data on the Crab nebula, we present numerical estimates of the diffusion coefficient V_0 and of the concentration n of the x-ray electrons, and these agree in order of magnitude with the estimates given in[13,14].

We propose to use the kinetic energy of magnetic pumping with account taken of the synchrotron radiation of the electrons in order to explain the observed x-radiation power of the pulsar located at the center of the Crab nebula.

2. DERIVATION OF ACCELERATION EQUATION

In the presence of an alternating magnetic field B(t) that varies with conservation of the adiabatic invariants, $p_z = \text{const}$ and $p_\perp^2/B(t) = \text{const}$, the equation for the distribution function of the fast particles f(p, t), averaged over the turbulence pulsations, can be written in the quasilinear approximation, as follows from[13,15], in the form

$$\frac{\partial f}{\partial t} + p\sin\theta \frac{\dot{B}}{2B}\left(\frac{\partial f}{\partial p}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{p}\right) = \hat{S}f, \tag{2}$$

where

$$\hat{S} = \frac{e^2}{m^2c^2}\frac{1}{\varepsilon\sin\theta}\frac{\partial}{\partial\theta}\sin\theta|P(\theta)|\frac{\partial}{\partial\theta} \tag{3}$$

is the scattering operator

$$P(\theta) = \frac{\Phi(\omega_B/cp\cos\theta)}{cp\cos\theta},$$

p and ϵ are the dimensionless momentum and energy of the particle, expressed in units of mc and mc^2, respectively, $\omega_B = eB/mc$ (e is the charge and m the rest mass of the particle), and Φ is the spectral function of the turbulence. In writing down (2) we took into account the facts that $p_\perp$ = psin and $p_z = p\cos\theta$ (θ is the angle between p and B), and we assumed that the main turbulence scale L is much larger than the Larmor radii of the fast particles.

The right-hand side in (2) corresponds to the one-dimensional model of cyclotron resonance (to waves traveling along the field, generally speaking, in two directions), and is the zeroth approximation in the expansion of the quasilinear term in powers of c_A^2/v^2 ($c_A = B_0/\sqrt{4\pi\rho}$ is the Alfven velocity, ρ is the density of the cold plasma, and $v = cp/\epsilon$ is the particle velocity). In this approximation, as shown in[13], the resonant interaction of the waves and particles does not lead to turbulent acceleration, and reduces to pure scattering.

We shall solve Eq. (2) in a quasilinear approximation, averaging over the period $T = 2\pi/W$ of the variation of the magnetic field (1). To this end we represent the solution in the form

$$f(p,\theta,t) = \bar{f}(p,t) + f_1(p,\theta,t) + f_2(p,\theta,t), \tag{4}$$

where f_1 and f_2 are increments of $\bar{f}$, symmetrical and asymmetrical with respect to $\theta = \pi/2$ (the superior bar denotes averaging over the period T). By virtue of the general properties of scattering that leads to an isotropic distribution[12,13], the average of the symmetrical function over the solid angle, $\langle f_1\rangle$, is equal to zero, i.e.,

$$\langle f_1\rangle = \frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta f_1\,d\theta = 0. \tag{5}$$

The reason for breaking down the angle part into f_1 and f_2 is that according to[13] the isotropization for an initial symmetrical distribution is different than that for an asymmetrical one, so that

$$\hat{S}f_1 \neq 0,\ \hat{S}f_2 = 0. \tag{6}$$

Using expression (3) for the scattering operator, we can readily show that

$$\langle\hat{S}f_1\rangle = \frac{e^2}{2\varepsilon m^2c^2}\left\{\sin\theta|P(\theta)|\frac{\partial f_1}{\partial\theta}\right\}\Big|_0^{\pi} = 0. \tag{7}$$

Indeed, the turbulence intensity $h^2 \sim k\Phi(k)$ is limited at $\theta = 0$ and π, since, by assumption, the main scale of the turbulence is much larger than the Larmor radii of the fast particles, so that h^2 decreases with increasing $k = \omega_B/cp\cos\theta$, and the value of $\partial f_1/\partial\theta$ along the magnetic field is also bounded or else approaches infinity no faster than 1/sin[13].

