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TWO-LIQUID EFFECTS IN A NORMAL LIQUID

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Effects connected with the presence of weakly-damped phonons in a normal liquid are discussed. The thermomechanical effect is calculated. The propagation of shear oscillations is investigated. It is shown that at large distances the damping of the oscillations has a power-law character.

THERE are a number of liquids (hydrogen, He^3, He^4) that solidify at temperatures much lower than the Debye temperature Θ, defined in the usual manner in terms of the density and the speed of sound. For such liquids there exists a temperature region $T \ll \Theta$, in which the wavelength of the phonons with energy on the order of the temperature greatly exceeds the interatomic distance, and therefore the phonons constitute weakly-damped thermal excitations. The liquid thus consists of two weakly-coupled subsystems—phonons and the remainder of the liquid.

We emphasize that when it comes to liquid helium the condition $T \ll \Theta$ begins to be satisfied, generally speaking, at temperatures that are high compared with the temperature of the transition to the superfluid state (in the case of He^4) or with the Fermi-degeneracy temperature (in the case of He^3). For example, in He^4 at a pressure 25 atm and at $T = 3^\circ K$, the Debye temperature is equal to approximately 40°K, and in He^3 at 3 atm and $T = 2^\circ K$ the Debye temperature is $\Theta \approx 20^\circ K$. With further increase of pressure, the Debye temperature becomes even larger, while the temperature of the quantum degeneracy decreases with increasing pressure.

We shall assume that the only characteristic parameters of length and frequency in the liquid (if we disregard the phonons) are, respectively, the interatomic distance and the Debye frequency. By the same token, we exclude, in the case of liquid helium, the temperature region below the quantum-degeneracy temperature, where the mean free path of the rotons (in He^4) or of the Fermi excitations (in He^3) greatly exceeds the interatomic distance. We exclude, of course, also the region near the λ point and the critical gas-liquid point, where the correlation radius is large.

Like any thermal excitation, phonons make a certain contribution to the specific heat of the liquid. This contribution, however, is small compared with the total specific heat. In exactly the same way, the contribution of the phonons to the kinetic coefficients (viscosity, thermal conductivity, etc.), is small. Nonetheless, there are many phenomena that are entirely connected with the presence of weakly-damped phonons. This is clear from the fact that the usual hydrodynamic description of the liquid ceases to be, generally speaking, valid if the characteristic dimension of the problem is of the order of or less than the phonon mean free path, and not of the interatomic distance. Let us consider, for example, the flow of a liquid through a thin capillary. Since the phonons are diffusely scattered from the walls, their average velocity is smaller than the average flow velocity of the remainder of the liquid. The situation here is analogous to the flow of a superfluid liquid, the phonons playing the role of the normal part. In both cases there arises a thermomechanical effect, described by similar formulas in very thin capillaries.

In ordinary hydrodynamics, the shear (and the temperature) oscillations attenuate exponentially in the interior of the liquid. Owing to the presence of the phonons, these oscillations also propagate to distances greatly exceeding the penetration depth, and at large distances the damping has a power-law character.

We make one more important remark. As will be shown below, the main contribution to the thermodynamic effect in sufficiently broad capillaries and to the oscillation amplitude at sufficiently large distances is made by phonons with arbitrarily low frequencies. Since phonons with sufficiently low frequencies always attenuate weakly, the formulas obtained below are valid at large distances in any liquid at any temperature.

1. EQUATIONS OF MOTION

The phonon distribution function $n(\mathbf{r}, \mathbf{p}, t)$ satisfies the kinetic equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{r}} = I, \tag{1}$$

where the Hamiltonian is $H = \epsilon + \mathbf{p}\cdot\mathbf{v}$, $\mathbf{v}$ is the velocity of the liquid, $\epsilon = cp$ is the energy of the phonon in the liquid at rest, and c is the speed of sound. The "collision integral" I is due in our case mainly to absorption and emission of the phonons by the liquid, and the processes of collisions between the phonons at $T \ll \Theta$ are much less probable. We can therefore write

