# *THEORY OF DOMAIN STRUCTURE OF UNIAXIAL FERROMAGNETS*

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Domain structure is investigated in a uniaxial ferromagnet with small anisotropy. It is shown that the 90-degree boundary assumed in the model of Landau and Lifshitz cannot be realized. This means that the model of Landau and Lifshitz does not guarantee a minimum of the energy. For thin films, a two-dimensional structure is proposed, with a smaller energy of exit of domains to the surface (anisotropy energy). The meaning of this circumstance is not completely clear, since it is impossible to take account of the energy due to formation of domain boundaries inclined to the axis of easy magnetization. This difficulty is a matter of principle (the concept of surface tension in general losses meaning for inclined boundaries), and therefore the problem of domain structure in thin films (unbranched domain structure) cannot be posed correctly within the framework of macrotheory. The problem of greatest interest is that of branched domain structure in bulk specimens. A plane model of such a structure is constructed. At a depth $h = 0.117\,l$ ($l$ is the thickness of the plate), each domain splits. Thereafter a further splitting occurs. The process continues until the dimensions of the domains that are being formed become comparable with the thickness of the domain wall, $\delta$. It is shown that the dependence of the domain width $a$ on the plate thickness $l$ has the form $a \sim \delta^{1/3} l^{2/3}$. This result is supported by experiment.

## 1. INTRODUCTION

THE basic results of the theory of domain structure were obtained in the well-known papers of Landau and Lifshitz[1,2] (see also[3]). It was shown that in the case of a sufficiently thin ferromagnetic plate (film), the domains far inside the specimen are plane-parallel, uniform layers. The dimensions of the layers are determined by the condition of minimization of the sum of the energy of surface tension on the boundaries of separation of the phases and the energy of exit of the domains to the surface. The latter arises through distortion of the structure near the surface of the specimen, at distances of the order of the domain width $a$. This energy, calculated for unit surface area of the plate, is proportional to the domain width $a$ and is independent of the plate thickness $l$, whereas the energy of surface tension is proportional to $l/a$. Therefore the domain width is proportional to $\sqrt{l}$. For a ferromagnet with small anisotropy ($U_{an} = \frac{1}{2}\beta M^2 \sin^2\theta$, where $\theta$ is the angle between the moment $\mathbf{M}$ and the axis of easy magnetization (the $z$ axis); $\beta/4\pi \ll 1$), the structure assumed in[1] was one with closed flux (see Fig. 1).

It was shown in[4] that the flux-closure condition is only approximate (it is correct only in the limit $\beta/4\pi \to 0$). Near the surface of the specimen, there appears a magnetic field $H \sim \beta M$, which orients the magnetization at the surface almost perpendicular to the easy axis. This field penetrates even beyond the boundaries of the specimen and can be measured. We mention, incidentally, that such a field (of order $10^4$ Oe) has been repeatedly observed on the surface of monocrystals of cobalt[5–7], for which $\beta/4\pi \approx \frac{1}{3}$, and cannot be explained within the framework of a closed-flux theory.

In the same paper, the conditions for coexistence of phases in magnets were established:

$$\mathbf{H}_t = \text{const},\quad B_n = \text{const},\quad \Phi'(\mathbf{H}_t, B_n) = \text{const},$$
$$\Phi'(\mathbf{H}_t, B_n) = -\frac{1}{4\pi}\int_0^{\mathbf{H}} \mathbf{B}\,d\mathbf{H} + \frac{H_n B_n}{4\pi}. \tag{1}$$

The magnetostatic problem of the theory of domain structures reduces to the solution of Maxwell's equations curl $\mathbf{H} = 0$ and div $\mathbf{B} = 0$ with the boundary conditions (1).

In[4] it was assumed that in the limit $\beta/4\pi \to 0$, the domain structure has the Landau-Lifshitz form (see Fig. 1). It will be shown below that a 90-degree boundary cannot be realized even in this limit[1)]. This means that the Landau-Lifshitz model does not guarantee a minimum of the energy (see Sec. 3). For thin films[2)], we shall obtain a structure with a smaller exit energy (anisotropy energy). The meaning of this circumstance is not completely clear to us, since it is impossible to take account of the energy due to formation of domain boundaries inclined to the axis of easy magnetization. This difficulty is a matter of principle (the concept of surface tension in general loses meaning for inclined boundaries). Furthermore, we were not able to get rid completely of 90-degree boundaries, so that our solution also does not guarantee a minimum of energy.

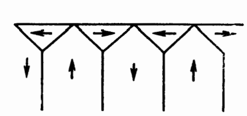

FIG. 1

[1)]We are considering the case of uniaxial ferromagnets. In crystals of cubic symmetry, a 90-degree wall is possible [2,8]. The domain structure shown in Fig. 1 has often been observed in iron (see, for example, the review [9]).

[2)]We apply the term "film" to a plate in which a plane domain structure is realized; the dependence of a on $l$ has the form $a \sim \sqrt{l}$.

