FLUCTUATIONS, LONG-RANGE ORDER, AND SUPERFLUIDITY

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Long-range order over very large distances in three-dimensional systems, in particular, Bose condensation in the level with momentum $p = 0$, does not exist for all ratios between the linear dimensions of a system. However, this does not lead to a dependence of the thermodynamic functions on these ratios. It is shown that superfluid persistent currents can exist in rings without ODLRO for $T = 0$ and for $T > 0$. In particular, the possibility of such currents is indicated within the framework of the exactly soluble one-dimensional model of Lieb and Liniger.

A superfluid system is an example of a system with long-range order. If the long-range order is preserved over arbitrarily large distances, then nonvanishing quasi-averages exist in the system.[1] The latter is quite frequently regarded as a necessary condition for the existence of such a system as a superfluid liquid, ferromagnetic substances, or crystals. This idea eliminates the possibility of the existence of one- and two-dimensional ordered systems since the long wavelength fluctuations destroy the long-range order over large distances,[2,3] which makes the existence of quasiaverages impossible.[4]

However, the long wavelength fluctuations destroy the long-range order not only in one- and two-dimensional systems, but also in three-dimensional systems having markedly different macroscopic dimensions (see the Appendix), which is in agreement with the previously well-known facts concerning the impossibility of Bose condensation and the vanishing of the quasi-averages in such systems.[5-9] Therefore the number of particles N_0 in the level with momentum $p = 0$ is not an additive thermodynamic function since the density of such particles $n_0 = N_0/V$ (V denotes the volume of the system) depends on the shape of the surface bounding the system. In this connection, a theory with a precipitated condensate and the perturbation theory constructed within its framework lead to nonadditivity of the thermodynamic functions and to anomalous size effects[9] associated with this.

Such effects are not present in the theory of collective variables,[10,11] which does not use the concept of condensation in the level $p = 0$. Here, as is shown in Sec. 1, the value of n_0 is determined by the shape of the system's surface. At the same time the theory of collective variables leads to the very same energy for the ground state and the same quasi-particle spectrum as does perturbation theory for three-dimensional systems, provided off-diagonal long-range order (ODLRO) exists over arbitrarily large distances and $n_0 \neq 0$.

Since the presence or absence of ODLRO does not determine the thermodynamic properties of a superfluid system, the question arises of the necessity of ODLRO for the appearance of superfluid properties. In connection with this, in the present article the necessary conditions for the existence of metastable superfluid currents in rings for $T = 0$ and for $T > 0$ are examined within the framework of the quasi-particle model of a Bose liquid (Sec. 2) and for the case of the exactly soluble one-dimensional model of Lieb and Liniger[12-14] (Sec. 3). It is shown that these conditions do not require the existence of ODLRO and nonvanishing quasi-averages. However, for the existence of persistent currents at all temperatures below the λ-transition point with the rather large critical velocities observed experimentally, it is necessary to introduce an assumption about the small probability for a transition between microstates with different values of the superfluid velocity. Such an assumption is sufficient for the existence of one-dimensional superfluidity, and in a somewhat different form it was already made earlier in connection with arguments in favor of the existence of one-dimensional superfluidity and superconductivity.[15,16] In Sec. 3 arguments are presented in favor of the plausibility of such an assumption in the one-dimensional model of Lieb and Liniger. In this connection, an expression is obtained for the normal mass of the liquid which is analogous to the expression obtained in the quasi-particle model.

1. ODLRO AND THE MOMENTUM DISTRIBUTION OF THE PARTICLES IN A NONIDEAL BOSE GAS

In systems where ODLRO and quasi-averages cannot exist, in particular, in one-dimensional systems, the theory of perturbations using the concept of a condensate in the level $p = 0$ leads to integrals for the density of the particles in excess of the condensate which diverge at small momenta. At the same time, a comparison with the exactly soluble one-dimensional model of Lieb and Liniger[12,13] for a Bose gas with a δ-function interaction shows that perturbation theory correctly gives the first two terms of the expansion for the energy of the ground state and correctly predicts the phonon spectrum of the excitations. This indicates that the incorrectness of perturbation theory in systems where ODLRO does not exist is confined to an incorrect determination of the occupation of the single-particle levels in the region of small p. Since this incorrectness is associated with the separating out of the condensate into the level $p = 0$, the momentum distribution of the particles is of interest in the theory of collective variables which was developed for a Bose gas in the articles by Bogolyubov and Zubarev[10] and by Bohm and Salt,[11] where such a separation is not made. For three-dimensional systems where ODLRO exists, this theory actually leads to a finite density of the condensate.[17] However, as we now show, this does not always occur.

