SLOW COLLISIONS IN A SYSTEM OF THREE BODIES INTERACTING IN ACCORD WITH COULOMB'S LAW. IV. MESIC-ATOM PROCESSES IN HYDROGEN

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A general approach, consisting of expanding the wave function of three bodies interacting in accordance with Coulomb's law in a series in the solutions of the Schrödinger equation of the two-center problem, is used to calculate concrete reactions occurring in a mixture of hydrogen isotopes with a meson taking part. The cross sections and the interception constants are obtained for μ^--meson transitions between the levels of the hyperfine structure of mesic atoms and for the isotopic-exchange reactions. These processes occur when mesic atoms of hydrogen isotopes collide with nuclei of the hydrogen atoms. The method of phase functions is used for the calculations. The results of the calculations are compared with earlier calculations and with the experimental data.

INTRODUCTION

WHEN μ^- mesons get into a mixture of hydrogen isotopes, they are decelerated and are captured by the Coulomb field of the nuclei, forming mesic atoms, which then go over quite rapidly to the ground state. In liquid hydrogen, all these processes last $\sim 10^{-12}$ sec, which is much shorter than the proper lifetime of the μ^- mesons ($\tau = 2.2 \times 10^{-6}$ sec). During that time, at thermal velocities ($v_T = 2 \times 10^5$ cm/sec), the hydrogen mesic atom has time to experience $\sim 10^8$ collisions with the nuclei of other hydrogen atoms.

Besides elastic scattering, collisions also produce numerous mesic-atom processes, namely interception of the meson by nuclei of heavier hydrogen isotopes, transitions between the hyperfine-structure levels of the mesic atoms, formation of mesic molecules, catalysis of nuclear reactions, etc.[1,2]. In this paper we calculate the cross sections of the first two processes and present a comparison with experiment[3-6] and with the earlier calculations[1,7-9]. The method of perturbed stationary states, details of which were described in earlier papers by the authors[10,11], is used for the calculations.

GENERAL FORMULATION OF PROBLEM

In the method of perturbed stationary states, all the aforementioned problems are formulated in a unified manner and reduce to a solution of a coupled system of Schrödinger equations[11-13]. In the two-level approximation, the system takes the form

$$\begin{aligned}\left(\frac{d^2}{dR^2}+k_1^2-\frac{L(L+1)}{R^2}\right)\chi_1 &= K_{11}\chi_1+K_{12}\chi_2+2Q_{12}\frac{d\chi_2}{dR},\\ \left(\frac{d^2}{dR^2}+k_2^2-\frac{L(L+1)}{R^2}\right)\chi_2 &= K_{21}\chi_1+K_{22}\chi_2+2Q_{21}\frac{d\chi_1}{dR}.\end{aligned} \tag{1}$$

Here k_1 and k_2 are the momenta in the input and output channels of the reaction (we assume $k_1 \leq k_2$ throughout), and $k_{ij} = K_{ij}(R) - K_{ij}(\infty)$ and $Q_{ij} \neq Q_{ij}(R)$ are certain effective potentials, the concrete form of which is determined by the peculiarities of the problem. For each given value of the orbital angular momentum L, the system of Schrödinger equations (1) is equivalent to a system of nonlinear differential equations of first order for the elements $t_{ij}^L(R)$ of the reaction matrix $T^L(R)$[14,15]:

$$\begin{aligned}\frac{d}{dR}t_{ij}^L(R) &= -a_{i\alpha}(K_{\alpha\beta}\tilde{a}_{\beta j}+2Q_{\alpha\beta}\tilde{a}_{\beta j}'),\\ t_{ij}^L(0) &= 0, \quad \alpha,\beta,i,j=1;2,\end{aligned} \tag{2}$$

where

$$\begin{aligned}a_{i\alpha} &= \delta_{i\alpha}u_\alpha + t_{i\alpha}^L(R)v_\alpha,\\ \tilde{a}_{\beta j} &= \delta_{\beta j}u_\beta + t_{\beta j}^L(R)v_\beta,\\ \tilde{a}_{\beta j}' &= \delta_{\beta j}u_\beta' + t_{\beta j}^L(R)v_\beta'.\end{aligned} \tag{3}$$

The functions u_α and v_α are expressed in terms of spherical Bessel functions

$$u_\alpha = \sqrt{\frac{\pi R}{2}}j_L(k_\alpha R), \qquad v_\alpha = -\sqrt{\frac{\pi R}{2}}n_L(k_\alpha R) \tag{3a}$$

and as $R \to \infty$ their asymptotic form is

$$u_\alpha = \frac{1}{\sqrt{k_\alpha}}\sin\left(k_\alpha R-\frac{\pi L}{2}\right), \qquad v_\alpha = \frac{1}{\sqrt{k_\alpha}}\cos\left(k_\alpha R-\frac{\pi L}{2}\right). \tag{3b}$$

The matrix T^L is defined by the condition $t_{ij}^L = t_{ij}^L(\infty)$.