The ambiguity in the surface tension is important only when the domain dimensions a cannot be considered negligibly small in comparison with the plate thickness $l$. At the same time, a domain structure with $a \sim \sqrt{l}$ is possible only at sufficiently small dimensions $l$: $l < l_c$ and, consequently, $a < a_c$. The ratio of the domain-wall thickness $\delta$ to $l_c$, and also the ratio $a_c/l_c$, are numerically small but independent of $\beta$ in the limit $\beta/4\pi \ll 1$[2].

The contribution of inclined boundaries to the total energy is of order $a/l$. It is not so small as to be completely negligible, and therefore the problem of the domain structure of thin films cannot be posed correctly within the framework of macrotheory.

With increase of the plate thickness $l$, progressive branching of domains near the surface becomes thermodynamically advantageous. This was first discovered by Landau in investigation of the intermediate state of superconductors[10]. The paper of Lifshitz[2] treated the initial stage of branching in ferromagnets.

In uniaxial ferromagnets, in distinction to superconductors, branching of the domains becomes advantageous at quite small dimensions $l$. Therefore the problem of greatest interest is that of the extremely branched structure. We shall construct here a plane model of such a structure for the case of a uniaxial ferromagnet with small anisotropy (Sec. 4). It will be shown that the dependence of the layer width a on the plate thickness $l$ changes: $a \sim l^{2/3}$. This dependence was observed for cobalt (see for example,[10]). The same dependence was obtained by Landau for superconductors[10].

## 2. THE MAGNETOSTATIC PROBLEM OF THE THEORY OF DOMAIN STRUCTURES

We shall now formulate the magnetostatic problem of the exit of domains to the surface. We suppose that there is no external magnetic field and that the axis of easy magnetization is perpendicular to the plane of the ferromagnetic plate. The exit-energy density is

$$\Phi = U_{\text{an}} + \frac{H^2}{8\pi} = \frac{1}{2}\beta M^2 \sin^2\theta + \frac{H^2}{8\pi}. \qquad (2)$$

In the limit of small anisotropy the second term can be neglected, since near the surface $H \sim \beta M$, whereas far inside the specimen $H \to 0$. Thus in the first approximation the field $\mathbf{H}$ can be left out of consideration, and the equations of magnetostatics reduce to the single equation

$$\operatorname{div} \mathbf{M} = 0, \quad (|\mathbf{M}| = M = \text{const}) \qquad (3)$$

with the boundary condition

$$M_n = 0 \qquad (4)$$

on the surface of the specimen; this guarantees closure of the magnetic flux in the specimen and continuity of it on the boundaries of separation of the phases.

From the mathematical point of view, the problem reduces to solution of the eikonal equation

$$\left(\frac{\partial A}{\partial x}\right)^2 + \left(\frac{\partial A}{\partial z}\right)^2 = 1; \quad \frac{\mathbf{M}}{M} = \operatorname{rot} \mathbf{A}, \quad \mathbf{A} = \{0, A(x, z), 0\}. \qquad (5)$$

The lines of force are the lines of constant value of the vector potential $\mathbf{A}$.

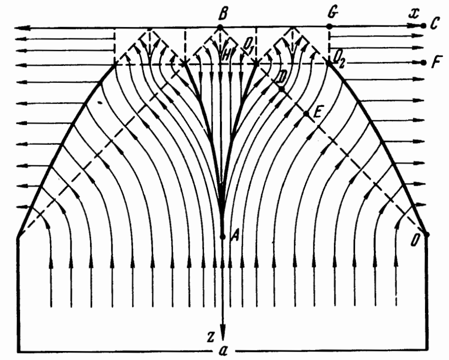


FIG. 2

After the magnetization distribution has been found in the zeroth approximation with respect to the parameter $\beta/4\pi \ll 1$, the magnetic field $\mathbf{H}$ can be found in the following manner.

The equation of state of the magnet (see[3])

$$\beta M \sin\theta \cos\theta = H_x \cos\theta - H_z \sin\theta \qquad (6)$$

is, for given $\theta(x, z)$, a relation between $H_x$ and $H_z$[3)].

A second relation is Maxwell's equation curl $\mathbf{H} = 0$. At the boundaries of separation of the phases, there are four equations for the four quantities $H_{x1}$, $H_{x2}$, $H_{z1}$, $H_{z2}$; the relations (6) for $\theta = \theta_1$ and for $\theta = \theta_2$ and the conditions $H_t$ = const and $\Phi'(H_t, B_n)$ = const. Thus, if we know, in the zeroth approximation, the position of the boundaries of separation and the distribution of the magnetization, we can find the magnetic field $\mathbf{H}$ on the boundaries of separation of the phases, and the equation curl $\mathbf{H} = 0$ together with the equation of state (6) uniquely determines $\mathbf{H}(x, z)$ in the whole volume of the specimen. In the next approximation there is a small (proportional to $\beta/4\pi$) displacement of the separation boundaries.