$$I = -(n - n_0)/\tau,$$

where $n_0 = \{e^{\epsilon/T} - 1\}^{-1}$ is the equilibrium distribution function and τ is the relaxation time of the phonons with the liquid. Using the well-known expression for the sound absorption coefficient (see[1]), we find

$$\tau = \frac{\alpha}{c^2}, \quad \alpha = \rho\hbar^2c^2\left\{\left(\frac{4}{3}\eta + \zeta\right) + \varkappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right\}^{-1}, \tag{2}$$

here ρ is the density of the liquid, c_p and c_V are the specific heats per unit mass, η and ζ are the first and second viscosity coefficients, and κ is the thermal conductivity coefficient.

The equations of motion of the liquid can readily be obtained in the usual manner (see[2]) by starting from the conservation laws. We note that at $v = 0$ the energy $\mathscr{E}$ and the momentum $\mathbf{j}$ per unit volume are equal to

$$\mathscr{E} = E(\rho, S) + \int \varepsilon n d\tau, \quad \mathbf{j} = \int \mathbf{p} n d\tau,$$

where $d\tau = d^3p/(2\pi\hbar)^3$ and E and S are the energy and entropy per unit volume of the liquid "without the phonons." With the aid of the Galileo transformation formulas we obtain, in an arbitrary coordinate system,

$$\mathscr{E} = \frac{\rho v^2}{2} + \mathbf{v} \int \mathbf{p} n d\tau + \int \varepsilon n d\tau + \mathrm{E}(\rho, S),$$

$$\mathbf{j} = \rho \mathbf{v} + \int \mathbf{p} n d\tau.$$

In the case considered by us, the total number of phonons is small, and we can therefore neglect the products of two or more integrals with respect to $d\tau$ throughout. Taking this circumstance into account, we obtain for the energy and for the momentum

$$\mathscr{E} = \frac{\rho u^2}{2} + \int \varepsilon n d\tau + E(\rho, S), \qquad (3)$$

$$\mathbf{j} = \rho \mathbf{u},$$

where we have introduced a new variable

$$\mathbf{u} = \mathbf{v} + \frac{1}{\rho} \int \mathbf{p} n d\tau,$$

which obviously plays the role of the "renormalized" velocity of the macroscopic motion of the liquid.

We seek the equations of motion in a form that ensures satisfaction of the mass and momentum conservation laws, and also the law governing the increase of the entropy

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0,$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_k} \{\rho u_i u_k + (-E + TS + \mu\rho)\delta_{ik} + \pi_{ik}\} = 0,$$

$$\frac{\partial \mathscr{S}}{\partial t} + \operatorname{div}(S\mathbf{u} + \mathbf{q}) = \frac{R}{T} \quad (R > 0). \qquad (4)$$

Here μ is the chemical potential, R is the dissipative function, and $\mathscr{S}$ is the total entropy per unit volume with allowance for the entropy of the phonons

$$\mathscr{S} = S + \int s\, d\tau, \quad s = (1 + n)\ln(1 + n) - n \ln n,$$

π_{ik} and q are the sought quantities due to the presence of phonons and to dissipation.

From (4) there should follow automatically an energy conservation law, i.e., an equation in the form

$$\dot{\mathscr{E}} + \operatorname{div} \mathbf{Q} = 0. \qquad (5)$$

Differentiating the first equation of (3) with respect to the time and using (4) and the thermodynamic identity $dE = TdS + \mu d\rho$, we obtain

$$\dot{\mathscr{E}} + \operatorname{div}\left\{\left(\frac{u^2}{2} + \mu\right)\rho\mathbf{u} + ST\mathbf{u} + u_k\pi_{ki} + T\mathbf{q}\right\}$$

$$= R + \pi_{ik}\frac{\partial u_i}{\partial x_k} + \mathbf{q}\nabla T + \frac{\partial}{\partial t}\int \varepsilon n\, d\tau - T\int \dot{s}\, d\tau.$$