The linearized kinetic equation for $f_2 + f_2$, which can readily be obtained with the aid of (2) and (4), turns out, after averaging over the solid angle and after using (5)--(7), to depend only on f_2:

$$\left\langle\frac{\partial f_2}{\partial t}\right\rangle + p\langle\sin^2\theta\rangle\frac{\partial\bar{f}}{\partial p}\frac{\dot{B}}{2B} = 0. \tag{8}$$

In the analysis that follows, we confine ourselves to the case of strong scattering, when the isotropization time τ_0 (the time of scattering of a particle with momentum p through an angle π) is much smaller than the period T. Subtracting (8) from the unaveraged (over the solid angle) linearized kinetic equation and assuming that in the case of strong scattering we have

$$\frac{\partial f_1}{\partial t} + \frac{\partial f_2}{\partial t} - \left\langle\frac{\partial f_2}{\partial t}\right\rangle \approx 0,$$

and that this equality can be violated only within negligibly short time intervals $\tau_0 \ll T$, we obtain the following equation for f_1:

$$p\frac{\partial\bar{f}}{\partial p}\frac{\dot{B}}{2B}(\sin^2\theta-\langle\sin^2\theta\rangle)=\hat{S}f_1. \quad (9)$$

The strong-scattering approximation is an expansion in powers of the parameter α/α_0, where α is the true anisotropy at the plasma stability limit, and α_0 is the anisotropy occurring in an alternating field when no account is taken of instability and scattering. The conditions under which this parameter is small will be given later (see Sec. 3) in the analysis of the oscillation increment connected with the anisotropy.

Although the condition $\tau_0 \ll T$ is a definite limitation, at the same time, in accord with[6] (p. 80), it is precisely under this condition that the Alfven cycle leads to acceleration, so that the use of this approximation is perfectly justified.

Using (3) and recognizing that $\langle \sin^2\theta\rangle = 2/3$, and integrating (9) twice with respect to θ, we obtain

$$f_1=\frac{\dot{B}}{2B}p\frac{\partial\bar{f}}{\partial p}Q(p,\theta), \quad (10)$$

where

$$Q(p,\theta)=-\frac{1}{3}\frac{m^2c^2}{e^2}\varepsilon\int\frac{\cos\theta\sin\theta\,d\theta}{|P(\theta)|}. \quad (11)$$

To obtain an equation for $\bar{f}(p, t)$, we substitute (4) in (2), average over the period T, and take into account the fact that $\overline{\dot{B}f_2/2B} = 0$, since, according to (8), $\dot{B}/2B \sim \partial f_2/\partial t$; retaining terms that are quadratic in the pulsations with the period T, we find in the quasilinear approximation

$$\frac{\partial\bar{f}}{\partial t}+p\sin^2\theta\overline{\frac{\dot{B}}{2B}\frac{\partial f_1}{\partial p}}+\sin\theta\cos\theta\overline{\frac{\dot{B}}{2B}\frac{\partial f_1}{\partial\theta}}=0. \quad (12)$$

Substituting (10) in (12), averaging (12) over the solid angle, and recognizing that according to (6) and (10) we have

$$\int_0^\pi\sin\theta\,Q(p,\theta)\,d\theta=0, \quad (13)$$

we obtain for $\bar{f}(p)$ after simple transformations an acceleration equation of the diffusion type,

$$\frac{\partial\bar{f}}{\partial t}=\frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2D(p)\frac{\partial\bar{f}}{\partial p}\right], \quad (14)$$

where

$$D(p)=\frac{m^2c^2p^2\varepsilon}{6e^2}\overline{\left(\frac{\dot{B}}{2B}\right)^2}\int_0^\pi\sin^3\theta\left[\int\frac{\cos\theta\sin\theta\,d\theta}{|P(\theta)|}\right]d\theta. \quad (15)$$

FORMULATION OF THE SELF-CONSISTENT PROBLEM OF THE WAVE AND PARTICLE SPECTRUM

Owing to the scattering of the particles by the plasma turbulence, part of the momentum accumulated during the time that the magnetic field (1) is increased as a result of the betatron acceleration ($p_\perp^2 \sim B$), is transferred to the parallel component of the momentum p_z. As a result, if the scattering time is small ($\tau_0 \ll T$), the loss of momentum during the time of decrease of the magnetic field (1) turns out to be smaller than the increase of the momentum during the time of the increase of the magnetic field, and the natural result is acceleration (see[6], p. 80).