According to the theory of collective variables, in configuration space the wave function of the ground state has the form

$$\Psi=\prod_{i>j} f(\mathbf{r}_i-\mathbf{r}_j), \tag{1}$$

where the indices i and j of the particles with coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$ take the values from 0 to N, and the function $f(\mathbf{r})$ is given by

$$f(\mathbf{r})=\exp\Big[\sum_{\mathbf{p}\neq 0}\varphi(p)\exp(i\mathbf{pr}/\hbar)\Big]. \tag{2}$$

The function $\varphi(p)$ has a $1/p$ singularity as $p\to 0$. If $f(\mathbf{r})\to 1$ for large values of r, then for a weakly nonideal Bose gas one can represent the function Ψ in the following form:

$$\Psi=\sum_{S} f^0(\mathbf{r}_1-\mathbf{r}_2)f^0(\mathbf{r}_3-\mathbf{r}_4)\ldots f^0(\mathbf{r}_{N-1}-\mathbf{r}_N), \tag{3}$$

where

$$f^0(\mathbf{r})=1+\sum_{\mathbf{p}\neq 0}\varphi(\mathbf{p})\exp(i\mathbf{pr}/\hbar), \tag{4}$$

and $\sum_S$ denotes the summation over all permutations of the indices of the particles. In the representation of second quatization

$$\Psi=\sum_{N_0,\ldots,N_\mathbf{p}}\frac{(a_0^+)^{N_0}}{(N_0/2)!}\prod_{\mathbf{p}\neq 0}\frac{(\sqrt{2\varphi(\mathbf{p})}\,a_\mathbf{p}^+a_{-\mathbf{p}}^+)^{N_\mathbf{p}}}{\sqrt{N_\mathbf{p}!}}\,|0\rangle, \tag{5}$$

where the summation goes over all $N_\mathbf{p}$ which satisfy the conditions $N_\mathbf{p}=N_{-\mathbf{p}}$ and $\sum_\mathbf{p} N_\mathbf{p}=N$.

If we assume that the fluctuations N_0 are small, then in (5) one can introduce the constant N_0 for all terms. This considerably simplifies operations on the wave function, which may be written in the following form:

$$\Psi=\prod_{\mathbf{p}\neq 0}\exp[\varphi(\mathbf{p})a_\mathbf{p}^+a_{-\mathbf{p}}^+]\frac{(a_0^+)^{N_0}}{(N_0/2)!}|0\rangle. \tag{6}$$

As a result we have obtained the wave function for the ground state of the Bogolyubov theory. Since the average number of particles $\langle N_\mathbf{p}\rangle$ in the level with momentum p is proportional to $\varphi(\mathbf{p})$, then $\langle N_\mathbf{p}\rangle$ has the same $1/p$ singularity as $\varphi(p)$. It is clear that if the $1/p$ singularity turns out to be nonintegrable (one-dimensional and also two-dimensional and three-dimensional systems with markedly different dimensions), then it is impossible to make the transition from (5) to (6). In this case the ratio N_0/N tends to zero, and evaluation of the average occupation numbers $\langle N_\mathbf{p}\rangle$ for small values of p is considerably complicated. Thus, an increase of one of the dimensions of the system leads to a smearing out of the condensate over a certain range of momenta near $p=0$.[1)] In connection with this, it is advisable to not relate the definition of the Bose condensate to the occupation numbers of one level or of a group of levels in momentum space.

A definition of Bose condensation which is useful for an analysis of superfluid properties in a one-dimensional Bose gas is given in Sec. 3. The concept of a spatially-inhomogeneous condensate has also been used in the literature;[19-21] according to this concept, in the condensed state a very large number of particles, proportional to the volume of the system, is found in one and the same single-particle state, but the wave function of this state (the wave function of the condensate) is not spatially constant.

In order to determine the condensate's wave function, a nonlinear Schrödinger equation was used, similar in form to the phenomenological equations of Ginzburg and Pitaevskiĭ[22] for the order parameter near the λ-point. The idea of a spatially inhomogeneous condensate is, in fact, also the starting point of attempts to construct a theory of superfluidity with the aid of the formalism of coherent states.[16]

It should be noted that the $1/p$ singularity for $T=0$ and the $1/p^2$ singularity for $T>0$ in the momentum distribution, which are characteristic for the theory with a separated-out condensate, can be obtained as a consequence of the fluctuations in the phase $\varphi(\mathbf{r})$ of the condensate's wave function $\psi(\mathbf{r})=Ae^{i\varphi(\mathbf{r})}$.

Everything said above enables us to conclude that perturbation theory with a separated-out condensate correctly determines the energy of the ground state and the excitation spectrum, but it does not adequately reflect the collective nature of the long wavelength excitations.[2)]

2. SUPERFLUIDITY IN THE QUASI-PARTICLE MODEL OF A BOSE LIQUID

Let us consider the problem of the existence of persistent currents for a system of N bosons in a ring of length L, which is large enough so that the properties of bosons in such a ring and in a rectangular box of height L are similar. For this purpose we isolate from the canonical ensemble of microstates of the system a more restricted ensemble of microstates with a given value of the projection P of the total momentum onto the circumference of the ring. If minima exist in the dependence of the free energy $F_0(P)$ of such an ensemble on P, then, as was shown in[8], persistent currents exist provided that the depth of such minima exceeds kT many times over.

Let us determine the free energy $F_0(P)$ in the quasi-particle model of a Bose liquid.[3)] In this case each microstate is determined by the velocity of the superfluid mass v_s and by the numbers of quasi-particles with momentum p and energy $\epsilon(p)$ in the

1)According to Girardeau [18] a generalized Bose condensation takes place in three-dimensional systems of arbitrary shape, i.e., the particles of the condensate have momenta in a region near p = 0 whose size tends to zero in the thermodynamic limit; however the number of levels in this region remains rather large.

2)Such an assertion agrees with the results of the work by Bohm and Salt, [11] where the distinction between collective and one-particle excitations is developed. In the region of small momenta, only the collective excitations permit the supplementary condition which appears in the theory of Bohm and Salt.