In practice, the system (2) is integrated up to the value $R = R_0$, and the contribution Δt_{ij}^L from the region $R_0 \leq R \leq \infty$ is estimated analytically[11,14]:

$$t_{ij}^L = t_{ij}^L(R_0) + \Delta t_{ij}^L. \tag{4}$$

In addition, since the matrix elements $K_{ij}(R)$ are singular when $R \to 0$, it is necessary to specify the asymptotic form

$$t_{ij}^L(R) = c_{ij}^L(k_ik_j)^{L+1/2}R^{2L+1}, \tag{5}$$

where the coefficients c_{ij}^L are constants determined by the form of $K_{ij}(R)$ from (2). The partial cross sections of the elastic and inelastic processes are calculated in accordance with the formulas[16]:

$$\sigma_{ij}^L = \frac{\pi}{k_i^2}(2L+1)\,|\delta_{ij}-S_{ij}^L|^2, \tag{6}$$

where the scattering matrix is

$$S^L = (1+iT^L)(1-iT^L)^{-1}. \tag{7}$$

Taking formulas (6) and (7) into account, the general expression for the cross sections σ_{ij} in terms of the

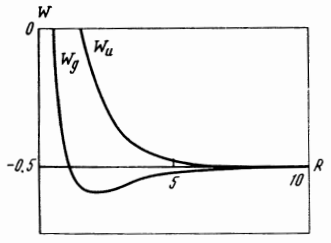

FIG. 1. Symmetrical and antisymmetrical terms $W_g(R)$ and $W_u(R)$ of a system consisting of two nuclei and a μ^-meson.

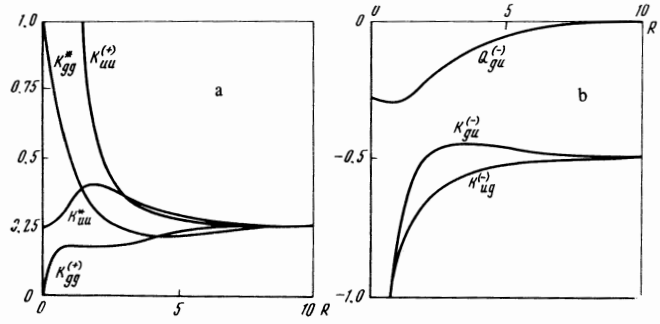

FIG. 2. a–Diagonal matrix elements of nuclear motion over the wave functions of the two-center problem $K_{gg}^{(+)} = K_{uu}^{(-)} = K_{gg}^{*} = K_{uu}^{*} = -\frac{1}{2}W_g = -\frac{1}{2}W_u = \frac{1}{4}$ as $R \to \infty$. b–Off-diagonal matrix elements $K_{gu}^{(-)} = K_{ug}^{(-)} = -\frac{1}{2}$ as $R \to \infty$.

elements t_{ij} of the matrix of the reaction takes the form

$$\sigma_{ij} = \frac{4\pi}{k_i^2} \frac{\delta_{ij} D^2 + t_{ij}^2}{(D-1)^2 + (t_{11} + t_{22})^2} \tag{8}$$

where $D = t_{11}t_{22} - t_{12}t_{21}$.

At small collision energies E, we can introduce the low-energy scattering parameters a_{ij}

$$t_{ij} = -a_{ij}\sqrt{k_i k_j}, \tag{9}$$

which are analogous to the scattering lengths a_g and a_u in the single-channel case[16].

If $k_1 \to 0$, then $D \sim k_1^2$, $k_2 \to k_0 = \sqrt{2M\Delta E}$, and formula (8) simplifies to

$$\sigma_{ij} = 4\pi \frac{k_j}{k_i} \frac{a_{ij}^2}{1 + k_0^2 a_{22}^2} \tag{10}$$

The matrix elements K_{ij} and Q_{ij} for all the types of problems which we shall solve henceforth are expressed in terms of the symmetrical terms $W_g(R)$ and the antisymmetrical terms $W_u(R)$ of the two-center problem, and also in terms of the matrix elements

$$\begin{gathered} K_{\alpha\beta} = K_{\alpha\beta}^{(+)} + \varkappa K_{\alpha\beta}^{(-)} + \varkappa^2 K_{\alpha\beta}^{*}, \\ Q_{\alpha\beta} = Q_{\alpha\beta}^{(+)} + \varkappa Q_{\alpha\beta}^{(-)}, \\ \varkappa = \frac{M_2 - M_1}{M_2 + M_1}, \qquad M_2 \geqslant M_1, \qquad (\alpha, \beta) \equiv (g, u), \end{gathered} \tag{11}$$

which were calculated in[17,18].

Figures 1 and 2 show plots of these functions in terms of the units $\hbar = e = m = 1$ (M_μ is the muon mass)

$$m = M_\mu \frac{M_1 + M_2}{M_1 + M_2 + M_p}. \tag{12}$$

The corresponding transition formulas are

$$\begin{aligned} K(R) &= A\tilde{K}A^{-1}, \\ Q(R) &= AQA^{-1}, \end{aligned} \tag{13}$$

where

$$\begin{gathered} \tilde{K}_{11} = 2MW_g(R) + K_{gg}(R) = 2M\tilde{W}_g, \\ \tilde{K}_{12} = K_{gu}(R), \quad \tilde{K}_{21} = K_{ug}(R), \\ \tilde{K}_{22} = 2MW_u(R) + K_{uu}(R) = 2M\tilde{W}_u, \\ M = \frac{M_0}{m}; \qquad \frac{1}{M_0} = \frac{1}{M_1} + \frac{1}{M_2}. \end{gathered} \tag{14}$$

The matrix A effects the transition from the set of molecular functions χ_g (symmetrical) and χ_u (asymmetrical) to the set χ_1 and χ_2

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = A \begin{pmatrix} \chi_g \\ \chi_u \end{pmatrix}, \tag{15}$$

the concrete form of which is determined by the peculiarities of the problem.