One can expect the cyclotron instability connected with the anisotropy of the angular distribution with respect to the velocities to be, in the presence of an alternating magnetic field, the source of the intense turbulence needed for effective acceleration. Such an anisotropy in an initially isotropic plasma is the result of the conservation of the adiabatic invariant $p_\perp^2/B = \text{const}$, and is connected with the two-dimensional contraction (expansion) of the Larmor orbits of the particle as a result of the periodic increase (decrease) of the magnetic field. This instability, due to cyclotron resonance between the waves and the particles at the Larmor frequency, occurs at rather low anisotropy when account is taken of the Doppler effect and at sufficiently high particle velocity[11].

According to[11,16] (see p. 188 of[16]), the expression for the increment of waves having circular polarization and traveling along the field is

$$\gamma=-\frac{2\pi^2e^2\omega}{c^2mk^2}\frac{\partial\omega}{\partial k}\cdot\int_0^\infty\left\{f\pm\frac{\omega_B}{\omega\varepsilon}\left(\frac{\partial f}{\partial p_\perp^2}-\frac{\partial f}{\partial p_z^2}\right)p_\perp^2\right\}dp_\perp^2,$$

$$|p_z|\approx\frac{\omega_B}{ck}, \quad (16)$$

where the upper sign corresponds to the Alfven wave and the lower to the fast magnetosonic wave. The quantity α, which characterizes the anisotropy of the distribution function and which determines the properties of the increment (16), is given by

$$\alpha=\frac{\partial f}{\partial p_\perp^2}-\frac{\partial f}{\partial p_z^2}=\frac{1}{2p^2\sin\theta\cos\theta}\frac{\partial f}{\partial\theta}. \quad (17)$$

Equation (17) can readily be proved if it is recognized that $p_\perp = p\sin\vartheta$, $p_z = p\cos\theta$.

Were there no scattering (no reaction of the waves on the particles), the corresponding anisotropy α_0 would be due only to the adiabatic variation of the momentum in the alternating field (a), and would be very large compared with the anisotropy α in the presence of strong scattering. Thus, in the case of strong scattering, when the angular distribution function is close to isotropic, there exists a small parameter $\alpha/\alpha_0 \ll 1$ in terms of which one can carry out the expansion, as already mentioned in the preceding section.

In the absence of scattering and at sufficiently high concentration of the fast particles, the increment (16) can periodically assume large positive values $\gamma_0 \gg W$. When $\gamma_0 \gg W$, the time $t_0 \sim 1/\gamma_0$ needed for the oscillations to increase by a factor e is much shorter than the period $T = 2\pi/W$, and it is precisely under this condition that the scattering can occur within a time $\tau_0 \ll T$. In other words, the condition $\gamma_0 \gg W$ is necessary for acceleration to be possible, for otherwise, when the instability develops slowly, the scattering of the particles does not lead to complete isotropization at the end of the cycle, and the Alfven acceleration may stop. In Sec. 5 we shall use the condition $\gamma_0 \gg W$ to estimate the minimum particle concentration at which Alfven acceleration is possible.

The scattering of the particles by the plasma turbulence leads to isotropization of the particle velocities and consequently to a damping of the resonant oscillations, until the magnetic pumping leads to a new cycle of energy redistribution. It can be assumed that the resultant oscillation increment $\gamma(\omega)$, describing the dynamics of the joint action of the magnetic pumping and of the resonance, is equal to zero under strong-scattering conditions, i.e., $\gamma(\omega) = 0$. In fact, within short time intervals ($\tau_0 \ll T$) the increment depends on the time and differs from zero. We shall neglect this dependence, however, since we assume the strong-scattering condition to be satisfied.