3)The stability of the metastable states in the quasi-particle model was considered earlier in [23] for T = 0.

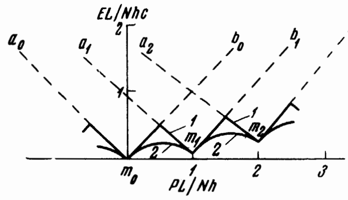

FIG. 1

system, which is moving with velocity v_S. At $T = 0$ the function $F_0(P)$ represents the least energy $E_0(P)$ for a given P.

Microstates without quasi-particles exist only for quantized values of the momentum $P_S = shN/L$ and velocity $v_S = sh/Lm$, where s is an integer. They can be obtained from the ground state by changing the momentum of each particle by s quanta h/L. The corresponding values $E_0(P)$ differ from the energy of the ground state by $P^2/2Nm$. For other values of the momentum, the microstates with minimum energy are realized by the creation of P/p quasi-particles with momentum p, minimizing the quantity $\epsilon(p)/p$. For different values of v_S the lines $a_S m_S b_S$ on Fig. 1 indicate the minimum energy in a weakly nonideal Bose gas, for which $\min[\epsilon(p)/p] = c$, where c denotes the velocity of sound. The energy $E_0(P)$ is indicated by the solid line 1. If the energy $P^2/2Nm$ is subtracted from $E_0(P)$ we obtain a periodic function of P with period hN/L.

The minima on the curve $E_0(P)$, determining the possibility of persistent currents, vanish at velocities $v_S = \min[\epsilon(p)/p]$ corresponding to the critical velocities according to Landau. However, in order for a state corresponding to a certain minimum to actually be long-lived, the probabilities for a transition to states with a smaller energy near a neighboring minimum, which corresponds to another value of v_S, must be small. Such transitions (for example, from the point m_2 to m_1) are associated with a change of the system's momentum by an amount of the order of N/L, and increase without any limit in the thermodynamic limit for two- and three-dimensional systems. Therefore, for such systems the quasi-particle model, satisfying the Landau criterion, leads to superfluidity at $T = 0$ independently of the existence of ODLRO.

For $T > 0$ the free energy $F_0(P)$ is determined by the expression

$$\exp\left(\frac{F_0(P)}{kT}\right)=\sum_s \exp\left(\frac{F(P,v_s)}{kT}\right), \tag{7}$$

where $F(P, v_S)$ denotes the free energy for an ensemble comprising microstates with a given P and with a given value of the superfluid velocity v_S.

Since it is sufficient to know $F_0(P)$ to within kT, on the right hand side of Eq. (7) one can leave only the one term corresponding to the smallest value of $F(P, v_S)$. The quantity $F(P, v_S)$ for $v_S \neq 0$ is related to $F(P, 0)$ by the relation

$$F(P,v_s)=F(P-P_s,0)+\frac{P^2-(P-P_s)^2}{2Nm}. \tag{8}$$

Using the usual methods of statistical physics, we obtain the following result for the ensemble of microstates with $v_S = 0$

$$F(P,0)=kT\sum_{\mathbf{p}\neq 0}\ln\left(1-\exp\frac{\varepsilon(\mathbf{p})-\mathbf{up}}{kT}\right)=F(0,0)+\frac{V\rho_n(u)u^2}{2}G(u), \tag{9}$$

where u denotes the velocity of the normal mass of the liquid in a reference system moving with velocity v_S, related to the momentum P for $v_S = 0$ by the relation

$$\sum_{\mathbf{p}\neq 0}\frac{\mathbf{pu}}{u}\left[\exp\left(\frac{\varepsilon(\mathbf{p})-\mathbf{pu}}{kT}\right)-1\right]^{-1}=V\rho_n(u)u=P, \tag{10}$$

$\rho_n(u)$ is the density, which depends on u, of the normal mass of the liquid.[24] The function G(u) in Eq. (9) varies from 1 as $u \to 0$ to 2 as u approaches the critical Landau velocity $\min[\epsilon(p)/p]$.

The free energy $F(P, v_S)$ has a minimum with respect to P at $u = -v_S$, i.e., where the normal velocity $v_n = v_S + u$ vanishes in the fixed coordinate system.

We present expressions for G(u) and $\rho_n(u)$ in systems of different dimensions for the approximation of the quasi-particle spectrum by the expression $\epsilon = cp$.

One-dimensional system:

$$\rho_n(u)=\frac{\pi(kT)^2}{3c^3\hbar}\left(1-\left(\frac{u}{c}\right)^2\right)^{-2},\quad G(u)=1+\left(\frac{u}{c}\right)^2. \tag{11}$$

Two-dimensional system:

$$\rho_n(u)=\frac{1{,}803}{\pi}\frac{(kT)^3}{c^4\hbar^2}\left(1-\left(\frac{u}{c}\right)^2\right)^{-5/2},$$
$$G(u)=\frac{2}{3}\left[4\left(\frac{u}{c}\right)^2-1+\left(1-\left(\frac{u}{c}\right)^2\right)^{5/2}\right]\left(\frac{c}{u}\right)^2. \tag{12}$$

Three-dimensional system:

$$\rho_n(u)=\frac{2\pi^2(kT)^4}{45c^5\hbar^3}\left(1-\left(\frac{u}{c}\right)^2\right)^{-3},$$
$$G(u)=1+\frac{3}{2}\left(\frac{u}{c}\right)^2-\frac{1}{2}\left(\frac{u}{c}\right)^4. \tag{13}$$