ISOTOPIC EXCHANGE PROCESSES

In a mixture of hydrogen isotopes, there occur the isotopic-exchange processes

$$p\mu^- + d \leftrightarrow p + d\mu^-, \tag{16a}$$

$$p\mu^- + t \leftrightarrow p + t\mu^-, \tag{16b}$$

$$d\mu^- + t \leftrightarrow d + t\mu^-, \tag{16c}$$

which are shown schematically in Fig. 3. The collision energy E is reckoned from the level E_1 of the lighter mesic atom, $E' = E + \Delta E$, and, accordingly,[1)]

$$k_1^2 = 2ME, \quad k_2^2 = 2ME' = k_1^2 + k_0^2, \quad k_0 = \sqrt{2M\Delta E} \tag{17}$$

(for example, in the isotope mixture (16a) the level E_1 corresponds to the system $p\mu^- + d$, and the level E_2 to the system $p + d\mu^-$). Each of the cross sections σ_{ij} describes the transition $E_i \to E_j$, for example, the cross section σ_{12} for the process (16a) corresponds to the interception reaction

$$p\mu^- + d \to p + d\mu^-. \tag{16d}$$

In such an approach, we defer the study of subthreshold effects and all the problems connected with the influence of the closed channel (σ_{11}) on the open channel (σ_{22}), for example, in the elastic scattering of $d\mu^-$ mesic atoms by protons of energy $E' < \Delta E$.

For the isotopic-exchange processes (16), the transition matrix A is given by

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{18}$$

It is chosen such that when $R \to \infty$, the system of equations (1) breaks up into two independent equations, the function χ_1 describing the system $p\mu^- + d$ (or $p\mu^-$

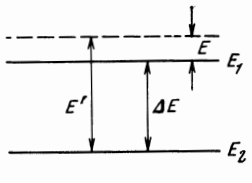

FIG. 3. Level scheme of the three-body system as $R \to \infty$: $E_1 = K_{11}(\infty)$ and $E_2 = K_{22}(\infty)$. The collision energy E is reckoned from the upper level of the system E_1.

[1)] In the two-level approximation, the isotopic level difference is $\Delta E = \kappa m/2M$, whereas the exact value is $\Delta E_0 = \kappa(M_1 + M_2)/2(M_1 + 1)(M_2 + 1)$ (all quantities are given in mesic-atom units $\hbar = e = M_\mu = 1$). However, since $\Delta E - \Delta E_0 \approx \frac{1}{2}\kappa(M_1^{-2} + M_2^{-2})$, the indicated error can be neglected at the calculation accuracy employed in this paper. Numerically, for the systems (13), ΔE is equal to 147, 200, and 50.4 eV, and ΔE_0 is equal to 135, 183, and 48.1 eV.

+ t, $d\mu^-$ + t), and the function χ_2 the system p + $d\mu^-$ (or p + $t\mu^-$, d + $t\mu^-$).

$$\begin{aligned} K_{11}(R) &= M(W_g + W_u) - {}^1/_2(K_{gu} + K_{ug}),\\ K_{12}(R) &= M(W_g - W_u) + {}^1/_2(K_{gu} - K_{ug}),\\ K_{21}(R) &= M(W_g - W_u) - {}^1/_2(K_{gu} - K_{ug}),\\ K_{22}(R) &= M(W_g + W_u) + {}^1/_2(K_{gu} + K_{ug}). \end{aligned} \tag{19}$$

The coefficients c_{ij}^L for the isotopic-exchange processes (16) are equal to

$$c_{11}^L = c_{22}^L = -c_{12}^L = -c_{21}^L,$$

$$c_{11}^L = -\frac{2L+1}{[(2L+1)!!]^2}\,\frac{(2L+1)^2+4-(2L+1)\sqrt{(2L+1)^2+8}}{8} \tag{20}$$

The results of numerical calculations for the processes (16a)–(61c) at collision energies 10^{-3}–100 eV are given in Tables I–III. Let us note a few distinguishing features of the results.

It is easily seen from (10) that at slow collisions ($k_1 \to 0$) we have $k_2 \approx k_0$ = const, and therefore

$$\sigma_{11} \approx \text{const}, \quad \sigma_{22} \approx \text{const},$$
$$\sigma_{12} \sim 1/k_1, \quad \sigma_{21} \sim k_1. \tag{21}$$

This allows us to introduce the transition constant λ by means of the formula

$$\lambda = \sigma_{12} v_1 \ \text{cm}^3\text{-sec}^{-1} \tag{22}$$

or else

$$\Lambda = \lambda n_0 \ \text{sec.}^{-1}, \tag{23}$$

where $n_0 = 4.25 \times 10^{22}$ cm^{-3} is the density of liquid hydrogen and v_1 is the initial collision velocity.

As $k_1 \to 0$ we have

$$\sigma_{12} \approx 4\pi \frac{k_0}{k_1}\,\frac{a_{12}^2}{1 + k_0^2 a_{22}^2}, \tag{24}$$

$$\lambda \approx 4\pi \frac{k_0}{M}\,\frac{a_{12}^2}{1 + k_0^2 a_{22}^2}. \tag{24a}$$

Formulas (21) indicate also the threshold behavior of the cross sections $\sigma_{21} \sim k_1$. The second columns of Tables I–III contain the numbers of the partial waves that contribute to the cross sections σ_{11}, σ_{12}, and σ_{21}. It follows from them that the region of pure s-scattering for the processes (16) extends all the way to a collision energy E ~ 10^{-2} eV. In this region, the cross sections σ_{ij} satisfy the formulas (10) and (24).