The equation for the plasma instability limit $\gamma(\omega) = 0$ will be used below to obtain self-consistent equations for the wave and particle spectrum. From (5)—(12) it follows that the angle function f_2 is not connected with the resonant interaction of the waves and particles, and makes no contribution to the dynamics of the acceleration in the case of strong scattering; we shall therefore take into account in the expression for the increment only that part of the anisotropy (17) which is connected with the angle function f_1. Taking this circumstance into account, using expressions (17), (10), and (11), and changing over in (16) to integration with respect to p, we find that the equation $\gamma = 0$ takes the form

$$\int\limits_{\omega_B/ck}^{\infty}\left\{\bar{f}\mp\frac{\omega_B}{\omega\varepsilon}\frac{m^2c^2}{6e^2}\frac{\dot{B}}{2B}\frac{\partial\bar{f}}{\partial p}\,|P(\theta)|^{-1}\sin^2\theta\right\}p\,dp=0,$$

$$\cos\theta=\omega_B/ckp. \tag{18}$$

Eq. (18) can easily be solved with respect to $\Phi(k)$, if it is recognized that at resonance ($\cos\theta = \omega_B/ckp$) the function

$$\Phi(\omega_B/cp\cos\theta)\equiv\Phi(k)$$

is independent of p. Taking $\Phi(k)$ outside the integral sign in (18), we obtain after simple transformations

$$\Phi(k)=\mp\frac{1}{3}\frac{m^2c^2\omega_B}{e^2\omega k}\frac{\dot{B}}{2B}$$

$$\times\left\{1+\lim_{p\to\infty}\left(\frac{\omega_B^2}{c^2k^2}-p^2\right)\bar{f}(p)\Big/2\int\limits_{\omega_B/ck}^{\infty}p\bar{f}dp\right\}. \tag{19}$$

Equations (19), (14), and (15) constitute the sought self-consistent system of equations for determining the particle spectrum defined by the distribution function $\bar{f}$, and the wave spectrum $\Phi(k)$.

4. SOLUTION OF SELF-CONSISTENT PROBLEM

We shall solve Eqs. (14), (15), and (19) by successive approximations, assuming as the zeroth approximation a wave spectrum in the form $\Phi(k) = \Phi_0/k^2$, at which the undetermined angle interval in (15) is given by

$$\int\frac{\cos\theta\sin\theta\,d\theta}{|P(\theta)|}=\frac{\omega_B^2}{\Phi_0cp}(C_0-b|\cos\theta|),$$

$$b=\begin{cases}+1, & 0<\theta<\pi/2\\ -1, & \pi/2<\theta<\pi.\end{cases} \tag{20}$$

We obtain the integration constant C_0 from Eq. (6), which is transformed with the aid of (10), (11), and (20) into

$$\int\limits_0^{\pi}\sin\theta(C_0-b|\cos\theta|)\,d\theta=0,$$

from which it follows that $C_0 = 1/2$. Substituting (20) (with $C_0 = 1/2$) in (15) and integrating with respect to the angle, we obtain

$$D(p)=D_0\varepsilon p,\quad D_0=\frac{m^2c^2\omega_B^2}{36e^2c\Phi_0}\overline{\left(\frac{\dot{B}}{2B}\right)^2}, \tag{21}$$

so that Eq. (14) takes the form

$$\frac{\partial\bar{f}}{\partial t}=\frac{D_0}{p^2}\frac{\partial}{\partial p}\left(p^3\varepsilon\frac{\partial\bar{f}}{\partial p}\right). \tag{22}$$

To find the next approximation for $\Phi(k)$, we must substitute the solution of (22) in (19). In the nonrelativistic case ($\epsilon = 1$) the solution of (22) is[17]

$$\bar{f}=\frac{A}{(D_0t)^3}\exp\left(-\frac{p}{D_0t}\right),$$

and in the ultrarelativistic case ($\epsilon = p$), the nonstationary solution of (22), as shown in[12], is

$$\bar{f}=Ap^{-3/2}\exp\left(-\ln^2p/4D_0t\right).$$

In the particular case of stationary acceleration ($\partial\bar{f}/\partial t = 0$), Eq. (22) with $\epsilon = 1$ is satisfied with the function $\bar{f} \sim p^{-2}$, and at $\epsilon = p$ by the function $\bar{f} \sim p^{-3}$. It is easily shown that in each of these four particular cases we have

$$\lim_{p\to\infty}\left(\frac{\omega_B^2}{c^2k^2}-p^2\right)\bar{f}(p)\Big/2\int\limits_{\omega_B/ck}^{\infty}p\bar{f}dp=0,$$

so that expression (19) takes the universal form

$$\Phi(k)=\mp\frac{m^2c^2\omega_B^2}{3e^2\omega k}\frac{\dot{B}}{2B} \tag{23}$$

and coincides with the initial approximation $\Phi(k) = \Phi_0/k^2$, for Alfven waves whose dispersion equation is $\omega = c_Ak$. Thus, the indicated solutions of (22) together with (21) and (23) (Alfven waves) are exact solutions of the self-consistent problem for the cases considered above.