## 3. DOMAIN STRUCTURE OF THIN FILMS

In this section we shall show that it is possible to construct a solution of the magnetostatic problem with an exit energy that is smaller than in the Landau-Lifshitz model; in it, the length of the boundaries per domain increases only by a quantity of order a, which is being assumed small in comparison with $l$. The proposed scheme of flux closure is shown in Fig. 2.

The lines of force begin to curve at a depth equal to half the width a of a domain (at $z = a/2$). For $z < a/2$ the lines of force are constructed out of segments of straight lines and arcs of circles; that is, we combine the simplest solutions of the eikonal equation: $A = \mathbf{r} \cdot \mathbf{n}$ and $A = |\mathbf{r} - \mathbf{r}_0|$. The bold lines show the boundaries of separation of the phases ($AO_1$ and $OO_2$). In the region $AO_1O$ the center of the circles is the point O; in the region $OO_1O_2$, the point $O_1$ ($x = z = O_1D = \gamma a/2$,

[3)]Usually this equation is treated as an equation for $\mathbf{M}$ at given $\mathbf{H}$ [3]. It is the condition for an extremum of the thermodynamic potential

$$\tilde{\Phi} = U_{\text{an}} - \mathbf{MH} - \frac{H^2}{8\pi} = -\frac{1}{4\pi}\int_0^{\mathbf{H}} \mathbf{B}\, d\mathbf{H}.$$

It must be mentioned that the values of $\theta$ for which the thermodynamic potential $\tilde{\Phi}$ has a maximum correspond to absolutely unstable states and must be rejected.

where $\gamma = 3 - 2\sqrt{2} = 0.172$). Then D lies on the circle with radius OA = a/2. In the regions $AHO_1$ and $OO_2GC$ the lines of force are straight lines, respectively parallel and perpendicular to the axis of easy magnetization.

In the region $AHO_1D$ the lines of force may be considered coordinate lines $\eta$ = const in an orthogonal system of coordinates $\eta\xi$, where the lines $\xi$ = const in the region $AO_1D$ are segments of radial straight lines, and in the region $AHO_1$ are continued by straight lines parallel to the x axis (for example, the line $DO_1H$). In these coordinates, the eikonal equation has the form

$$\left(\frac{\partial A}{\partial\eta}\right)^2+\left(\frac{\partial A}{\partial\xi}\right)^2=1,$$

and the solution depicted in Fig. 2 corresponds to the function $A = \eta$. Consequently, the distances from a point on the boundary of separation $AO_1$ to the straight line AH and to the arc AD are equal. Therefore the line $AO_1$ is the parabola $2ax = (z - a/2)^2$. We construct similarly the boundary $OO_2$, the parabola $2\gamma az = (x - \gamma a/2)^2$.

The region $BHO_2G$ consists of three squares. In Fig. 2 they are separated from each other by dotted lines. These squares are filled by lines of force in the same way as was the original square ABCO. On the surface there are formed new squares with side $\gamma^2 a/2$. The process continues until the dimensions of the squares being formed becomes comparable with the domain-wall thickness $\delta$.

The exit energy per unit area of plate surface (with allowance for the two sides of the plate) is

$$E=\frac{2\beta M^2}{a}\iint\limits_0^{a/2} dx\,dz\sin^2\theta(x,z). \tag{7}$$

The ratio of this energy to the energy obtained for the Landau-Lifshitz structure ($E_{LL} = \frac{1}{4}\beta M^2 a$) is

$$E/E_{LL}=0.858. \tag{8}$$

Thus the exit energy is less in our model than in the model with closure triangles. The significance of this fact is not completely clear to us, for the reasons presented below.

A. If the ratio of the domain width a to the plate thickness $l$ were small ($a/l \ll 1$), then our structure would be preferable, since the energy of surface tension (calculated for one domain) would be the same in both models (of equal $\Delta l$) to within terms of order $a/l$. Actually, even for quite small thicknesses $l$ the domain structure begins to branch[2,3] and ceases to be plane. In order to determine the critical values $l_c$ and $a_c$ at which the first branching occurs, it is necessary to know, in particular, the initial unbranched structure. The ratio $a_c/l_c$ in the limit of weak anisotropy is numerically small but independent of $\beta$[2]. This last fact is due to the fact that both the anisotropy energy and the energy of surface tension $\Delta = 2\beta\delta M^2$ are proportional to $\beta$. Here $\delta$ is the thickness of a domain wall, which in this problem is the only parameter of the dimensions of length, so that $a_c$ and $l_c$ are proportional to $\delta$, and the coefficients of proportionality are independent of $\beta$ in the limit $\beta/4\pi \to 0$.

Experiment shows that for cobalt $a_c/l_c = 0.12$; when $l < l_c$, that is when $a/l > a_c/l_c$, the relation $a \sim \sqrt{l}$ is obeyed[12]4). This means that the ratio $\delta/a_c < 0.015$, so that one can speak with very good accuracy of domains separated by an infinitely thin wall. At the same time there is no assurance that the additional energy due to the formation of domain boundaries of the type $AO_1$ and $OO_2$ will not exceed the gain of exit energy. The length of such boundaries is somewhat larger in our model than in the Landau-Lifshitz model (the sum of the boundary lengths in the region ABCO converges; it is a geometric progression with ratio $3\gamma = 0.516$).

