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FLUCTUATIONS IN NONEQUILIBRIUM STATIONARY STATE

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A general expression is obtained for the structural density of the fluctuations of the distribution function of a system of particles in a stationary non-equilibrium state described by a kinetic equation. The method developed by the authors, based on solving the equation directly for the correlator of the distribution-function fluctuations, is compared with the Langevin and with the "entropy" approaches in the theory of fluctuations in non-equilibrium systems. For an ideal Fermi gas under stationary external conditions that cause noticeable deviations from equilibrium, a functional is constructed, which yields the distribution of the probabilities of different macroscopic states of such a system. The functional reaches a maximum value in the stationary state determined from the solution of the corresponding kinetic equation.

1. INTRODUCTION

WE have recently derived, using a diagram technique, equations for the kinetics of fluctuations of a system of particles in a stationary non-equilibrium state.[1,2] For concreteness, we have considered in [1,2] the electrons in a semiconductor acted upon by a strong external electric field E, which causes the non-equilibrium states. However, the methods developed are sufficiently general and applicable to any system of particles, for example for an ion-electron plasma or to a gas of neutral molecules, provided the non-equilibrium state is described by a kinetic equation.

We show in the present paper that by using the equations derived in [1,2] it is possible to represent the special density of the fluctuations of the distribution function in simple form. The corresponding formula recalls the expression obtained earlier by Kadomtsev[3] in an analysis of fluctuations in a gas, although it differs from it under conditions when the system is sufficiently far from equilibrium. This difference will be discussed by us below.

In the theory of fluctuations, other methods are also used. The equivalence of the approach developed by us to the analysis of fluctuations in the nonequilibrium state[1] and to the approach proposed by Kogan and Shul'man[4] of quantities based on the introduction of random forces into the kinetic equation) was established recently by Kogan and Shul'man.[4a] 1).

We have succeeded in ascertaining the extent to which our results can be obtained by the "entropy" method. We have established that this method, developed for the calculation of fluctuations in the state of thermodynamic equilibrium (see, for example, the book of Landau and Lifshitz[5]), can be generalized to include non-equilibrium stationary states in the case when the electron-electron collisions are negligible and the stationary distribution $\overline{F}_p$ is determined only by the action of the electric field and the interaction of the electrons with the thermostat (for example in a semiconductor—with the phonons and impurities). It turns out that the probability of a given distribution F_p that deviates from the mean distribution $\overline{F}_p$ (stationary but not at all in equilibrium!) is expressed in terms of a simple functional of F_p and $\overline{F}_p$ (see (27)), just as in the equilibrium problem the probability of a certain distribution is determined by the entropy for a closed system or by the corresponding thermodynamic potential for an open system. In the more general case, when an important role is assumed by the additional correlation between the electron states, a correlation connected with the pair collisions, it is necessary, in order to determine such a probability, to know not only $\overline{F}_p$ but also the higher many-particle distribution functions.

2. THE CORRELATOR OF THE DISTRIBUTION-FUNCTION FLUCTUATIONS

Just as before,[1,2] we shall carry out the analysis using as a concrete example a system of nondegenerate electrons in a semiconductor, the electrons interact with one another, with the impurity ions, and also with a certain equilibrium external system, henceforth called thermostat for brevity. In a semiconductor, the role of the thermostat is usually played by the phonons. The number of electrons in the conduction band, N, is assumed specified.

The average electron distribution over the (quasi) momenta $\overline{F}_p$ is assumed to be stationary (but not at all in equilibrium). It is determined from the kinetic equation

$$\left(e\mathbf{E}\frac{\partial}{\partial \mathbf{p}} + I_{\mathbf{p}}\right)\overline{F}_{\mathbf{p}} + I_{\mathbf{p}}^{ee}\{\overline{F}, \overline{F}\} = 0. \tag{1}$$

Here I_p is the linear operator for the collisions of the electrons with the thermostat (and with the impurities), $I_p^{ee}\{\overline{F}, \overline{F}\}$ is the bilinear interelectron collision term. The function F_p is normalized by the condition $\sum_p \overline{F}_p = N$ (N is the number of particles). For simplicity, we disregard the spin of the particles.

The kinetic equation (1) determines the mean distribution of the electrons $\overline{F}_p$. The fluctuations about this mean value are described by the mean-square deviations from it $\overline{\delta F_p(t)\,\delta F_{p_1}(t_1)}$ (the bar denotes averaging).2) In

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1) We are grateful to Kogan and Shul'man for their preprint.