According to (23), (21), and (1), the final expressions for the Alfven-wave spectrum and the diffusion coefficient D_0, recognizing that $\dot{B}/2B \approx -\frac{1}{2}\beta W\sin Wt$ and putting $|\sin Wt| \approx (\overline{\sin^2 Wt})^{1/2} = 1/\sqrt{2}$, take the form

$$\Phi(k)=\frac{\Phi_0}{k^2},\quad \Phi_0=\frac{B_0^2\beta W}{6\sqrt{2}\,c_A}, \tag{24}$$

$$D_0=\frac{\sqrt{2}}{48}\beta W\frac{c_A}{c}. \tag{25}$$

It follows from (24) that the energy of the turbulent pulsations

$$\frac{h^2}{8\pi}=\frac{1}{8\pi}\int\Phi(k)\,dk=\frac{\beta WB_0^2}{48\sqrt{2}\pi\omega}$$

is much smaller than the magnetic energy $B_0^2/8\pi$, since $\beta < 1$, and the resonant frequencies are $\omega \gg W$. Thus, the condition $h^2 \ll B_0^2$ for quasilinearity[11] of the initial kinetic equation (2) at the plasma stability limit is well satisfied.

In[12], the following expression is obtained for the Fermi diffusion coefficient D_F in momentum space, as applied to the case of the most effective Fermi acceleration (reflection from long strong waves, when $\Phi(k) \sim k^{-\nu}$, $\nu > 2$):

$$D_F\approx\frac{c_A^2}{Lc}\frac{h_m}{B_0}, \tag{26}$$

where L is the main scale, and $h_m = B_m - B_0$ is the maximum amplitude of the turbulence. If the fundamental frequency of the Fermi reflections $\omega(L) = 2\pi c_A/L$ coincides with the frequency of the field (1), then at $h_m = \beta B_0/4\sqrt{2}$ the expressions for the diffusion coefficients (25) and (26) do not differ from each other. However, a difference remains in the turbulence spectrum $\Phi(k)$ needed for effective acceleration. In the case of two-dimensional adiabatic variation of the Larmor orbits, a larger turbulence energy is needed for the isotropization of the particle velocities than in the case of the one-dimensional adiabatic Fermi mechanism. Indeed, in the Alfven acceleration ($\nu = 2$) the average amplitude h of pulsations of scale λ, i.e., the quantity

$$h \sim \left(\int \Phi(k)\,dk\right)^{1/2} \sim \sqrt{\lambda^{\nu-1}},$$

decreases with decreasing λ more slowly than in the case of Fermi acceleration, when $\nu > 2$. This leads in the case $\nu = 2$ to an increase in the role of medium and small scales, i.e., to a stronger scattering, for it is precisely these scales that are responsible for the cyclotron-resonance interaction.

Finally, we present a comparison with the turbulence acceleration investigated in[13]. According to[13], the diffusion coefficient D_T in momentum space in the presence of the spectrum $\Phi(k) \sim k^{-2}$ is given by

$$D_T \approx \frac{c_A^2}{Lc}\left(\frac{h_m}{B_0}\right)^2, \tag{27}$$

so that if

$$\omega(L) = 2\pi c_A / L, \quad h_m = \beta B_0 / 4\sqrt{2},$$

then, according to (25)--(27),

$$D_0 / D_T = D_F / D_T \sim 1/\beta.$$

Thus, at the same form of the spectrum ($\nu = 2$), the Alfven diffusion coefficient D_0 is larger by a factor $1/\beta$ than D_T.

5. ACCELERATION OF ELECTRONS WITH ALLOWANCE FOR SYNCHROTRON RADIATION: CONDITIONS FOR THE FEASIBILITY OF ACCELERATION

Let us consider an ultrarelativistic stationary problem, in which account is taken of the synchrotron radiation in addition to the Alfven acceleration. When the losses for radiation are taken into account, the diffusion equation (14) takes the form (see[12])

$$\frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 D(p)\frac{\partial \bar f}{\partial p} + p^4 \eta \bar f\right] = 0, \tag{28}$$

where

$$\eta = \frac{\dot p}{p^2} = \frac{4e^4 B_0^2}{9m^3c^5}, \tag{29}$$

$\dot p$ is the rate of momentum loss into synchrotron radiation, averaged over the angle between the field **B** and **p**. Equations (28), (15), and (19) constitute a self-consistent system of equations for $\bar f$ and $\Phi(k)$, and it is necessary to put $\epsilon = p$ in (15), since ultrarelativistic acceleration is being considered.