In Fig. 2 the dashed lines indicate the free energy $F(P, v_S)$ for a three-dimensional system for three values of v_S ($s = 0, 1, 2$), and the solid line denotes the free energy $F_0(P)$. The value $F(0, 0) = F_0(0)$ is adopted as the zero reference for the free energy. Let us define the magnitude of the barriers $E_b(s)$ on the curve $F_0(P)$ to be equal to the difference of ordinates for the points t_S and m_S. The values of u on these segments of the curves $F_0(P)$ are inversely proportional to 1/L; therefore, for the definition of $\rho_n(u)$ and G(u) one can take the values $\rho_n(0) = \rho_{n0}$ and $G(0) = 1$. In this approximation we obtain

$$E_6(s)=\frac{h^2N}{4mL^2}\frac{\rho}{\rho_{n0}}\left(\frac{1}{2}-\frac{\rho_{n0}}{\rho}s\right)^2. \tag{14}$$

Barriers exist only for $s < (1/2)\rho/\rho_{n0}$, which corresponds to a critical velocity

$$v_{cr}<\frac{h}{2mL}\frac{\rho}{\rho_{n0}}, \tag{15}$$

which decreases in proportion to 1/L for large rings.

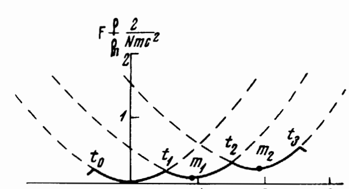

FIG. 2

In order for at least one minimum to exist on the curve $F_0(P)$ for $s = 1$, fulfilment of the condition

$$\rho_{n0} / \rho < 1/2. \tag{16}$$

is necessary. The lifetime of the current states increases without limit in the thermodynamic limit according to Eq. (14), provided $N/L^2 \rightarrow \infty$. This, as follows from Eq. (A.6), is one of the conditions guaranteeing ODLRO around the circumference of the ring. However, since in this connection the second condition (A.6) can be violated, then very large barriers E_b can exist even without ODLRO around the circumference of the ring, but only for two- and three-dimensional systems. In any case, the use of ODLRO as the necessary condition for superfluid currents unconditionally leads to the inequality (15).

From the estimates made above of the stability of the current states, it follows that superfluid currents exist only far away from the λ-transition point (condition (16)), and the critical velocities decrease like $1/L$ (inequality (15)). Neither of these is confirmed experimentally.[25]

An additional factor contributing to the stability of the current states, which was not taken into account above, may be the small probability of a transition between microstates with different values of v_s, which is necessary, as already indicated above, for the existence of one-dimensional superfluidity at $T = 0$. In this case the metastable ensemble of states is characterized by the free energy $F(P, v_s)$ but not by $F_0(P)$. The critical velocities are determined by processes which violate the stability of the ensemble of microstates with a given value of v_s for sufficiently large values of u. In particular, the formation of vortices is among such processes.

Thus, the quasi-particle model of a Bose liquid agrees with the experimentally observed superfluid phenomena only upon making an additional assumption about the small probability for a transition between microstates with different superfluid velocities v_s. This same assumption leads to one-dimensional superfluidity for $T = 0$ and $T > 0$.

3. SUPERFLUIDITY IN THE EXACTLY SOLUBLE MODEL OF LIEB AND LINIGER

The exactly soluble model of Lieb and Liniger makes it possible to verify, at a microscopic level, how valid the conclusions are which follow from the quasi-particle model. According to articles[12-14] each microstate of a system of N bosons with a δ-function interaction

$$V(r_1 - r_2) = \frac{h^2}{m} a\delta(r_1 - r_2) \tag{17}$$

corresponds to a set $\{I\}$ of nonidentical quantum numbers I_i $(i = 1, 2, \ldots, N)$ which are integers for N odd and half-integers for N even. Each set $\{I\}$ determines a set N of nonidentical quasimomenta p_i, so-called because without being the momenta of particles they are related to the energy and total momentum by the same relations

$$P = \sum_i p_i, \quad E = \sum \frac{p_i^2}{2m},$$

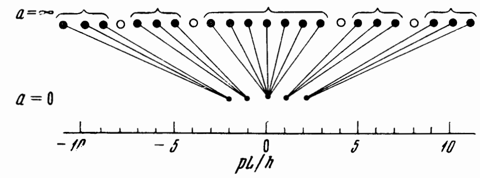

FIG. 3

as the ordinary momenta. To each integer (half-integer) for odd (even) N which is not included in the set $\{I\}$, there corresponds a certain quasimomentum of a hole. Thus, certain effective one-particle states exist with different quasimomenta, where the distribution of the particles in these states obey Fermi-Dirac statistics. However, the spectrum of the quasimomenta of the particles and holes not only depends on the interaction constants, but it also depends on the set $\{I\}$. This dependence vanishes for the strong interaction $a \rightarrow \infty$, when the spectrum of the quasimomenta $p_i = hI_i/L$ coincides with the spectrum of the momenta of noninteracting fermions. In the limit $a \rightarrow \infty$ the bosons become impenetrable, i.e., a collision between them always reverses the direction of their motion.