To calculate the energy of a boundary of type $AO_1$ and $OO_2$ is in principle impossible, since the concept of surface tension loses meaning for boundaries inclined to the easy axis. To explain this, we suppose that we know the structure of a domain wall.

All quantities inside a domain wall change only in the direction perpendicular to the separation boundary (along the $\xi$ axis). The values of $H_t$ and $B_n$ inside the domain wall do not change, whereas the values of $H_n$ and $B_t$ change, taking for $\xi \to \pm\infty$ the asymptotic values $H_n^{\pm}$ and $B_t^{\pm}$. As has already been indicated (see (1)), the values of the thermodynamic potential $\Phi'$ at $\xi \to \pm\infty$ must coincide. This is a condition for existence of a solution of the wall problem (see[4]).

The surface tension $\Delta$ is the contribution of a domain wall to the complete thermodynamic potential

$$\tilde{\Phi}=\int\left[U_{an}-\mathbf{MH}-\frac{H^2}{8\pi}+\frac{\alpha}{2}\left(\frac{\partial M_i}{\partial x_k}\right)^2\right]dV,$$

where the last term in the integrand is the energy of nonuniformity, which vanishes far from the wall (in the one-dimensional problem). In general it is permissible to neglect the energy of nonuniformity far from the walls if $a \gg \delta$. In the absence of an external field, one may calculate instead of the thermodynamic potential $\tilde{\Phi}$ the thermodynamic potential $\Phi$:

$$\Phi=\tilde{\Phi}+\int\frac{\mathbf{HB}}{8\pi}dV,$$

since the second term in this case vanishes.

If at the boundaries of separation of phases, in addition to conditions (1), the condition $H_{n1} = H_{n2}$ is also satisfied—as is always true far inside a specimen of ellipsoidal shape in a uniform external field $H_0$[4] (inside the wall, $H_n$ may vary as usual)—then from the equality $\Phi_1' = \Phi_2'$ it follows that $\tilde{\Phi}_1 = \tilde{\Phi}_2$. In this case the surface tension can be defined as follows:

$$\Delta=\int\limits_{-\infty}^{\infty}d\xi[\tilde{\Phi}(\xi)-\tilde{\Phi}(\pm\infty)].$$

But if $H_{n1} \neq H_{n2}$, then likewise $\tilde{\Phi}_1 \neq \tilde{\Phi}_2$, and the quantity $\Delta$ cannot be defined in this way. In fact, the position of a boundary of separation is determined only to within the wall thickness $\delta$. Therefore in case $\tilde{\Phi}_1 \neq \tilde{\Phi}_2$ it is not possible to separate uniquely from the quantity $\int d\xi\tilde{\Phi}(\xi)$ that part that is due to the formation of a domain boundary. The indeterminacy in the

4)In superconductors the situation is much more favorable. The ratio $a_c/l_c$ is there so small [13] that the branched structure assumed by Landau [10] is never observed. Experimentally, all that is realized is the unbranched structure also calculated by Landau [3,14]. Again, there are no literally small parameters in this problem.

surface energy is of order $\delta(\tilde{\Phi}_1 - \tilde{\Phi}_2)$. If this quantity is small, then the surface tension can have an approximate meaning.

In a uniaxial ferromagnet, the condition $\tilde{\Phi}_1 = \tilde{\Phi}_2$ is satisfied only for boundaries parallel to the axis of easy magnetization[4]: For large angles of inclination, the indeterminacy in the surface energy is of the order of the surface tension far inside the specimen[5].

Thus in our problem, the concept of surface tension on inclined boundaries has no meaning. The surface energy ceases to be local; and in order to determine the advantages of various domain structures, it is necessary to consider the microscopic problem, that is to solve the static equation of Landau and Lifshitz[1] throughout the whole volume of the specimen. This seems to us impossible.

In the theory of thin magnetic films, there are frequent attempts to calculate the surface tension on inclined walls. From our point of view, this has no meaning.