Choosing as the zeroth approximation $\Phi(k) = \Phi_0/k^2$ and calculating in accordance with formula (15) the function $D(p)$ corresponding to this approximation, we find in analogy with the analysis of the preceding section that $D(p) = D_0p^2$, where D_0 is determined from (21). When $D(p) = D_0p^2$, the solution of (28), leading to a limited plasma density, is

$$\bar f = Ae^{-\mu p}, \quad \mu = \eta / D_0. \tag{30}$$

It is easy to show that for $\bar f$ in the form (30), the second term in the curly bracket of (19) vanishes, so that in the case of Alfven waves ($\omega = kc_A$) the first approximation for $\Phi(k)$ coincides with the initial zeroth approximation and assumes the universal form (24). Thus, expression (30) is the exact solution of the problem, and, in accordance with (25),

$$D_0 = \frac{\sqrt{2}}{48}\beta W \frac{c_A}{c}.$$

It is possible to present another expression for D_0. The intensity of the synchrotron radiation $dJ \sim \bar f p^4 dp$, according to (30), has a maximum at $\mu = 4/p$, so that the quantity $D_0 = \eta/\mu$ can be expressed in terms of the effective radiation frequency $\omega_{\text{eff}} \sim \omega_B p^2$ just as in the case of the Fermi acceleration (formula (56) of[12]), namely:

$$D_0 = \frac{\eta p}{4} \approx \frac{1}{9}\frac{e^2\omega_B^2}{mc^3}\sqrt{\frac{\omega_{\text{eff}}}{\omega_B}}. \tag{31}$$

According to expressions (30) and (31) and the results of[12], the spectra of the particles in the ultrarelativistic case, with account taken of the Lorentz-friction force, coincides fully for the Alfven and Fermi accelerations.

The constant A in (30) can be expressed in terms of the concentration n of the particles with momentum $> p$, namely

$$A = n \Big/ \int_p^\infty p^2 e^{-\mu p}dp = \frac{n\mu^3}{2(1 + \mu p + {}^1\!/_2\mu^2p^2)e^{-\mu p}}. \tag{32}$$

Using (30) and (32), and carrying out the corresponding integration, we obtain the following expressions for the energy density w_e of electrons with momentum $> p$ and for the intensity J of their synchrotron radiation:

$$w_e = Amc^2\int_p^\infty p^3 e^{-\mu p}\,dp = \frac{3mc^2n}{\mu}Q(\mu p), \tag{33}$$

$$J = Amc^2\eta\int_p^\infty p^4 e^{-\mu p}\,dp = \frac{12mc^2\eta n}{\mu^2}M(\mu p), \tag{34}$$

where

$$Q(z) = \frac{1 + z + {}^1\!/_2z^2 + {}^1\!/_6z^3}{1 + z + {}^1\!/_2z^2}, \quad M(z) = Q(z) + \frac{{}^1\!/_{24}z^4}{1 + z + {}^1\!/_2z^2}. \tag{35}$$

It follows from (33) and (35) that

$$J = \frac{4\eta w_e M(\mu p)}{\mu Q(\mu p)} \sim \eta w_e p, \tag{36}$$

where account is taken of the fact that $zQ(z)/M(z) \sim 4$ for all $z = \mu p > 4$.

We note that expressions (33)--(36) are equally valid for the Alfven mechanism and for the Fermi acceleration.