In Fig. 3 the black circles denote the quasimomenta of the particles, and the open circles denote the quasimomenta of the holes as $a \rightarrow \infty$ for a certain microstate. One can divide the spectrum of quasimomenta into continuous bands, inside of which holes are not present. These bands are indicated in the figure by curly brackets. Upon switching off the interaction for a given set $\{I\}$ the quasimomenta of each such band tend to a certain single-particle momentum of an ideal Bose gas, and the number of quasimomenta in such a band becomes equal to the number of particles with this momentum. Such a deformation of the quasimomentum distribution, associated with the transition from the case $a = \infty$ to the case $a = 0$, is shown in Fig. 3. The distance between the limiting momenta, associated with $a = 0$, for two adjacent bands is equal to the number of holes between the bands multiplied by h/L.

Now let us define the concept of a condensate for interacting bosons. If upon switching off the interaction a certain microstate of the interacting system goes over into a microstate of an ideal Bose gas containing a condensate, i.e., containing a large number N_0 of particles in a level with momentum $p_s = sh/L$, then in the initial microstate of the interacting system N_0 particles are also found in the condensate with a velocity $v_s = p_s/m$. Such a definition of the condensate means that for an arbitrarily strong interaction, in the ground state all of the particles are found in the condensate with a velocity $v_s = 0$, since in this case all of the quasimomenta form a single band which is symmetrically distributed with respect to the point $p = 0$. In analogy with a Fermi gas, one can call this band the Fermi sphere. For nonvanishing total momentum in the state with the least energy, a single hole appears which moves with increasing P from the right boundary of the Fermi sphere to the left boundary (a type II spectrum according to the terminology of Lieb[13]). The energy $E_0(P)$ of such states is indicated in Fig. 1

by the solid line 2 for $a = \infty$. It coincides with the corresponding curve 1 in the quasi-particle model for P close to the quantized values $P_s = shN/L$, when the hole is located close to the Fermi boundary.

For $T > 0$ there exists an ensemble of excited states with a certain number of particles outside of the Fermi sphere and a certain number of holes inside it; however a considerable fraction of the particles remains inside the continuous band without holes, which corresponds to the condensate.

If the velocity of the condensate is associated with the superfluid velocity, then in order to determine the free energy $F(P, v_s)$ for $v_s = 0$ it is necessary to include in the ensemble only microstates with zero velocity of the condensate. One can show that for such microstates the number of particles with quasimomenta $p > 0$ must be equal to the number of particles with quasimomenta $p < 0$, that is

$$\frac{1}{h}\int_0^\infty \rho(p)\,dp = \frac{1}{h}\int_{-\infty}^0 \rho(p)\,dp = \frac{n}{2}, \tag{18}$$

where $Lh^{-1}\rho(p)dp$ is the number of particles with quasimomenta in the interval from p to $p + dp$.

In order to determine $F(P, 0)$ we shall seek the most probable distribution $\rho(p)$ of the particles with respect to the quasimomentum, repeating the procedure used in[14] in order to determine the free energy of an ordinary canonical ensemble. However, in finding the extremal distribution, the single supplementary condition of[14]

$$\frac{1}{h}\int_{-\infty}^\infty \rho(p)\,dp = n$$

which guarantees a given average value for the number of particles, is now replaced by three additional conditions, viz., the two conditions (18) and the condition which guarantees a given average value for the momentum P:

$$\frac{L}{h}\int_{-\infty}^\infty p\rho(p)\,dp = P. \tag{19}$$

Consequently three Lagrangian multipliers appear, determined by the conditions (18) and (19): the two chemical potentials μ_+ and μ_- for particles with $p > 0$ and $p < 0$, and the velocity u. As a result we obtain the following equations:

$$\rho(p)\left(1 + \exp\left(\frac{\varepsilon(p)}{kT}\right)\right) = 1 + \frac{\hbar a}{\pi}\int_{-\infty}^{+\infty} \frac{\rho(q)\,dq}{(\hbar a)^2 + (p-q)^2}, \tag{20}$$

$$\varepsilon(p) = -\mu(p) + \frac{p^2}{2m} - up - \frac{kT\hbar a}{\pi}\int_{-\infty}^{\infty} \frac{\ln[1 + \exp(-\varepsilon(q)/kT)]\,dq}{(\hbar a)^2 + (p-q)^2} \tag{21}$$

where

$$\mu(p) = \begin{cases} \mu_+, & p > 0 \\ \mu_-, & p < 0 \end{cases},$$

$$F(P,0) = \frac{\mu_+ + \mu_-}{2}N - L\frac{kT}{h}\int_{-\infty}^{\infty} \ln\left[1 + \exp\left(-\frac{\varepsilon(p)}{kT}\right)\right]dp + Pu. \tag{22}$$

For small values of u expression (22) coincides in form with the analogous expression (9) in the quasi-particle model:

$$F(P,0) = F(0,0) + L\rho_{n0}u^2/2, \tag{23}$$

where $\rho_{n0} = dP/du|_{u=0}$ denotes the density of the normal mass for $u = 0$.