Expressing the time derivatives of n with the aid of the kinetic equation (1), those of ϵ with the aid of the identity $d\epsilon = d\rho\partial\epsilon/\partial\rho + dS\partial\epsilon/\partial S$ and Eqs. (4) in terms of the spatial derivatives, we transform the last equation, with the required degree of accuracy, into

$$\dot{\mathscr{E}} + \operatorname{div}\left\{\left(\frac{u^2}{2} + \mu\right)\rho\mathbf{u} + ST\mathbf{u} + u_k\pi_{ki} + T\mathbf{q} + \int \varepsilon n \frac{\partial H}{\partial \mathbf{p}} d\tau\right.$$

$$\left. - T\int s\frac{\partial H}{\partial \mathbf{p}} d\tau\right\} = R + \int \varepsilon I\, d\tau - T\int \frac{ds}{dn} I\, d\tau \qquad (6)$$

$$+ \frac{\partial u_i}{\partial x_k}\left\{\pi_{ik} - \int p_i \frac{\partial \varepsilon}{\partial p_k} n\, d\tau - \delta_{ik}\int\left(\rho\frac{\partial\varepsilon}{\partial\rho} + S\frac{\partial\varepsilon}{\partial S}\right) n\, d\tau\right\}$$

$$+ \nabla T\left\{\mathbf{q} - \int s\frac{\partial H}{\partial \mathbf{p}} d\tau\right\}.$$

Comparing (6) and (5) and taking into account the condition that R be positive, we obtain the unknown quantities

$$\mathbf{Q} = \left(\frac{u^2}{2} + \mu\right)\rho\mathbf{u} + ST\mathbf{u} + u_k\pi_{ki} + T\mathbf{q} + \int \varepsilon n\frac{\partial H}{\partial \mathbf{p}} d\tau - T\int s\frac{\partial H}{\partial \mathbf{p}} d\tau,$$

$$\pi_{ik} = \int p_i \frac{\partial\varepsilon}{\partial p_k} n\, d\tau + \delta_{ik}\int\left(\rho\frac{\partial\varepsilon}{\partial\rho} + S\frac{\partial\varepsilon}{\partial S}\right) n\, d\tau$$

$$- \eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right) - \zeta\delta_{ik}\frac{\partial u_l}{\partial x_l}, \qquad (7)$$

$$\mathbf{q} = \int s\frac{\partial H}{\partial \mathbf{p}} d\tau - \frac{\varkappa\nabla T}{T},$$

$$R = T\int \frac{ds}{dn} I\, d\tau - \int \varepsilon I\, d\tau + \frac{\varkappa(\nabla T)^2}{T} + \frac{\eta}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right)^2$$

$$+ \zeta\left(\frac{\partial u_l}{\partial x_l}\right)^2.$$

The phonon part of the dissipative function is positive, since the difference

$$T\frac{ds}{dn}I - \varepsilon I = T\frac{n - n_0}{\tau}\left(\ln\frac{1 + n_0}{n_0} - \ln\frac{1 + n}{n}\right)$$

is positive for all positive n and n_0.

Formulas (1) and (4) constitute the complete system of equations of motion of the liquid with allowance for the phonons. We shall need a corresponding linearized system. Putting $n = n_0 + \chi\partial n_0/\partial\epsilon$ and calculating the integrals with respect to $d\tau$ of the equilibrium distribution function, we can write the linear system, after simple transformations, in the form

$$\dot{\rho} + \rho \operatorname{div} \mathbf{u} = 0,$$

$$\rho\dot{u}_i = -\frac{\partial P}{\partial x_i} + \eta\Delta u_i + \left(\zeta + \frac{\eta}{3}\right)\frac{\partial}{\partial x_i}\operatorname{div}\mathbf{u}$$

$$- \frac{\rho}{c}\left(\frac{\partial c}{\partial \rho}\right)_\sigma \int \frac{\varepsilon^3 d\varepsilon\, do}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon}\frac{\partial\chi}{\partial x_i} - \int \frac{\varepsilon^3 d\varepsilon\, \partial o}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon} n_i n_k \frac{\partial\chi}{\partial x_k}, \qquad (8)$$