B. Besides the defect of the theory indicated above, in our problem there is also another difficulty, of less basic character. As has already been stated, the fields $H_1$ and $H_2$ on the boundaries of separation of phases must be uniquely determined from the relations (6) for $\theta = \theta_1$ and $\theta = \theta_2$, the Maxwell boundary condition $H_t$ = const, and the condition for coexistence of phases $\Phi_1' = \Phi_2'$. The number of equations in this case is equal to the number of unknowns, so that the problem has been correctly posed. Nevertheless, in our model there are on the boundaries sections on which these equations do not have solutions corresponding to stable states. Thus, for example, at point $O_2$ there must be realized a so-called 90-degree wall; the angle of inclination of the boundary is $\pi/4$, while the direction of the magnetization changes by $\pi/2$. In the Landau-Lifshitz model there are also such walls near the surface. The angles $\theta_1$ and $\theta_2$ are equal respectively to $\pi$ and to $\pi/2$. On substituting these angles into Eq. (6), we obtain the two relations

$$H_{x1} = H_{z2} = 0.$$

The condition $H_{t1} = H_{t2}$ in this case has the form

$$H_{z1} = H_{x2}.$$

The thermodynamic potential $\Phi'$ can be expressed in the form

$$\Phi' = U_{\mathrm{an}} - \mathbf{M}_t\mathbf{H}_t - \frac{H_t^2}{8\pi} + \frac{H_n^2}{8\pi}.$$

On taking account of the fact that

$$\mathbf{M}_{t1}\mathbf{H}_{t1} = -\frac{MH_{z1}}{2}, \quad \mathbf{M}_{t2}\mathbf{H}_{t2} = \frac{MH_{x2}}{2},$$

we get

$$H_{z1} = H_{x2} = \beta M/2.$$

The state $M_{X2} = M$, $H_{X2} = \beta M/2$, $H_{Z2} = 0$ corresponds, as is not difficult to show, not to a minimum but to a maximum of the thermodynamic potential $\tilde{\Phi}$ for given H; that is, it is absolutely unstable. This means that the 90-degree wall, which was assumed in the Landau-Lifshitz model and also plays a part in our theory, is impossible[6].

The boundary condition $\Phi_1' = \Phi_2'$ can be obtained from the equations of microtheory[4]. In the present model, this becomes the static equations of Landau and Lifshitz[1], which themselves are derived from the condition of minimum thermodynamic potential $\tilde{\Phi}$. If we obtain a solution that does not satisfy the boundary condition $\Phi_1' = \Phi_2'$, then that solution does not correspond to a minimum of the energy. This relates both to our model and to the Landau-Lifshitz model. In our case the situation seems to us more advantageous, since there are sections of the inclined walls on which the boundary condition $\Phi_1' = \Phi_2'$ is known to be satisfied. Thus, for example, near point A, which the boundary $AO_1$ approaches with zero inclination, the presence of a stable solution is obvious. It is very probable that by slightly changing the structure shown in Fig. 2, one can obtain a solution that satisfies the boundary conditions.

Thus the question of the domain structure of thin films remains unclear. Furthermore, the very posing of the question within the framework of macrotheory is unclear, since even if we obtain a solution satisfying the boundary conditions, we cannot take account of the energy due to the formation of inclined boundaries. Because of the fact that the solution of the magnetostatic problem is nonunique, one cannot show which of these solutions corresponds to the least energy. The difficulty arises from the fact that the problem can be formulated as a problem without small parameters (the parameter $\beta/4\pi$ is only a common multiplier in the complete energy and drops out of the equations). In this connection it seems to us very important to obtain reliable experimental data on the domain structure of thin films with small uniaxial anisotropy. At present such data are lacking, although the problem is being investigated rather intensively.

## 4. PLANE MODEL OF A BRANCHED STRUCTURE

As has already been indicated, for sufficiently large values of $l$ the domain structure begins to branch. In the limit $l/\delta \to \infty$ an infinitely branched structure is formed. We shall study here a plane model of such a structure, shown in Fig. 3.

[5] In the case $\beta/4\pi \gg 1$, the indeterminacy in the surface tension is small (or order $4\pi\delta M^2$) in comparison with $\Delta \sim \delta\beta M^2$, if the angle of inclination of the boundary is not very close to $\pi/2$. In this case the surface energy has an approximate meaning. In superconductors, this difficulty in general is absent, since in the separation boundaries $B_n = 0$ and consequently $\tilde{\Phi}_1 = \tilde{\Phi}_2$.

[6] In the literature it is often assumed that the formation of a locally unstable state is necessary for attainment of a minimum of the complete thermodynamic potential $\tilde{\Phi}$ of the body (see, for example, [8,9]). Allusion is made to the fact that Maxwell's equations make the problem nonlocal. It can be shown, however, that from the condition of minimization of the complete thermodynamic potential $\tilde{\Phi}$ of the body there follows a local condition of minimization of the density of the thermodynamic potential $\Phi$, considered as a function of $\mathbf{M}$ for a given local value of $\mathbf{H}$. To demonstrate the instability of triangular closure domains, it is sufficient to consider infinitely small perturbations of the form

$$\delta\mathbf{H} = 0, \ \delta\mathbf{M} = 1/4\pi \ \mathrm{rot}\, \delta\mathbf{A}, \ \delta\mathbf{A} = \{\delta A(x), 0, 0\},$$

where $\delta A(x)$ is an arbitrary function of three variables ($\delta M_z \neq 0$), localized inside the triangular domains; the x axis is chosen along the direction of $\mathbf{M}$ in the triangles. Such a perturbation decreases the anisotropy energy without disturbing the equations curl $\mathbf{H} = 0$ and div $\mathbf{B} = 0$.