To estimate the minimum concentration $n_{\min}$ of fast particles, at which Alfven acceleration is possible, it is necessary, as noted in Sec. 3, to calculate the increment $\gamma_0(\omega)$ of the resonant oscillations in the absence of scattering. The expression for $\gamma_0(\omega)$ can be obtained by using (16) and (17) and taking into account the fact that in the absence of scattering the angle part $f_\theta = f_1 + f_2$ of the distribution function (4) is determined only by the adiabatic variation of the momentum in the alternating field (1). The breakdown of the function f_θ into f_1 and f_2 then becomes meaningless, so that the linearized kinetic equation for f_θ, after averaging over the slow period T, can be readily shown to take the form

$$\frac{\partial f_\theta}{\partial t}+\sin^2\theta\frac{\partial \bar f}{\partial p}\frac{\dot B}{2B}=0.$$

Putting $\dot B/2B \approx -\frac{1}{2}\beta W \sin Wt$ and substituting the solution of this equation in (17), we find that

$$\frac{\partial \bar f}{\partial p_\perp^2}-\frac{\partial \bar f}{\partial p_z^2}=\frac{\beta}{2p}\frac{\partial \bar f}{\partial p}(-\cos Wt+C). \quad (37)$$

Since the anisotropy (37) should be positive when $\cos Wt \geq 0$, i.e., when $B \geq B_0$, and for fast particles we have $\partial \bar f/\partial p < 0$ on the tail of the distribution, it follows that the integration constant is $C = 0$. We use for $\bar f$ expression (30), which takes into account the synchrotron radiation. Substituting (30) and (37) in (16), taking (32) into account, and using the dispersion equation of the Alfven waves $\omega = kc_A$, we obtain after integration the following expression for the increment

$$\gamma_0=A_0 nxe^{-x}(1+x)\left\{\frac{\beta c}{2c_A}x\cos Wt\left[1+\frac{x^2\,\mathrm{Ei}(-x)}{(1+x)e^{-x}}\right]-1\right\}, \quad (38)$$

where $\mathrm{Ei}(-x)$ is the integral exponential function and where

$$A_0=\frac{4\pi^3e^2}{m\omega_B}\left(\frac{c_A}{c}\right)^2\left(1+\mu p+\frac{\mu^2p^2}{2}\right)^{-1}e^{-\mu p}, \quad (39)$$

$$x=\frac{\omega_B}{\omega}\frac{c_A}{c}\mu. \quad (40)$$

From the resonance condition

$$p\cos\theta=\omega_B c_A/\omega c$$

we obtain, after the averaging over θ,

$$p^2/3=(\omega_B c_A/\omega c)^2,$$

so that (40) can be written in the form $x = \mu p/3$.

It can be shown that the increment (38) has a maximum at $x = 2.3$, i.e., at $\mu p = \sqrt{3x} \approx 4$. Within the resonant-frequency band, the momentum changes from p_0 to p_{max}. By stipulating that γ_0 be much larger than the frequency W of the alternating field in this band, i.e., by stipulating $n \gg n_{min}$, we can find the value of n_{min} from the equation $\gamma_0(n_{min}, p_{max}) = W$. This method will be used in Sec. 6 to estimate the minimum concentration of the x-ray electrons in the Crab nebula.

We can formulate one more obvious condition for the feasibility of Alfven acceleration, namely:

$$t\geqslant T, \quad t=\frac{9}{4}\frac{m^3c^5}{e^4B_0^2}\sqrt{1-\frac{v^2}{c^2}}=\frac{1}{\eta p}, \quad (41)$$

where t is the time during which the electron rotating in the field B_0 loses an energy equal to its total energy to radiation[18]; p is the dimensionless momentum and η is defined by (29). Indeed, in the opposite case the particle loses its entire energy to radiation before isotropization of the momenta takes place, so that the Alfven cycle does not lead to acceleration.

6. APPLICATION OF THE RESULTS OF THE THEORY TO THE CRAB NEBULA

As is well known, the power of the synchrotron radiation of the Crab nebula is $\mathscr{P} = 9 \times 10^{37}$ erg/sec, and its x-ray spectrum extends from $h\nu \sim 2$ keV to $h\nu \sim 500$ keV[14,19,20]. According to[14,21] the average magnetic field in the radiating regions is $B_0 \approx 9 \times 10^{-4}$ Oe, the characteristic period of the condensation of the magnetic field B in the direction perpendicular to B, in the so-called "twist," is $T = \lambda/c_A \sim 10^7$ sec, where $\lambda \sim 10^{17}$ sec is the average distance between the "twist" and where the Alfven velocity is $c_A \sim (0.1-0.2)c$.