$$\rho T\dot{\sigma} - \varkappa\Delta T + \int \frac{\varepsilon^3 d\varepsilon\, do}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon}\{\chi + c(\mathbf{n}\nabla)\chi\} = 0,$$

where $\mathbf{n} = \mathbf{p}$ and do is the solid-angle element in the direction of $\mathbf{n}$. Equation (8) contains the renormalized entropy per unit mass of the liquid, σ, equal to

$$\sigma = \frac{S}{\rho} + \frac{2\pi^2}{45}\frac{T^3}{\rho\hbar^3 c^3},$$

and the renormalized pressure

$$P(\rho, \sigma) = P_0\left(\rho, \frac{S}{\rho}\right) + \frac{\pi^2}{90}\frac{T^4}{(\hbar c)^3} + \frac{\pi^2}{30}\frac{T^4}{(\hbar c)^3}\frac{\rho}{c}\left(\frac{\partial c}{\partial \rho}\right)_\sigma,$$

where $P_0 = -E + TS + \mu\rho$ is the pressure without allowance for the contribution of the phonons, so that the deviation from the equations of ordinary hydrodynamics lies only in the presence of terms connected with the non-equilibrium character of the phonon gas.

2. THERMOMECHANICAL EFFECT

Let us consider the flow of a liquid between two parallel planes separated by a distance a. If we neglect the phonons, then the usual Poiseuille flow takes place:

$$u \equiv u_x(z) = \frac{1}{2\eta}\frac{\partial P}{\partial x}\left[z^2 - \left(\frac{a}{2}\right)^2\right], \quad \frac{\partial T}{\partial x}(z) = \text{const}. \tag{9}$$

We have chosen a coordinate system such that the boundaries of the liquid correspond to $z = \pm a/2$. The rate of flow and the temperature gradient are directed along the x axis.

The non-equilibrium part of the phonon distribution function satisfies the equation

$$\frac{\partial\chi}{\partial z} + \frac{\chi}{n_z c\tau} = \frac{\varepsilon}{c} n_x \frac{\partial u}{\partial z} + \frac{\varepsilon}{T}\frac{n_x}{n_z}\frac{\partial T}{\partial x}, \tag{10}$$

and the velocity and temperature in the right-hand side can be set equal to the quantities (9) since, as already noted, the contribution of the phonons to all phenomena is small.

The general solution of (10) is

$$\chi = \frac{\varepsilon\tau}{\eta}\frac{\partial P}{\partial x} n_x n_z (z - n_z c\tau) + \frac{\varepsilon}{T} n_x c\tau \frac{\partial T}{\partial x} + F(\mathbf{p}) e^{-z/n_z c\tau}, \tag{11}$$

where $F(\mathbf{p})$ is an arbitrary function determined by the law of reflection of the phonons from the walls. We shall assume the reflection to be diffuse, and then the following conditions should be satisfied[1)]

$$\chi|_{z=a/2}(n_z < 0) = 0, \quad \chi|_{z=-a/2}(n_z > 0) = 0,$$

whence we obtain

$$F(n_z < 0) = -\varepsilon\tau e^{a/2n_z c\tau}\left\{\frac{n_x n_z}{\eta}\frac{\partial P}{\partial x}\left(\frac{a}{2} - n_z c\tau\right) + \frac{n_x c}{T}\frac{\partial T}{\partial x}\right\},$$

$$F(n_z > 0) = -\varepsilon\tau e^{-a/2n_z c\tau}\left\{-\frac{n_x n_z}{\eta}\frac{\partial P}{\partial x}\left(\frac{a}{2} + n_z c\tau\right) + \frac{n_x c}{T}\frac{\partial T}{\partial x}\right\}.$$