The region of collision energies at which the condition $a_{ij} \approx$ const is satisfied is even narrower than the region of pure s-scattering, and is bounded by the condition E ≤ 10^{-3} eV. The condition $\lambda \approx$ const, which is customarily used in the analysis of the experiments (16a) and (16b) on mesic-atom scattering, was satisfied up to collision energies E ~ 1 eV, in spite of the fact that the s-wave approximation is violated in this case (see Tables I and II). We see also from Table III that λ for the reaction (16c) is anomalously small and the contribution of the p wave must be taken into account already at E ~ 10^{-2} eV. In this case the two-level approximation is expected to give a less reliable result than in the case of reactions (16a) and (16b).

The number of partial waves contributing to the cross section σ_{22} at a collision energy E < 100 eV is approximately constant and is equal to 5, 6, and 7 for processes (16a)–(16c), respectively.

Our calculations of the transition constant λ are compared in Table IV with the results of earlier calculations and with the experimental data.

Table I. Cross sections σ_{ij} (in units of 10^{-20} cm^2) and the constant λ for the pμ^- + d system

E, eV	L_{max}	σ_{11}	σ_{12}	σ_{21}	λ, 10^{13} cm^3-sec^{-1}
10^{-3}	0	6.7	$7.3\cdot10^2$	$5.0\cdot10^{-3}$	3.9
10^{-2}	0	7.0	$2.3\cdot10^2$	$1.5\cdot10^{-2}$	3.9
0,1	1	7.9	71	$4.8\cdot10^{-2}$	3.8
1,0	2	9,7	21	0.14	3.6
10	3	10	8,5	0,54	4.6
100	6	6,7	5,4	2,2	9.1

M = 6.14; M_μ/m = 1.04; k_0 = 0.577; $k_1 = 4.76 \times 10^{-2}\sqrt{E(eV)}$; $\sigma_{22} \approx 1.8 \times 10^{-19}$ cm^2.

Table II. Cross sections σ_{ij} (in units of 10^{-20} cm^2) and the constant λ for the pμ^- + t system

E, eV	L_{max}	σ_{11}	σ_{12}	σ_{21}	λ, 10^{13} cm^3-sec^{-1}
10^{-3}	0	3.3	$3.5\cdot10^2$	$1.7\cdot10^{-3}$	1.8
10^{-2}	0	3.6	$1.1\cdot10^2$	$5.5\cdot10^{-3}$	1.7
0,1	1	4.6	34	$1.7\cdot10^{-2}$	1,7
1,0	2	7.1	11	$5,6\cdot10^{-2}$	1.8
10	3	9.9	5,9	0,28	3,0
100	6	7.0	3,6	1.2	5.8

M = 6,85; M_μ/m = 1,03; k_0 = 0,706; $k_1 = 5,00\cdot10^{-2}\sqrt{E, (eV)}$; $\sigma_{22} = 2,0\cdot10^{-19}$ cm^2.

Table III. Cross sections σ_{ij} (in units of 10^{-20} cm^2) and the constant λ for the dμ^- + t system

E, eV	L_{max}	σ_{11}	σ_{12}	σ_{21}	σ_{22}	λ, 10^{13} cm^3-sec^{-1}
10^{-3}	0	2.6	4.1	$8.1\cdot10^{-5}$	18	$1.6\cdot10^{-2}$
10^{-2}	1	3.2	1.4	$2.9\cdot10^{-4}$	18	$1.8\cdot10^{-2}$
0,1	1	5.3	0.85	$1.7\cdot10^{-3}$	18	$3.4\cdot10^{-2}$
1,0	2	11	1,6	$3.2\cdot10^{-2}$	18	0.21
10	4	17	6.5	1.1	17	2.6
100	6	12	7.0	4.7	13	8.9

M = 10,9; M_μ/m = 1,02; k_0 = 0,446; $k_1 = 6,29\cdot10^{-2}\sqrt{E, (eV)}$.

Table IV. Transition constant λ for the isotopic exchange processes, in units of 10^{-13} cm^3 sec^{-1}

Reaction	Source				
	Dzhelepov et al. [8]	Bleser et al. [3]	Cohen et al. [9]	Zel'dovich and Gershtein [1]	Present work
$p\mu^- + d \to p + d\mu^-$	2.8±0,9	3.4±0.3	3,3	3,4	3,9
$p\mu^- + t \to p + t\mu^-$	—	—	—	1.5	1.7
$d\mu^- + t \to d + t\mu^-$	—	—	—	$1.2\cdot10^{-2}$	$1.8\cdot10^{-2}$

TRANSITIONS BETWEEN HYPERFINE-STRUCTURE LEVELS

For hydrogen mesic atoms, the hyperfine splitting ΔE of the ground state greatly exceeds the average thermal-collision energy (E ≈ 0.02 eV):

$$\Delta E = \begin{cases} 0.183 \text{ eV for } p\mu^- \\ 0.049 \text{ eV for } d\mu^- \\ 0.241 \text{ eV for } t\mu^-. \end{cases} \tag{25}$$

This leads to a complication of the picture of the scattering processes in the systems $p\mu^- + p$, $d\mu^- + d$, and $t\mu^- + t$, and to the need for taking the influence of the spin into account for their description. The system of resultant equations then coincides with the system (1), and was obtained by Gershteĭn[8]. Since ΔE is sufficiently small, we can confine ourselves in (1) to the s-wave ($L = 0$) if $E < 1$ eV. If $E > 1$ eV, this approximation is not valid and it is necessary to take into account the partial cross sections σ_{ij}^{L} with $L \neq 0$. At these collision energies, however, the influence of the spin on the scattering processes can already be neglected, and the thus simplified problem was solved in an earlier paper by the authors.[11]

Allowance for the spin interaction between the meson and the nuclei leads to confusion of the equations for the wave functions χ_g and χ_u. In addition, since the nuclei are identical, it is necessary to take into account the statistics of the scattered particles. In the approximation of pure s-scattering, a system of two nuclei with spins J_1 and J_2 and a μ^- meson with spin $S = 1/2$ is characterized by a total angular momentum $J = J_1 + J_2 + S$. If we neglect the spin interaction of the nuclei, then the levels are also classified in accord with the value of the total angular momentum $F = J_1 \pm 1/2$ of the meson-plus-nucleus system.