Equating the density of the magnetic energy in the Crab nebula $w_m = B_0^2/8\pi = 3.3\cdot10^{-3}$ erg/cm^3 to the energy density of the electrons (33), and equating the radiation power $\mathscr{P} = 9\cdot10^{37}$ erg/sec to the quantity $P = VJ$, where V is the volume of the radiating region and where the intensity of the radiation J is determined by formula (34), we obtain equations for the diffusion coefficient $D_0 = \eta/\mu$ and the particle concentration n. Assuming that $p = 1.8 \times 10^7$ is the dimensionless momentum of the electrons producing a radiation with energy 2 keV in a field $B_0 = 0 \times 10^{-4}$ Oe, and that the possible volume of the "twist" is $V = 2 \times 10^{52}$ cm^3[14], we find from the indicated equations that $D_0 = 1.8 \times 10^{-8}$ sec^{-1} and $n = 4 \times 10^{-10}$ cm^{-3}, where n is the number of particles with $p > 1.8 \times 10^7$.

According to[14], the concentration of the optical electrons of the Crab nebula is $n_0 = 5 \times 10^{-9}$ cm^{-3}. As expected, in accordance with the kinetic analysis we have $n < n_0$.

A diffusion coefficient of the same order can be obtained from formula (25) by putting $W = 2\pi/T = 2\pi \times 10^{-7}$ sec^{-1} and $c_A = 0.2c$, and recognizing that according to Shklovskiĭ's estimate (p. 289 of[14]) the maximum relative increase of the magnetic field in the "twist" is $\beta = 3.5$. We note that the Fermi diffusion coefficient (31) at $h\nu_{eff} \sim 30$ keV and $B_0 = 9 \times 10^{-4}$ Oe also coincides with the value $D_0 = 1.8 \times 10^{-8}$ sec^{-1} obtained above.

Let us estimate the minimum concentration of the fast particles n_{min}, at which Alfven acceleration ensuring synchrotron radiation of the Crab nebula up to energies $h\nu \sim 500$ keV is possible. We obtain the value of n_{min} from the equation $\gamma_0(n_{min}, p_{max}) = W$, where γ_0 is the increment (expression (38)) in the absence of resonance, and p_{max} is the dimensionless momentum of the electrons causing radiation of 500 keV. Putting in (38) $W = 2\pi \times 10^{-7}$ sec^{-1}, $\beta = 3.5$, $c_A = 0.1$ c, $D_0 = 1.8\cdot10^{-8}$ sec^{-1} and using (38) and (39), we find that $n_{min} = 2.6 \times 10^{-10}$ cm^{-3}. Since $n = 4 \times 10^{-10}$ cm^{-3} is close to n_{min}, we can state that the hardest x-radiation of the Crab nebula corresponds to the limit of applicability of the investigated type of acceleration.

For the pulsar located at the center of the Crab nebula, for which we know the period $T = 0.033$ sec and the power of x-radiation $\mathscr{P} = 5 \times 10^{35}$ erg/sec in the range $h\nu = 1.5-10$ keV[20,22], the formulas obtained in Sec. 5, including the limiting properties of the Alven acceleration, can be used to estimate the average magnetic field B_0 in the radiating region surrounding the pulsar, the volume V of this region, and the concentration n of the x-ray electrons. A preliminary estimate of these quantities is: $B_0 \sim 1000$ Oe, $V \sim 10^{29}$ cm^3, and $n \sim 10^6$ cm^{-3}.

Assuming that the energy density of the electrons responsible for the x-ray and optical radiation of the pulsar are approximately the same, and that both types of radiation come from the same volume V, we get from (37)

$$\mathscr{P}_X / \mathscr{P}_0 \sim (\nu_X / \nu_0)^{1/2}, \tag{42}$$

where $\mathscr{P}_X$ and $\mathscr{P}_0$ are the synchrotron-radiation powers at the x-ray and optical frequencies ν_X and ν_0, respectively. Assuming for the pulsar in the Crab nebula $\nu_X \sim 10^{18}$ Hz and $\nu_0 \sim 10^{14}$ Hz, we get from (42) $\mathscr{P}_X/\mathscr{P}_0 \sim 100$, which agrees with the available data[20,22].

Note added in proof (19 October 1970). The authors have learned that the results of Sec. 4 (formulas (24) and (25)) were obtained independently and by a different method by D. B. Melrose, Astrophysical Space Science 4, 165 (1969).

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