Substituting now the obtained function χ in the second equation of (8) and integrating, we get

$$u(z) = \frac{1}{\eta}\frac{\partial P}{\partial x}\frac{z^2}{2} + c_1 z + c_2 + \frac{2\pi}{\eta}\frac{\partial T}{\partial x}\int_0^\infty \frac{\varepsilon^4\, d\varepsilon}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon} \times \frac{c^2\tau^2}{T}\int_0 dn_z n_z^2(1 - n_z^2)\,\mathrm{ch}\left(\frac{z}{n_z c\tau}\right) e^{-a/2n_z c\tau}, \tag{12}$$

where we have neglected the small corrections to the term proportional to $\partial P/\partial x$; c_1 and c_2 are integration constants. They should be determined from the condition that the velocity v vanish on the boundaries. It is easy to verify, however, that one can use the conditions $u(\pm a/2) = 0$ accurate to terms of order of the ratio of the phonon wavelength to the width a of the gap (inclusion of such terms would be an exaggeration of the accuracy, since we assume the phonon motion to be quasiclassical).

Let us write down the expression obtained in this manner for the total mass flow through the gap

$$J = \rho\int_{-a/2}^{a/2} u\, dz = -\frac{\rho}{\eta}\frac{\partial P}{\partial x}\frac{a^3}{12} + \frac{\rho}{\eta T}\frac{\partial T}{\partial x}\Lambda, \tag{13}$$

where

$$\Lambda = 2\pi c^2\int_0^\infty \frac{\varepsilon^4\, d\varepsilon}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon}\tau^2\int_0^1 dn_z n_z^2(1 - n_z^2)\left\{\left(n_z c\tau - \frac{a}{2}\right) - \left(n_z c\tau + \frac{a}{2}\right)e^{-a/n_z c\tau}\right\} = S_{\text{ph}}\frac{Ta^3}{12}\Phi\left(\frac{c\alpha}{aT^2}\right).$$

Here S_{ph} is the phonon entropy per unit volume of the liquid, equal to $(2\pi^2/45)(T/\hbar c)^3$, and

$$\Phi(x) = \frac{135}{2\pi^4}x^2\int_0^\infty dt\,[(e^t - 1)(1 - e^{-t})]^{-1}\int_0^1 n_z^2(1 - n_z^2)\,dn_z \times\left\{\left(\frac{1}{2} - \frac{n_z x}{t^2}\right) + \left(\frac{1}{2} + \frac{n_z x}{t^2}\right)e^{-t^2/n_z x}\right\}.$$

Let us calculate also the non-equilibrium part of the entropy flux

$$\mathbf{q}' = \int \frac{ds}{dn}\frac{\partial n_0}{\partial\varepsilon}\chi\frac{\partial H}{\partial\mathbf{p}}d\tau - \frac{\varkappa\nabla T}{T}.$$

Substituting here (11) and neglecting the contribution of the phonons to the flux, which is proportional to the temperature gradient, we get

$$q_x'(z) = -\frac{\varkappa}{T}\frac{\partial T}{\partial x} + \frac{2\pi c}{T\eta}\frac{\partial P}{\partial x}\int_0^\infty \frac{\varepsilon^4\, d\varepsilon}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon}\tau\left\{-\frac{2}{15}c\tau + \int_0^1 (1 - n_z^2) n_z\, dn_z\,\mathrm{ch}\left(\frac{z}{n_z c\tau}\right)e^{-a/2n_z c\tau}\left(\frac{a}{2} + n_z c\tau\right)\right\}.$$

The heat flux through the gap is equal to

$$Q = T\int_{-a/2}^{a/2} q_x'\, dz = -a\varkappa\frac{\partial T}{\partial x} + \frac{1}{\eta}\frac{\partial P}{\partial x}\Lambda. \tag{14}$$

Thus, the presence of weakly-damped phonons in the liquid gives rise to a term proportional to the temperature gradient in the expression for the mass flux (the thermomechanical effect), and to a term proportional to the pressure gradient in the expression for the heat flux (the mechanocaloric effect). Formulas (13) and (14) are in full agreement with the principle of the symmetry of the kinetic coefficients.