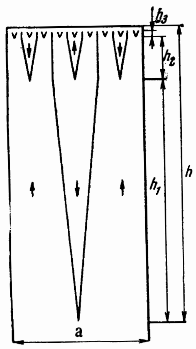


FIG. 3

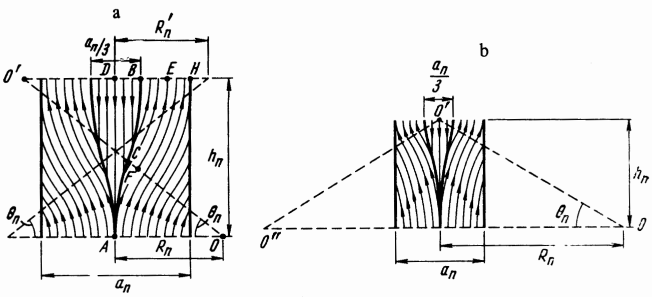


FIG. 4

At a depth h, which will be calculated below, each domain splits. With approach to the surface, the width of the new domains increases, until it becomes equal to a/3. At this point a new splitting occurs. The process continues until the dimensions of the domains that are forming becomes comparable with the thickness δ of a domain wall.

The concentration of the opposite phase in the original domain (of width a) after the n-th splitting is determined by the recurrence relation

$$c_n = {}^2/_3 c_{n-1} + {}^1/_3(1 - c_{n-1}), \quad c_1 = {}^1/_3,$$

that is,

$$c_n = {}^1/_3 + {}^1/_9 + \ldots + 1/3^n.$$

For $n \to \infty$, we get $c_\infty = {}^1/_2$. This signifies closure of the magnetic flux in the specimen; that is, vanishing of the component of the magnetization normal to the surface (averaged over distances of order δ).

Possible splitting schemes are shown in Fig. 4. The thin solid lines represent the lines of force. The bold lines show the boundaries of separation of the phases. We consider first a more general scheme, depicted in Fig. 4a. In the region ACO, the lines of force are arcs of circles whose centers are at the point O. In the region BCGH, the center of the circles is the point O′. In the central domain, the lines of force are parallel to the axis of easy magnetization. The magnetization **M** is parallel (or antiparallel) to the axis of easy magnetization everywhere on the lines OA and O′H; that is, at the beginning and the end of each stage of splitting.

It follows from the condition of conservation of flux that a point on the separation boundary ACB must be equidistant from the straight line AD and from the line AFE that is the continuation of this straight line in its role as a line of force (in the latter case, we mean the distance along a radial straight line; for example, CF for point C). In particular, DB = BE = $a_n/6$. Therefore

$$\frac{a_n}{3} = DE = O'E - R_n' = OO' - (R_n + R_n') = (R_n + R_n')\left(\frac{1}{\cos\theta_n} - 1\right).$$

We have, furthermore,

$$h_n = (R_n + R_n')\,\mathrm{tg}\,\theta_n.$$

It will be shown below that the angles $\theta_n$ are small (of order $(\delta/a_n)^{1/2}$). Therefore

$$R_n + R_n' = 2a_n/3\theta_n^2, \tag{9}$$

$$h_n = 2a_n/3\theta_n, \tag{10}$$

where

$$a_n = a/3^{n-1}. \tag{11}$$

The parameter $R_n$, $R_n'$, $h_n$, and $\theta_n$ are determined by the condition of minimization of the total energy due to the splitting.

We note that in polar coordinates with centers at the points O and O′, the equations of the lines AC and BC have the form

$$r\cos^2 {}^1/_2\theta = R_n, \tag{12}$$

$$r'\cos^2 {}^1/_2\theta = R_n' + {}^1/_6 a_n. \tag{12'}$$

Therefore the anisotropy energy of a section of height $h_n$ is

$$U^{(n)} = \beta M^2 \int_0^{\theta_n} \sin^2\theta\, d\theta \left[\int_{\rho_1}^{\rho_2} r\,dr + \int_{\rho_3}^{\rho_4} r'\,dr'\right], \tag{13}$$

where

$$\rho_1 = \frac{R_n - a_n/2}{\cos\theta}, \quad \rho_2 = \frac{R_n}{\cos^2\theta/2}, \quad \rho_3 = \frac{R_n' + a_n/6}{\cos^2\theta/2}, \quad \rho_4 = \frac{R_n' + a_n/2}{\cos\theta}.$$

On allowing for the fact that $\theta_n \ll 1$ and neglecting terms of order $a_n^2\theta_n^3$, we get

$$U^{(n)} = {}^1/_2\beta M^2[{}^1/_3\theta_n^3 a_n(R_n + {}^2/_3 R_n') + {}^1/_{10}\theta_n^5(R_n'^2 - R_n^2)]. \tag{14}$$

By use of the relation (9), this expression can be transformed into the form

$$U^{(n)} = {}^1/_{18}\beta M^2[{}^5/_3 a_n^2\theta_n - {}^1/_{10}(R_n - R_n')a_n\theta_n^3]. \tag{15}$$