Equations (20)–(22) are most easily solved in the limit of a strong interaction.[14] Correct to terms of second order in n/a, for $kT \ll \mu_\pm$ we obtain

$$\mu_\pm = \frac{h^2n^2}{8m}\left(1 - \frac{16}{3}\frac{n}{a}\right) \mp u\frac{hn}{2}\left(1 - 2\frac{n}{a}\right), \tag{24}$$

$$\rho_{n0} = \left(1 + 2\frac{n}{a}\right)\int_{-\infty}^{\infty} \frac{\exp(\varepsilon(p)/kT)\,d\varepsilon(p)/du}{(1 + \exp(\varepsilon(p)/kT))^2}\,dp = \left(1 + 2\frac{n}{a}\right)\frac{\pi(kT)^3}{3c^3\hbar}, \tag{25}$$

where $c = (hn/2m)(1 - 2n/a)$ is the velocity of sound.

For $n/a = 0$ the obtained expression agrees with ρ_{n0} in the quasi-particle model, in spite of the fact that the elementary excitations obey Fermi-Dirac statistics in the first case and Bose-Einstein statistics in the second case. The decrease by a factor of two of the normal mass for $n/a = 0$ due to statistics is compensated by the same increase of the normal mass due to the contribution of the holes.

It is considerably more difficult to obtain a solution of Eqs. (20)–(22) in the case of small values of n/a, i.e., for a weakly nonideal Bose gas. We shall confine our attention to qualitative considerations, enabling us to assume that in this case the normal density will have the same value as in the quasi-particle model. According to[13], each particle in excess of the condensate, i.e., each particle lying outside the broad continuous band of quasimomenta corresponding to the condensate (see Fig. 3), can be associated with an elementary excitation having an energy $\epsilon = cp$. The energy of such excitations accumulates additively if their number is small in comparison with N.

Upon switching off the interaction the quasimomenta of the continuous bands contract to a certain limiting value p; therefore one can equate the number of particles in excess of the condensate in each such band to the number of quasi-particles with momentum p in the quasi-particle model. A microstate is described well by the quasi-particle model if the number of quasi-particles is small in comparison with N. However, for small momenta the equilibrium density of quasi-particles

$$n(p) = \left(\exp\left(\frac{cp}{kT}\right) - 1\right)^{-1} \approx \frac{kT}{cp}$$

increases without any limit whereas the corresponding density of particles $\rho(p)$ in quasimomentum space does not, according to Eq. (20), exceed n/a. Therefore, the density of quasi-particles $n(p)$ in momentum space does not agree with the density of particles $\rho(p)$ in quasimomentum space for $n(p) > n/a$, i.e., for

$$p < \frac{kTa}{cn} = \frac{mkT}{h}\sqrt{\frac{a}{n^3}}. \tag{26}$$

In the region $kT \lesssim nah^2/m$, where superfluidity and Bose condensation occur in the sense defined above, this region of small momenta does not affect the value of the density of the normal mass.

Thus, one-dimensional superfluidity exists if the probability of a transition between microstates of the system having different values of v_s is small. Such transitions may occur due to the interaction with the surrounding medium (thermostat). It is natural to assume that the interaction energy can be represented by a sum of single-particle potentials

$$H_{\text{int}} = \sum_{i=1}^{N} V(r_i)$$

where $V(r_i)$ depends on the coordinates r_i of the bosons of the system and on variables which characterize the state of the thermostat.

For such an interaction the probability of a transition per unit time between states with momentum and energy P_1, E_1 and P_2, E_2 is equal to $W(P_1E_1; P_2E_2)h|M|^2/L$, where M is the matrix element of the operator

$$\sum_{i=1}^{N} \exp\left(\frac{ipr_i}{\hbar}\right), \quad p = P_2 - P_1,$$

and $W(P_1E_1; P_2E_2)$ is determined by the properties of the thermostat. For the one-particle problem the quantity $\int W(P_1E_1; P_2E_2)dP_2$ determines the reciprocal lifetime in the state with momentum P_1 and energy $E_1(P_1)$.

In a gas of impenetrable bosons ($a = \infty$), just as in an ideal Fermi gas, nonvanishing values $|M| = 1$ exist for transitions between microstates which are obtained one from the other by a change of the quasimomentum (momentum) of only one particle. In particular, $|M| = 1$ for a transition between two neighboring minima on the curve $E_0(P)$ since during such a transition one particle goes from one Fermi boundary to another. In order to determine the lifetime of the metastable state corresponding to a certain minimum of the curve $E_0(P)$ for $T = 0$, it is necessary to take into consideration transitions into all states with smaller energies near the neighboring minimum. The number of such states for the minimum corresponding to the velocity v_s is equal to s to within a coefficient of the order of unity; therefore the reciprocal lifetime of such a state for impenetrable bosons is equal to $\sim W(P_sE_0(P_s); P_{s-1}E_0(P_{s-1}))mv_s$ and becomes small only for very small v_s.

However, in connection with a decrease of a one can expect a rapid decrease of $|M|$ for transitions between states with different velocities of the condensate, since according to perturbation theory for small a such transitions have $|M| \sim a^{N_0-1}$ where N_0 is the number of particles in the condensate. For transitions between minima $|M| \sim a^{N-1}$ and one can anticipate that $|M|$ does not exceed the value

$$\bar{M} = \gamma^{N-2}, \tag{27}$$

where $\gamma = a/(a+n)$ varies from zero to unity. For these transitions the exact values of M were calculated for N = 2 and N = 3. The calculations showed

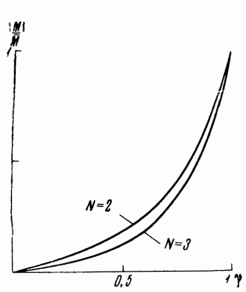

FIG. 4

that $(|M|/\bar{M})_3 < (|M|/\bar{M})_2 < 1$ where $(|M|/\bar{M})_2$ and $(|M|/\bar{M})_3$ are the values of $|M|/\bar{M}$ for N = 2 and N = 3 shown in Fig. 4. If the inequality $|M|/\bar{M} < 1$ remains valid for arbitrary N, then the probability of a transition decreases exponentially as $N \to \infty$ and long-lived current states for T = 0 exist for finite values of a right up to the velocity where the barriers on the curve $E_0(P)$ vanish, i.e., up to the velocity of sound.