In what follows, particular interest attaches to transitions between the hyperfine-structure levels of mesic atoms, i.e., the transitions

$$F_1 = J_1 + 1/2 \to F_2 = J_1 - 1/2.$$

The level scheme of Fig. 3 remains valid also in this case, but now the level E_1 corresponds to the upper state of the hyperfine structure with angular momentum F_1, and the level E_2 to the lower state with angular momentum F_2.

Just as in the case of isotopic-exchange reactions, in order to calculate the cross sections $\sigma_J(F_1 \to F_2)$ for a specified total angular momentum J, it is necessary to find the matrix A of the transition from the molecular functions χ_g and χ_u, with definite values of the spin of two nuclei, to the atomic functions χ_1 and χ_2, with specified value of the spin F of the meson-plus-nucleus system. The effective potentials K_{ij} of the problem can then be calculated readily by means of formula (13).

At the known value (25) of the hyperfine splitting and at specified values of the masses $M = 10-30$, the momentum k_0 is very small ($\sim 10^{-2}$). Therefore formulas (10) and (24a) simplify to[2]:

$$\sigma_{ij} \approx 4\pi a_{ij}^2 \frac{k_j}{k_i}, \tag{26}$$

$$\lambda(J) \approx 4\pi a_{12}^2 \frac{k_0}{M}, \tag{26a}$$

where $\lambda(J)$ is the constant of transitions from the level $F_1 = J_1 + 1/2$ to the level $F_2 = J_1 - 1/2$ in the state with total angular momentum J.

[2] All the preceding formulas are written in the units $e = \hbar = m = 1$. It will be convenient to use henceforth the parameters a_{ij} in the mesic-atom system of units $e = \hbar = M_\mu = 1$. In this case, to obtain the dimensional quantities it is necessary to multiply formulas (10), (24), and (26) by $a_\mu^2 = 6.55 \times 10^{-22}$ cm^2, and formulas (24a) and (26a) by $\alpha c a_\mu^2 = 1.43 \times 10^{-13}$ cm^3-sec^{-1}.

Numerical calculations show that at a collision energy $E \leq 10^{-2}$ eV (for the system $t\mu^- + t$ at $E \leq 10^{-3}$ eV) we have $a_{ij} \approx$ const, and the matrix of the coefficients $a = \{a_{ij}\}$ is connected with the diagonal matrix $\tilde{a}$ by a relation analogous to (13):

$$a = A\tilde{a}A^{-1}, \quad \tilde{a} = \begin{pmatrix} a_g & 0 \\ 0 & a_u \end{pmatrix}. \tag{27}$$

Here a_g and a_u are the scattering lengths in the even and odd channels, respectively, calculated without allowance for the hyperfine structure in an earlier paper[9].

THE REACTION $p\mu^- + p$

For this system we have

$$J_1 = J_2 = 1/2, \quad M = 4.69, \quad M_\mu/m = 1.06,$$
$$k_0 = 1.80 \cdot 10^{-2}, \quad k_1 = 4.20 \cdot 10^{-2}\sqrt{E(\text{eV})}. \tag{28a}$$

The transition matrix A is[8]:

$$A = \frac{1}{2}\begin{pmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{pmatrix}, \tag{29a}$$

and the matrix elements K_{ij} in Eqs. (1), which describe this process, take the form

$$Q_{ij} = 0, \quad K_{11} = \frac{M}{2}(3\bar{W}_g + \bar{W}_u),$$
$$K_{12} = K_{21} = -\frac{\sqrt{3}}{2}M(\bar{W}_g - \bar{W}_u), \tag{30a}$$
$$K_{22} = \frac{M}{2}(\bar{W}_g + 3\bar{W}_u).$$

As $R \to \infty$ the function χ_1 represents a system comprising a proton and an incoming mesic atom $p\mu^-$ in the upper state E_1 of the hyperfine structure, with angular momentum $F_1 = 1$, while the function χ_2 represents the same system in the lower state E_2 with angular momentum $F_2 = 0$. (The transition $F_1 \to F_2$ is possible here only in a state with total angular momentum $J = 1/2$, since the state $J = 3/2$ does not contain a level with angular momentum $F_2 = 0$.)

The values of the coefficients c_{ij} in formula (5) for the initial integration conditions are respectively

$$c_{11} = -1/8, \quad c_{12} = c_{21} = -\sqrt{3}/8, \quad c_{22} = -3/8. \tag{31a}$$

The results of the calculations for the $p\mu^- + p$ system are listed in Table V. Attention is called to the strong dependence of the cross section σ_{22} on the collision energy.