Since the sum $R_n + R_n'$ for given $\theta_n$ is constant, the smallest value of $U^{(n)}$ is attained when

$$R_n' = 0, \quad R_n = 2a_n/3\theta_n. \tag{16}$$

Thus the splitting scheme that is realized is that shown in Fig. 4b, and the anisotropy energy $U^{(n)}$ is

$$U^{(n)} = {}^4/_{45}\beta M^2 a_n^2\theta_n. \tag{17}$$

The energy of surface tension, $E^{(n)}$, is[7)]

$$E^{(n)} = 3h_n\Delta,$$

where $\Delta = 2\beta\delta M^2$, δ being the thickness of the domain wall; that is,

$$E^{(n)} = 4\beta\delta M^2 a_n/\theta_n. \tag{18}$$

[7)] In the preceding section it was shown that the concept of surface tension, strictly speaking, loses meaning for boundaries inclined to the axis of easy magnetization, since the surface energy is determined only to within an accuracy $\delta(\tilde{\Phi}_1 - \tilde{\Phi}_2)$. For boundaries parallel to the easy axis, $\tilde{\Phi}_1 = \tilde{\Phi}_2$ [4]. In the case under consideration, the inclination of the boundaries is small (of order $\theta_n$), and the indeterminacy in the surface energy is small in comparision with Δ.

The angle $\theta_n$ is determined by the condition of minimization of the value of $U^{(n)} + \frac{2}{3} E^{(n)}$.

The second term includes only the energy of the boundaries of the central domain. Thus in the given stage, we shall minimize the energy that is due to the splitting.

After simple calculations we get

$$\theta_n^2 = 30\delta / a_n. \tag{19}$$

The total energy of a region of width $a_n$ and height $h_n$ is

$$\Phi^{(n)} = U^{(n)} + E^{(n)} = \frac{2\sqrt{10}}{3\sqrt{3}} \beta M^2 a_n^{3/2} \delta^{1/2}. \tag{20}$$

On using the fact that

$$\sum a_n^{3/2} = \frac{a^{3/2}}{1 - 1/3\sqrt{3}},$$

we get

$$\sum_n 3^{n-1} \Phi^{(n)} \frac{5}{2} \lambda_1 \beta M^2 a^{3/2} \delta^{1/2}, \quad \lambda_1 = \frac{8}{3\sqrt{10}(\sqrt{3}-1)} = 1.15, \tag{21}$$

$$h = \sum_n h_n = \frac{1}{4} \lambda_2 \frac{a^{3/2}}{\delta^{1/2}}, \quad \lambda_2 = \frac{8}{\sqrt{10}(3\sqrt{3}-1)} = 0.60. \tag{22}$$

The total energy of the specimen, per domain of width a, is

$$\Phi = 2 \sum_n 3^{n-1} \Phi^{(n)} + 2\beta\delta M^2 (l - 2h) = 2\beta M^2 a \left(2\lambda a^{1/2} \delta^{1/2} + \frac{\delta l}{a}\right),$$

$$\lambda = \frac{1}{4}(5\lambda_1 - \lambda_2) = 1.29 \tag{23}$$

where $l$ is the thickness of the plate.

It is still necessary to minimize the energy per unit area of the surface of the plate, that is $\Phi/a$. This gives

$$a = \delta^{1/3} l^{2/3} / \lambda^{2/3}, \tag{24}$$

$$\Phi / a = 6\lambda^{2/3} \beta M^2 \delta^{2/3} l^{1/3}, \tag{25}$$

$$h = \frac{\lambda_2}{4\lambda} l = 0.117\, l, \; h_1 = h\left(1 - \frac{1}{3\sqrt{3}}\right), \quad h_n = \frac{h_1}{(3\sqrt{3})^{n-1}}, \tag{26}$$

$$\theta_n = 3^{n/2}\sqrt{10}(\delta/\lambda l)^{1/3}. \tag{27}$$

It is easily shown that

$$h - \sum_{n'=1}^{n'=n} h_{n'} = \frac{h}{(3\sqrt{3})^n}.$$

If this quantity is of order of magnitude $\delta$, that is if $(3\sqrt{3})^n \sim l/8\delta$, then $a_n \sim \delta$, whereas $\theta_n \sim 1$. Thus the total number of splittings is

$$n_{max} \sim \frac{2}{3} \ln (l/\delta).$$

Figure 2 corresponds to the case $l/\delta \sim 2 \times 10^3$. It is seen that the number of splittings is already quite small. With increase of $l$, the ratio $a/l$ decreases, that is the angles of inclination of the boundaries decrease, while the number of splittings increases.

It is easy to find the field H that turns the magnetization M out of the easy axis. The corresponding problem was stated in general form in Section 2. To the first order in the angle of inclination $\theta$, we have

$$H_x = \pm\beta M\theta, \quad H_z = 0, \tag{28}$$

which agrees with the relation (6). A similar result was obtained by Lifshitz[2].