APPENDIX

FLUCTUATIONS IN ORDERED SYSTEMS

Let us show that the long wavelength fluctuations of the parameter of degeneracy in ordered systems destroy the long-range order over very large distances in three-dimensional systems with strongly differing, but macroscopic in all directions, dimensions. The degeneracy parameter for a crystal is the displacement of the lattice sites, for a ferromagnetic substance it is the rotation of the magnetic moment, and for a superfluid system it is the phase of the wave function.

Let us divide the volume of the system up into cells with the radius vector to the center of the cell denoted by **R** and the value of the degeneracy parameter denoted by $\varphi(\mathbf{R})$. We take the dimensions of a cell to be macroscopic, however small in comparison with the wavelengths of the fluctuations so that $\varphi(\mathbf{R})$ can be regarded as continuous. The free energy of the entire sample only depends on the difference of $\varphi(\mathbf{R})$ in neighboring cells, i.e., on $\nabla\varphi(\mathbf{R})$. Therefore, the formulation of the problem is completely equivalent to the formulation of the problem about fluctuations of the density at the critical point, and one can use the results of Klein and Tisza[26] and write down expressions for the quantities characterizing the fluctuations of $\varphi(\mathbf{R})$ with respect to its average value, which we take equal to zero, $\langle\varphi(\mathbf{R})\rangle = 0$, and also expressions for the correlations between the fluctuations at different points of the system:

$$\langle\varphi(\mathbf{R}_1)\varphi(\mathbf{R}_2)\rangle = \frac{T}{AV}\sum_{\mathbf{k}}{}' \frac{\cos(\mathbf{k}(\mathbf{R}_1-\mathbf{R}_2))}{k^2}, \tag{A.1}$$

$$\langle\varphi(\mathbf{R})^2\rangle = \frac{T}{AV}\sum_{\mathbf{k}}{}' \frac{1}{k^2}, \tag{A.2}$$

$$\langle(\varphi(\mathbf{R}_1)-\varphi(\mathbf{R}_2))^2\rangle = 2(\langle\varphi(\mathbf{R})^2\rangle - \langle\varphi(\mathbf{R}_1)\varphi(\mathbf{R}_2)\rangle) = \frac{2T}{AV}\sum_{\mathbf{k}}{}' \frac{1-\cos(\mathbf{k}(\mathbf{R}_1-\mathbf{R}_2))}{k^2}, \tag{A.3}$$

where A is a constant which depends on the form of the ordered system and T denotes the temperature. The system is chosen in the form of a parallelepiped with edges $L_x > L_y > L_z$ and volume $V = L_xL_yL_z$. The components of the vector **k** take values which are multiples of $2\pi/L_x$, $2\pi/L_y$, and $2\pi/L_z$, and the symbol $\langle\ \rangle$ denotes averaging over the space of the functions $\varphi(\mathbf{R})$. The values k = 0 and k greater than a certain value q do not appear in the summation over **k**. In order to evaluate these sums, we shall use the method employed in[8-9], in which we divide the summations over k in Eqs. (A.1)–(A.3) into three terms: a one-dimensional sum over $k_x \neq 0$ for $k_y = k_z = 0$, a two-dimensional sum over $k_y \neq 0$ and k_x for $k_z = 0$, and a three-dimensional sum over k_x, k_y, and $k_z \neq 0$. As a result of calculation of these summations in the limit $L_x \to \infty$, $L_y \to \infty$, $L_z \to \infty$ we obtain the following result for large $R_1 - R_2$

$$\langle\varphi(\mathbf{R})^2\rangle = \frac{T}{2A}\left(\frac{\pi}{3}\frac{L_x}{L_yL_z} + \frac{\ln(qL_y)}{\pi L_z} + \frac{q}{\pi^2}\right), \qquad (A.4)$$

$$\langle(\varphi(\mathbf{R}_1) - \varphi(\mathbf{R}_2))^2\rangle = \frac{T}{A}\left(\frac{|x|}{L_yL_z} + \frac{\ln(q\rho)}{\pi L_z} + \frac{q}{\pi^2}\right), \qquad (A.5)$$