THE REACTION $d\mu^- + d$

For this system we have

$$J_1 = J_2 = 1, \quad M = 9.12, \quad M_\mu/m = 1.03,$$
$$k_0 = 1.3 \cdot 10^{-2}, \quad k_1 = 5.77 \cdot 10^{-2}\sqrt{E\,(\text{eV})}. \tag{28b}$$

The function χ_1 corresponds to a system comprising a deuteron plus an incoming $d\mu^-$ mesic atom in a state with angular momentum $F_1 = 3/2$, and the function χ_2 corresponds to this system in the state with angular momentum $F = 1/2$. In s-scattering, the total angular momentum J of the three-particle system is conserved, and transitions $F_1 \to F_2$ are possible only if both levels F_1 and F_2 belong to a multiplet with a definite value of J. For the system $d\mu^- + d$, two cases

Table V. The cross section σ_{ij} (in units of 10^{-19} cm^2) and the parameters a_{ij} for the $p\mu^-$ + system

E, eV	σ_{11}	σ_{12}	σ_{21}	σ_{22}	a_{11}	$a_{12}=a_{21}$	a_{22}
10^{-3}	6.5	58	0.32	$2.0\cdot10^{-3}$	−8.9	7.2	−0.14
10^{-2}	6.3	18.6	0.97	$4.6\cdot10^{-3}$	−8.8	7.2	−0.13
0.1	5.6	6.7	2.4	$3.1\cdot10^{-2}$	−8.3	7.1	$5\cdot10^{-3}$
0.5	4.0	3.7	2.7	0.14	−7.3	6.6	0.44

are possible: J = 3/2 and J = 1/2; these cases must be considered separately.

When J = 3/2, the transition matrix[8] is

$$A=\frac{1}{\sqrt{6}}\begin{pmatrix}1 & \sqrt{5}\\ -\sqrt{5} & 1\end{pmatrix}. \qquad (29b)$$

The matrix elements are

$$K_{11}(R)=\frac{M}{3}(\tilde{W}_g+5\tilde{W}_u),$$
$$K_{12}(R)=K_{21}(R)=-\frac{\sqrt{5}}{3}M(\tilde{W}_g-\tilde{W}_u), \qquad (30b)$$
$$K_{22}(R)=\frac{M}{3}(5\tilde{W}_g+\tilde{W}_u).$$

The initial-condition coefficients are

$$c_{11}=-\frac{5}{12},\quad c_{12}=c_{21}=-\frac{\sqrt{5}}{12}\quad c_{22}=-\frac{1}{12}. \qquad (31b)$$

At a collision energy E = 10^{-2} eV, the following values are obtained for the cross sections:

$$\sigma_{11}=1.2\cdot10^{-19}\,\text{cm}^2,\quad \sigma_{12}=1.5\cdot10^{-20}\,\text{cm}^2;$$
$$\sigma_{21}=2.5\cdot10^{-21}\,\text{cm}^2,\quad \sigma_{22}=2.5\cdot10^{-19}\,\text{cm}^2. \qquad (32a)$$

We note that the interception cross section σ_{12} is anomalously small compared with the analogous cross section for the process (16a). This is due to the fact that the scattering lengths a_g and a_u, which determine, via the parameter a_{12}, the interception cross section, have like signs.

For J = 1/2 we have

$$A=\frac{1}{\sqrt{3}}\begin{pmatrix}\sqrt{2} & 1\\ -1 & \sqrt{2}\end{pmatrix}, \qquad (29c)$$
$$K_{11}(R)=2/3M(2\tilde{W}_g+\tilde{W}_u),$$
$$K_{12}(R)=K_{21}(R)=-\frac{2\sqrt{2}}{3}M(\tilde{W}_g-\tilde{W}_u),$$
$$K_{22}(R)=2/3M(\tilde{W}_g+2\tilde{W}_u). \qquad (30c)$$

The coefficients c_{ij} are given by

$$c_{11}=-\frac{1}{6},\quad c_{12}=c_{21}=-\frac{\sqrt{2}}{6},\quad c_{22}=-\frac{1}{3}. \qquad (31c)$$

The cross sections at E = 10^{-2} eV are equal to

$$\sigma_{11}=2.0\cdot10^{-19}\,\text{cm}^2,\quad \sigma_{12}=2.4\cdot10^{-20}\,\text{cm}^2,$$
$$\sigma_{21}=4.1\cdot10^{-21}\,\text{cm}^2,\quad \sigma_{22}=1.3\cdot10^{-19}\,\text{cm}^2. \qquad (32b)$$

THE REACTION $t\mu^- + t$

This system is analogous in many respects to the $p\mu^-$ + p system, and differs from it formally only in the values

$$M=13.5,\quad M_\mu/m=1.02,$$
$$k_0=3.44\cdot10^{-2},\quad k_1=7.00\cdot10^{-2}\sqrt{E(\text{eV})}. \qquad (28c)$$

Actually, however, it turns out that at this value of M the conditions $a_{ij}\approx$ const are valid only up to a collision energy E ≤ 10^{-3} eV. The results of numerical calculations for this process are given in Table VI.

Table VI. Cross sections σ_{ij} (in units of 10^{-20} cm^2) and the parameters a_{ij} for the $t\mu^-$ + t system

E, eV	σ_{11}	σ_{12}	σ_{21}	σ_{22}	a_{11}	$a_{12}=a_{21}$	a_{22}
10^{-3}	13	$1.7\cdot10^2$	0.7	2.9	−4.0	3.6	1.9
10^{-2}	11	53	2.1	3.0	−3.7	3.6	1.9
0.1	5.7	17	5.1	3.8	−2.5	3.5	2.0
0.5	0.90	8.3	5.6	6.5	−0.87	3.0	2.9

Table VII. The parameters a_{ij} of low-energy scattering

	a_{11}	$a_{12}=a_{21}$	a_{22}	a_g	a_u
$p\mu^-+p$	−8.9	7.2	−0.31	−13.3	3.7
$d\mu^-+d$ ($J=1/2$)	4.7	−1.1	3.5	5.5	3.1
$d\mu^-+d$ ($J=3/2$)	3.5	0.86	5.1	5.5	3.1
$t\mu^-+t$	−4.2	3.6	1.9	−6.5	2.4