As seen from (13), in order for mass transport to be absent, the pressure difference ΔP and the temperature difference ΔT at the ends of the gap should be connected by the relation

$$\frac{\Delta P}{\Delta T} = S_{\text{ph}}\Phi\left(\frac{c\alpha}{aT^2}\right).$$

For $a \ll c\alpha/T^2$, using the limiting value $\Phi(\infty) = 1$, we obtain

$$\Delta P / \Delta T = S_{\text{ph}}.$$

The last formula differs from the known formula for a superfluid liquid only in that the entropy is replaced by the phonon entropy. This is perfectly natural, for in the case of a superfluid liquid, the normal part carrying the total entropy does not flow in practice through a narrow gap, and in our case the transport of phonons, and consequently of the phonon entropy, through a gap that is thin compared with the phonon free path is exceedingly difficult.

In the opposite limiting case $a \gg c\alpha/T^2$, noting that $\Phi(x \to 0) = (2/\pi^2)(x/\pi)^{3/2}$, we get

$$\frac{\Delta P}{\Delta T} = \frac{4}{45}\left\{\frac{\pi a}{\rho c}\left[\frac{4}{3}\eta + \zeta + \varkappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right]\right\}^{-3/2}. \tag{15}$$

[1)] We have neglected the absorption of the phonons at the walls. The corresponding relaxation time is of the order of $(a/c)\sqrt{\Theta/\epsilon}$ and is large compared with τ, provided the thickness is $a \gg c\tau\sqrt{\epsilon/\Theta}$. We shall show below that interest attaches to the values $a \sim c\tau$, so that the absorption at the wall can actually be neglected.

It is important to note that in the last case of large thicknesses a, the main contribution to Λ is made by phonons with a mean free path on the order of a. The energy of such phonons is small compared with the temperature. In order for formula (15) to be valid it is therefore not necessary to satisfy the inequality $T \ll \Theta$. All that is needed is that relation (2) be applicable to phonons with energy of the order of $(c\alpha/a)^{1/2}$. For this reason, formula (15) describes the thermomechanical effect in any liquid at any temperature, provided the thickness a is large compared with $c\alpha/(\hbar\omega_0)^2$, where ω_0 is the frequency above which an appreciable dispersion of sound begins. It must be borne in mind, incidentally, that when a increases the wavelengths of the phonons that determine the effect increase, and the reflection of the phonons from the walls ultimately ceases to be diffuse.

3. PROPAGATION OF SHEAR OSCILLATIONS

Let the liquid be in contact with a flat solid surface executing tangential oscillations with frequency ω. If we neglect the phonons, then, as is well known, the shear oscillations that arise in the liquid (the viscous wave) attenuate exponentially at distances on the order of the depth of penetration $\delta \sim a_0(\Theta/\hbar\omega)^{1/2}$, where a_0 is the interatomic distance. When phonons are taken into account, motion occurs in the liquid at distances that are large compared with δ.

Let the solid surface be the xy plane, let the oscillation direction be the x axis, and let the liquid occupy the region $z > 0$. We are interested only in values $z \gg \delta$. The function χ then satisfies the homogeneous equation

$$cn_z\frac{\partial\chi}{\partial z} + \left(\frac{1}{\tau} - i\omega\right)\chi = 0,$$

from which it follows immediately that

$$\chi = F(\mathrm{p})\exp\left\{-\frac{z}{n_z c\tau} + \frac{i\omega z}{cn_z}\right\},$$

where F(p) is an arbitrary function. From the required finiteness as $z \to \infty$, we obtain the equality $F(n_z < 0) = 0$. For phonons with $n_z > 0$ the distribution function at $z = 0$ should correspond to motion of the gas of phonons as a unit, with the velocity of the surface V, i.e., it should equal $n_0(\epsilon - p_x V)$. It follows therefore that $F(n_z > 0) = -p_x V$. Substituting now the function χ in the second equation of (8) and taking into account the fact that when $z \gg \delta$ the liquid can be regarded as ideal $(\eta = 0)$, we obtain the velocity of motion

$$u = \frac{1}{i\omega\rho}\int\frac{\varepsilon^3\,d\varepsilon\,do}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon}n_x n_k\frac{\partial\chi}{\partial x_k}$$