If we take into account that

$$|\partial\theta / \partial z| \sim \theta_n^2 / a_n, \quad H_z \sim H_x \theta_n,$$

it is easy to see that (28) does not contradict the condition curl H = 0 and the Maxwell boundary condition $H_{t1} = H_{t2}$. The condition for coexistence of phases, $\Phi_1' = \Phi_2'$, for small angles of inclination of the boundaries has the form[4]

$$H_{z1} + H_{z2} = 0,$$

which is also satisfied. The condition $B_n$ = const is satisfied in the zeroth approximation (to within $H_{x1} - H_{x2} \sim \beta M \theta_n \sim \beta M(\delta/l)^{1/3}$). In the next approximation, it is necessary to allow for displacement of the boundaries of separation of phases. We shall not write down the formulas for the field H to the second order in $\theta$. The energy of the magnetic field, $\int dVH^2/8\pi$, as has already been pointed out, is small in comparison with the anisotropy energy, and we shall not allow for it. Outside the specimen, the field H decreases rapidly over distances of order $\delta$.

We shall now briefly discuss the proposed model. At each stage of the splitting, we introduced two variational parameters ($R_n'$ and $\theta_n$). The parameters $a_{n+1}/a_n$ were not varied. The assumption that $a_{n+1}/a_n = \frac{1}{3}$ seems natural to us, but we are not able to prove it. The fact is that in the mathematical literature problems with boundaries not fixed have not been studied at all, and it is not clear by how many independent parameters the general solution is determined. We can assert rigorously only that the energy of the branched structure is not larger than what we have obtained.

It is tempting to estimate the critical dimensions $l_c$ and $a_c$ at which the unbranched structure ceases to be stable. As an unbranched structure we take the structure of Landau and Lifshitz, for which $\Phi/a = \beta M^2(2\delta l)^{1/2}$. Then $l_c = 3^6 \cdot 2^3 \lambda^4 \delta$ and $a_c = 3^3 \cdot 2^3 \lambda^2 \delta$; that is $a_c/l_c = 0.022$, whereas experiment shows that for cobalt, $a_c/l_c \approx 0.12$[12]. Our result, however, must not be taken too seriously; for, first, in order to determine $a_c$ and $l_c$ it is necessary to investigate the stability of the unbranched structure with respect to infinitely small perturbations[2], and, second, the very question of the domain structure of thin films with small uniaxial anisotropy is unclear. The Landau-Lifshitz structure does not guarantee a minimum of the energy, although it evidently gives for the energy a value that is close to the correct one.

The results obtained are valid also for ferromagnets of cubic symmetry. In this case also the anisotropy energy, for small inclinations from the easy axis, can be written in the form $U_{an} = \frac{1}{2}\beta M^2 \theta^2$. The energy of surface tension $\Delta$ can be represented in the form $\Delta = 2\beta\delta M^2$, where $\delta$ is a coefficient of proportionality that does not coincide with the thickness of a domain wall. In cubic ferromagnets, the critical dimensions $l_c$ and $a_c$ are appreciably larger because of the fact that the energy of the unbranched structure is very small[2]. For that very reason, cubic ferromagnets are especially suitable for observation of a simple domain structure with closure triangles.

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[1] L. Landau and E. Lifshitz, Physik. Z. Sowjetunion 8, 153 (1935).
[2] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. 15, 97 (1945) [J. Phys. USSR 8, 337 (1944)].
[3] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), GITTL, M. 1957 (Translation: Addison-Wesley, 1960).
[4] I. A. Privorotskiĭ, Zh. Eksp. Teor. Fiz. 56, 2129 (1969) [Soviet-Phys.-JETP 29, 1145 (1969)].
[5] L. H. Germer, Phys. Rev. 62, 295A (1942).
[6] M. Blackman and E. Grünbaum, Proc. Roy. Soc. (London) 241A, 508 (1957); 245A, 408 (1958).
[7] B. P. Bilenskiĭ, in the collection Fizika magnitnykh plenok, Materialy mezhdunarodnogo simpoziuma (Physics of Magnetic Films, Data of an International Symposium), Irkutsk, 1968, p. 145.
[8] L. Néel, Cahiers de Phys. 25, 1 (1944); J. Phys. Radium 5, 241 (1944).
[9] C. Kittel, Rev. Mod. Phys. 21, 541 (1949).
[10] L. Landau, Zh. Eksp. Teor. Fiz. 13, 377 (1943).
[11] J. Kaczér, Zh. Eksp. Teor. Fiz. 46, 1787 (1964) [Sov. Phys.-JETP 19, 1204 (1964)].
[12] B. Wyslocki and W. J. Zietek, Phys. Lett. 29A, 114 (1969).
[13] E. M. Lifshitz and Yu. V. Sharvin, Dokl. Akad. Nauk SSSR 79, 783 (1951).
[14] L. D. Landau, Zh. Eksp. Teor. Fiz. 7, 371 (1937).

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