DISCUSSION OF RESULTS

The calculation results show that in calculating the cross sections for the transition between the levels of the hyperfine structure of hydrogen in mesic atoms up to collision energies E ~ 1 eV it is possible to confine oneself to pure s-scattering. For this case there are simple formulas (26) which make it possible to express the cross sections in terms of the parameters calculated in the present paper for low-energy scattering. At low collision energies, when the condition $a_{ij}\approx$ const is satisfied, the parameters a_{ij} are expressed in simple fashion with the aid of formulas (27) in terms of the scattering lengths a_g and a_u, which were calculated by the authors earlier[9]. The concrete form of these formulas is analogous to expressions (30), once the substitutions $2M\tilde{W}_g \to a_g$, $2M\tilde{W}_u \to a_u$, $k_{ij} \to a_{ij}$ are made.

Substitution of these expressions into (26) leads to the Gershteĭn formulas[8] for the cross section σ_{ij}. The region of applicability of these formulas is determined by the condition $a_{ij}\approx$ const, and differs for different cross sections and for different processes.

For all processes of symmetrical charge exchange with allowance for the hyperfine splitting, the condition a_{12} = const is well satisfied in the collision-energy region E < 0.5 eV. The condition a_{11} = const is satisfied for the processes $p\mu^-$ + p and $d\mu^-$ + d up to collision energies E ≈ 10^{-2} eV, and for the process $t\mu^-$ + t

Table VIII. The constant λ, units of 10^{-13} cm^3-sec^{-1}, of the transitions between the hyperfine-structure levels at different collision energies

E, eV	$p\mu^- + p$	$d\mu^- + d$	$t\mu^- + t$
10^{-3}	1.2	$1.1 \cdot 10^{-2}$	0.20
10^{-2}	1.2	$1.2 \cdot 10^{-2}$	0.20
0.1	1.4	$1.9 \cdot 10^{-2}$	0.21
0.5	1.7	$3.2 \cdot 10^{-2}$	0.22
Gershtein [8]	0.5	$1.7 \cdot 10^{-3}$	—

up to $E \approx 10^{-3}$ eV. The parameter a_{22} has a real meaning only for the process $d\mu^- + d$, for which the hyperfine splitting ΔE is sufficiently small. For the processes $p\mu^- + p$ and $t\mu^- + t$, the value of a_{22} obtained by formula (9) should be regarded only as a formal parameter, which has no special physical meaning. Its use, however, is convenient, inasmuch as $a_{22} \approx$ const in the collision-energy region $E \leq 0.5$ eV (with the exception of the process $p\mu^- + p$, see Table V).

Outside the indicated energy region, it is necessary to use the general formula (8) for the cross sections σ_{ij}. For $p\mu^- + p$, and especially for $d\mu^- + d$, the region of applicability of Table VII can be extended by using the following expansion for the diagonal elements t_{ii}:

$$k_i t_{ii}^{-1} = -\frac{1}{a_{ii}} + \frac{3\pi M}{2a_{ii}^2} + \frac{3M}{a_{ii}} k_i^2 \ln \frac{9Mk_i^2}{32}. \quad (33)$$

From this expansion we can calculate the matrix elements t_{11} for the process $p\mu^- + p$, and also t_{11} and t_{22} for the process $d\mu^- + d$; after substituting them in (8) we can obtain analytic expressions for the cross sections σ_{ij} up to collision energies $E \sim 0.1$ eV[3].

TRANSITION CONSTANTS

The constants $\lambda(J)$ and $\Lambda(J)$ of the transitions between the levels of the hyperfine structure in states with definite values of the total angular momentum J differ from the physically-measured transition constants $\lambda = \lambda(F_1 \to F_2)$ by a factor equal to the statistical weight of the levels with values of the angular momentum F_1 in the mixture of states with different values of the total angular momentum J of the three-particle system. The corresponding formulas have been written out in Gershteĭn's papers[8]. For the processes $p\mu^- + p$, $t\mu^- + t$ and $d\mu^- + d$ they take the respective forms

$$\lambda = \lambda(1 \to 0) = {}^1/_3\lambda \quad (J = {}^1/_2), \quad (34)$$

$$\lambda = \lambda({}^3/_2 \to {}^1/_2) + {}^1/_3\lambda(J = {}^3/_2) + {}^1/_6\lambda(J = {}^1/_2). \quad (35)$$

It is seen from Table VIII that the condition λ = const, which is usually employed in the analysis of the experimental data, is satisfied only in a narrow energy region $E < 10^{-2}$ eV. When $E < 10^{-2}$ eV and relations (27) and (26) are taken into account, the expressions (34) and (35) go over into the Gershteĭn formulas[8]

$$\lambda_p \approx \frac{\pi}{4}(a_g - a_u)^2 \frac{k_0}{M} \cdot \alpha c a_\mu^2 \, cm^3\text{-}sec^{-1}, \quad (36)$$

$$\lambda_d \approx \frac{\pi}{3}(a_g - a_u)^2 \frac{k_0}{M} \cdot \alpha c a_\mu^2 \, cm^3\text{-}sec^{-1}, \quad (37)$$

The values obtained in[8] are respectively $\lambda_p = 0.5 \times 10^{-13}$ and $\lambda_d = 1.7 \times 10^{-16}$.

We take special notice of the fact that the transition constant λ_d exceeds the earlier estimate by one order of magnitude. This is of importance for experiments on μ capture in deuterium and for the catalysis of the dd reaction at different deuterium concentrations (the so-called Gershteĭn—Wol'fenshteĭn effect[3]).