$$= -i\frac{\pi V}{\rho c^2\omega}\int_0^\infty\frac{\varepsilon^4\,d\varepsilon}{(2\pi\hbar c)^3}\frac{\partial n_0}{\partial\varepsilon}\left(\frac{1}{\tau} - i\omega\right)\int_0^1 dn_z(1 - n_z^2)$$

$$\times\exp\{-z/n_z c\tau + i\omega z/cn_z\}.$$

If $z \ll c\alpha/T^2$, then $\exp(-z/n_z c\tau)$ can be replaced by unity, and after integrating over the energies we obtain

$$u = \frac{2\pi^4 i}{21}V\frac{T^6}{\rho\hbar^3c^5\alpha\omega}\left(1 - \frac{7i}{20\pi^2}\frac{\omega\alpha}{T^2}\right)\int_0^1 dn_z(1 - n_z^2)e^{i\omega z/cn_z}.$$

From this we have for $z \ll c/\omega$

$$u = \frac{4\pi^4 i}{63}V\frac{T^6}{\rho\hbar^3c^5\alpha\omega}\left(1 - \frac{7i}{20\pi^2}\frac{\omega\alpha}{T^2}\right), \tag{17}$$

and for $z \gg c/\omega$

$$u = -\frac{4\pi^4 i}{21}V\left(\frac{T^2}{\hbar c\omega}\right)^3\left(1 - \frac{7i}{20\pi^2}\frac{\omega\alpha}{T^2}\right)\frac{e^{i\omega z/c}}{\alpha\rho z^2}. \tag{18}$$

In the opposite limiting case $z \gg c\alpha/T^2$, the main contribution is made by phonons with energies that are small compared with the temperature. We can therefore put $\partial n_0/\partial\epsilon = -T/\epsilon^2$, after which we can readily integrate with respect to the energies. As a result we get

$$u = \frac{3i}{64}V\frac{T}{\rho\omega}\left(\frac{\alpha}{\pi\hbar^2}\right)^{3/2}(cz)^{-5/2}\int_0^1 n_z^{5/2}(1 - n_z^2)dn_z\left(1 - \frac{2i\omega z}{3cn_z}\right)e^{i\omega z/cn_z}.$$

The last formula goes over into

$$u = \frac{3i}{616}V\frac{T}{\rho\omega}\left(\frac{\alpha}{\pi\hbar^2}\right)^{3/2}(cz)^{-5/2} \tag{19}$$

when $z \ll c/\omega$ and into

$$u = -\frac{V}{16}\frac{Tc^2}{\rho\omega^2}\left(\frac{\alpha}{\pi\hbar^2}\right)^{3/2}(cz)^{-7/2}e^{i\omega z/c} \tag{20}$$

when $z \gg c/\omega$.

Since δ is always small compared with c/ω, there exist three frequency regions:

1) $\omega \gg T^2/\alpha$. In this region, u is first independent of z with increasing z ($\delta \ll z \ll c/\omega$, formula (17)), after which it decreases like z^{-2} (formula (18)) and then like $z^{-7/2}$ (formula (20)).

2) $\delta \ll c\alpha/T^2 \ll c/\omega$. Here the velocity is first independent of z, then varies like $z^{-5/2}$, and later like $z^{-7/2}$.

3) $\delta \gg c\alpha/T^2$. The velocity decreases first like $z^{-5/2}$ and then like $z^{-7/2}$.

One can make the same remarks with respect to formulas (19) and (20) as were made in the preceding section with respect to formula (15). Their applicability is not limited by the condition $T \ll \Theta$, since the phonons that matter are those for which $c\tau \sim z$.

In conclusion we note that perfectly analogous effects should also be observed when temperature waves propagate in a liquid. In general, it can be stated that whenever ordinary hydrodynamics leads to an exponential damping of the perturbations, the influence of the phonons always causes the appearance of power-law "tails." In particular, such power-law "tails" should be observed on the trailing edge of a shock wave in a liquid.

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[1] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Dynamics), Gostekhizdat, 1953 [Addison-Wesley, 1958].

[2] I. M. Khalatnikov, Vvedeniye v teoriyu sverkhtekuchesti (Introduction to the Theory of Superfluidity), Nauka, 1965.

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206