where $x = R_{1x} - R_{2x}$ and $\rho^2 = x^2 + (R_{1y} - R_{2y})^2$

If the edges L_x, L_y, and L_z of the parallelepiped are of the same order and $L_x \sim L_y \sim L_z \to \infty$, then $\langle \varphi(R)^2 \rangle$ and $\langle (\varphi(R_1) - \varphi(R_2))^2$ remain finite for any arbitrary points inside the sample. However, the unlimited growth of the quantity L_x/qL_yL_z or $\ln(qL_y)/qL_z$ leads to an unlimited growth of $\langle \varphi(R)^2 \rangle$. The quantity $\langle(\varphi(R_1) - \varphi(R_2))^2\rangle$, characterizing the correlation between $\varphi(R_1)$ and $\varphi(R_2)$, remains the same as in a cubic sample only for $|x| < L_yL_zq$ and $\rho < \exp(qL_z)/q$. For larger values of x and ρ the quantity $\langle(\varphi(R_1) - \varphi(R_2))^2\rangle$ increases without limit, which indicates a disappearance of the correlation. The off-diagonal long-range order (ODLRO) in superfluid systems also vanishes for such large values of x and ρ. Thus, very large fluctuations of the degeneracy parameter associated with large values of L_x/qL_yL_z or $\ln(qL_y)/qL_x$ lead to the "removal" of the degeneracy and the vanishing of the quasi-averages.[27] However it is clear that in a crystalline solid with dimensions satisfying just one of the cited inequalities, none of the measurements of the thermodynamic functions observe any difference between the properties of such a solid and an ordinary crystal of cubic shape. Large displacements do not lead to the destruction of the binding between atoms since the average deformations are the same as in a cubic sample.[4)]

From formula (A.5) it also follows that the long-range order, which vanishes along the very longest dimension, may be preserved along the remaining directions.[29] It is not difficult to verify that ODLRO is preserved with respect to a certain direction along all corresponding lengths L, i.e., $\langle(\varphi(R_1) - \varphi(R_2))^2\rangle$ remains finite provided the conditions

$$\frac{N}{L^2} \to \infty, \quad \frac{N}{LL'\ln(qL)} \to \infty, \qquad (A.6)$$

are satisfied in the thermodynamic limit ($N \to \infty$, $N/V = \text{const}$), where L' denotes either of the two remaining dimensions of the system.

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4) Arguments against the use of conservation of long-range order over arbitrarily large distances as a necessary condition for the existence of a crystal were given at one time by Frenkel'. [28] The author thanks A. I. Ansel'm for calling his attention to these statements by Ya. I. Frenkel',

[1] N. N. Bogolyubov, Quasi-averages in Problems of Statistical Mechanics, JINR Preprint, Dubna, 1963.
[2] L. D. Landau, Zh. Eksp. Teor. Fiz. 7, 627 (1937).
[3] Richard A. Ferrell, Phys. Rev. Letters 13, 330 (1964).
[4] P. C. Hohenberg, Phys. Rev. 158, 383 (1967).
[5] J. M. Ziman, Phil. Mag. 44, 548 (1953).
[6] David A. Krueger, Phys. Rev. Letters 19, 563 (1967).
[7] David A. Krueger, Phys. Rev. 172, 211 (1968).
[8] É. B. Sonin, Zh. Eksp. Teor. Fiz. 56, 963 (1969) [Sov. Phys.-JETP 29, 520 (1969)].
[9] É. B. Sonin, Zh. Eksp. Teor. Fiz. 57, 1411 (1969) [Sov. Phys.-JETP 30, 765 (1970)].
[10] N. N. Bogolyubov and D. N. Zubarev, Zh. Eksp. Teor. Fiz. 28, 129 (1955) [Sov. Phys.-JETP 1, 83 (1955)].
[11] D. Bohm and B. Salt, Rev. Mod. Phys. 39, 894 (1967).
[12] Elliott H. Lieb and Werner Liniger, Phys. Rev. 130, 1605 (1963).
[13] Elliott H. Lieb, Phys. Rev. 130, 1616 (1963).
[14] C. N. Yang and C. P. Yang, J. Math. Phys. 10, 1115 (1969).
[15] William A. Little, Phys. Rev. 156, 396 (1967).
[16] J. S. Langer, Phys. Rev. 167, 183 (1968).
[17] D. N. Zubarev, Zh. Eksp. Teor. Fiz. 29, 881 (1955) [Sov. Phys.-JETP 2, 745 (1956)].
[18] M. D. Girardeau, J. Math. Phys. 6, 1083 (1965).
[19] L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. 40, 646 (1961) [Sov. Phys.-JETP 13, 451 (1961)].
[20] Tai Tsun Wu, J. Math. Phys. 2, 105 (1961).
[21] Eugene P. Gross, J. Math. Phys. 4, 195 (1963).
[22] V. L. Ginzburg and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. 34, 1240 (1958) [Sov. Phys.-JETP 7, 858 (1958)].
[23] Warren D. Grobman and Marshall Luban, Phys. Rev. 147, 166 (1966).
[24] I. M. Lifshitz and M. I. Kaganov, Uchenye zapiski XGU (Science Notes, Khar'kov Univ.), 49; Tr. fiz. otd. fizmat. fak-ta (Proc. Phys. Div. of Phys.-Math. Dept.) 4, 23 (1953).
[25] William E. Keller, Helium-3 and Helium-4, Plenum Press, 1969.
[26] M. J. Klein and L. Tisza, Phys. Rev. 76, 1861 (1949).
[27] David Jasnow and Michael E. Fisher, Phys. Rev. Letters 23, 286 (1969).
[28] Ya. I. Frenkel', Kineticheskaya teoriya zhidkosteĭ (Kinetic Theory of Fluids), Izd. Akad. Nauk SSSR, 1945.
[29] Gideon Carmi, J. Math. Phys. 9, 174 (1968).

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162