Table IX. Cross section for elastic scattering of $p\mu^-$ mesic atoms in the lower state of the hyperfine structure

	Dzhelepov et al. [4]	Alberigi et al. [5]	Cohen et al. [9]	Zel'dovich and Gershteĭn [1]	Present work
σ_{22}, 10^{-21} cm^2	167±30	7.6±0.7	8.2	1.2	2.5

ELASTIC SCATTERING IN THE LOWER STATE OF THE HYPERFINE STRUCTURE

The calculation scheme employed in the present paper is not suited for the calculation of the cross sections σ_{22} at collision energies $E' < \Delta E$. They can be estimated, however, by means of Gershteĭn's formulas[8], and the corresponding calculations are given in our earlier paper[11]. The applicability of Gershteĭn's formulas at thermal collision energies $E' \sim 10^{-2}$ eV has been verified by the results of this paper. For the processes $p\mu^- + p$, $t\mu^- + t$, and $d\mu^- + d$, respectively, these formulas are

$$\sigma_{22} = \pi(a_g + 3a_u)^2,$$

$$\sigma_{22} = \frac{4\pi}{3}\left[\left(\frac{a_g + 3a_u}{3}\right)^2 + 2\left(\frac{5a_g + a_u}{6}\right)^2\right]. \quad (38)$$

In Tables IX and X, these values are compared with the earlier calculations[8,9] and with the experimental results[3-6].

COLLISIONS AT HIGH ENERGIES

At collision energies $E > 1$ eV, it is necessary to take into account the contribution made by the partial waves with $L \neq 0$ to the cross sections σ_{ij}. In a rigorous analysis of the problem, this calls for taking into account the spin-orbit interaction during the motion of the nuclei, and entails considerable difficulties. However, experience with the earlier calculations[9] has shown that at relatively high collision energies it is possible to neglect both the influence of the spin-orbit interaction and the hyperfine splitting of the levels ΔE, and all that need be taken into account is the identity of the scattered particles. In this case, only the total symmetrized cross section is meaning-

[3] It should be noted that in calculating the matrix elements t_{ii} by means of Eq. (33), it is necessary to substitute in it the parameters a_{ii} in units of the problem, $a_m = \hbar^2/me^2$, which differ from the tabulated values by a factor m/M_μ. If the momenta k_i are defined by formulas (28), then the cross sections in (8) are obtained in units of $a_m^2 = (M_\mu/m)^2 \times 6.55 \times 10^{-22}$ cm^2.

Table X. Cross section for the elastic scattering of $d\mu^-$ mesic atoms in the lower state of the hyperfine structure $\sigma_{22} = {}^1/_3\sigma_{22}(J = {}^1/_2) + {}^2/_3\sigma_{22}(J = {}^3/_2)$

	Dzhelepov et al. [5]	Alberigi et al. [6]	Dzhelepov et al. [4]	Zel'dovich and Gershteĭn [1]	Cohen et al. [9]	Present work
σ_{22}, 10^{-19} см^2	1,5±0,5	0,55±0,20	4,15±0,29	3,3	3.5	1.8

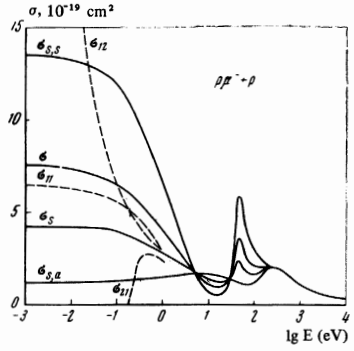

FIG. 4. Energy dependence of the cross sections of different processes in the system $p\mu^-$ + p. When E < 1 eV it is necessary to take into account the hyperfine structure of the $p\mu^-$ mesic atom and only the cross sections σ_{ij} have a real meaning. When 1 < E < 150 eV, the hyperfine splitting can be neglected, but it is still necessary to take into account the particle statistics: $\sigma_{s,s}$—cross section for scattering in the singlet state of the protons, $\sigma_{s,a}$—in the triplet state, σ_s—statistical mixture of both states. When E > 150 eV, all these cross sections are equal to the cross section σ without allowance for the particle spins.

ful[10] (we shall henceforth consider only the process $p\mu^-$ + p)

$$\sigma_s(k) = {}^1/_4\sigma_{s,s}(k) + {}^3/_4\sigma_{s,a}(k), \qquad (39)$$

where $\sigma_{S,S}(k)$ is the cross section for the scattering in the singlet state of two protons, and $\sigma_{S,a}(k)$ the cross section in the triplet state. When $E > 10^3$ eV, it also becomes unnecessary to take into account the statistics of the protons, for in this region $\sigma_S(k) \approx \sigma(k)$, i.e., to the total cross section without allowance for the particle spins.

Figure 4 shows plots of the corresponding cross sections. It is of interest to note that reactions of this type possess resonances, the origin of which was discussed earlier[9].

CONCLUSION

The present results make it possible to determine the accuracy and limits of applicability of the earlier calculations. The employed calculation method can be generalized and made more precise. This pertains principally to allowance for the higher states, which do not take part directly in the interception processes, but which introduce corrections $\sim 1/M^2$ in the effective potentials K_{ij} of the problem. This influence can usually be neglected, but in resonance situations, when the scattering lengths are large (for example, a_g in the $p\mu^-$ + p process), allowance for these corrections may turn out to be important.

Besides physical applications, the foregoing calculations are also of methodological interest, since they justify the applicability of the method of perturbed stationary states for slow collisions in a system of three bodies interacting in accordance with Coulomb's law.

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