2) A rigorous quantum-statistical definition of these mean values is given in [1,2].

the stationary state, this quantity depends only on the difference $\tau = t - t_1$. As shown in [1,2], when $\tau > 0$ this quantity satisfies the equation

$$\left(\frac{\partial}{\partial\tau} + \mathcal{I}_{\mathbf{p}}\right)\overline{\delta F_{\mathbf{p}}(\tau)\,\delta F_{\mathbf{p}_1}} = 0. \tag{2}$$

Here $\mathcal{I}_{\mathrm{p}}$ is the linearized operator of the kinetic equation (1):

$$\mathcal{I}_{\mathbf{p}} \equiv e\mathbf{E}\frac{\partial}{\partial\mathbf{p}} + I_{\mathbf{p}} + I_{\mathbf{p}}^{ee}\{\overline{F}\}; \tag{3}$$

$I_{\mathrm{p}}^{ee}\{\overline{F}\}$ is a linear operation acting on the momentum function ψ_{p} in the following manner:

$$I_{\mathbf{p}}^{ee}\{\overline{F}\}\psi_{\mathbf{p}} \equiv I_{\mathbf{p}}^{ee}\{\overline{F}, \psi\} + I_{\mathbf{p}}^{ee}\{\psi, \overline{F}\}. \tag{4}$$

By definition

$$I_{\mathbf{p}}^{ee}\{\overline{F}, \psi\} \equiv \sum_{\mathbf{k}\mathbf{k}_1\mathbf{p}_1}\left(W_{\mathbf{k}\mathbf{k}_1}^{\mathbf{p}\mathbf{p}_1}\overline{F}_{\mathbf{p}}\psi_{\mathbf{p}_1} - W_{\mathbf{p}\mathbf{p}_1}^{\mathbf{k}\mathbf{k}_1}\overline{F}_{\mathbf{k}}\psi_{\mathbf{k}_1}\right), \tag{4a}$$

where $W_{\mathrm{pp}_1}^{\mathrm{kk}_1}$ is the probability of the transition of two electrons from the states $\mathbf{k}$ and $\mathbf{k}_1$ into the states p and p_1 as a result of their collision.

The initial condition in (2) is the equal-time correlator $\overline{\delta F_{\mathrm{p}}\delta F_{\mathrm{p}_1}}$. This quantity has a clear physical meaning: in its terms is expressed the two-particle equal-time electron distribution function, equal to

$$\overline{F_{\mathbf{p}}F_{\mathbf{p}_1}} = \overline{F}_{\mathbf{p}}\overline{F}_{\mathbf{p}_1} + \overline{\delta F_{\mathbf{p}}\delta F_{\mathbf{p}_1}}. \tag{5}$$

The principal term in this expression is the first term; the second is of relative order 1/N. To describe the fluctuations, however, we need just the second term. From the quantity $\overline{\delta F_{\mathrm{p}}\delta F_{\mathrm{p}_1}}$ we can separate, in turn, the "diagonal part":

$$\overline{\delta F_{\mathbf{p}}\delta F_{\mathbf{p}_1}} = \overline{F}_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}_1} + \tilde{\varphi}_{\mathbf{p}\mathbf{p}_1} \tag{6}$$

(so that $\tilde{\varphi}_{\mathrm{pp}_1} \equiv \varphi_{\mathrm{pp}_1} - N(\partial F_{\mathrm{p}}/\partial N)(\partial F_{\mathrm{p}_1}/\partial N)$, where φ is a function introduced in [1]).

According to [1,2], the function $\tilde{\varphi}_{\mathrm{pp}_1}$ satisfies the equation

$$(\mathcal{I}_{\mathbf{p}} + \mathcal{I}_{\mathbf{p}_1})\tilde{\varphi}_{\mathbf{p}\mathbf{p}_1} = -I_{\mathbf{p}\mathbf{p}_1}^{ee}\{\overline{E}, \overline{F}\} \tag{7}$$

and the normalization condition

$$\sum_{\mathbf{p}_1}\tilde{\varphi}_{\mathbf{p}\mathbf{p}_1} = -\overline{F}_{\mathbf{p}}. \tag{8}$$

The right-hand side of (7) differs from the usual inter-electron collision term in the absence of one summation:

$$I_{\mathbf{p}\mathbf{p}_1}^{ee}\{\overline{F}, \overline{F}\} = \sum_{\mathbf{k}\mathbf{k}_1}\left(W_{\mathbf{k}\mathbf{k}_1}^{\mathbf{p}\mathbf{p}_1}\overline{F}_{\mathbf{p}}\overline{F}_{\mathbf{p}_1} - W_{\mathbf{p}\mathbf{p}_1}^{\mathbf{k}\mathbf{k}_1}\overline{F}_{\mathbf{k}}\overline{F}_{\mathbf{k}_1}\right), \tag{9}$$

i.e.,

$$I_{\mathbf{p}}^{ee}\{\overline{F}, \overline{F}\} = \sum_{\mathbf{p}_1} I_{\mathbf{p}\mathbf{p}_1}^{ee}\{\overline{F}, \overline{F}\}. \tag{9a}$$

The spectral density of the fluctuation correlator is

$$(\delta F_{\mathbf{p}}\delta F_{\mathbf{p}_1})_\omega = \int_{-\infty}^{\infty} e^{i\omega\tau}\overline{\delta F_{\mathbf{p}}(\tau)\,\delta F_{\mathbf{p}_1}}\,d\tau \tag{10}$$

and consequently

$$(\delta F_{\mathbf{p}}\delta F_{\mathbf{p}_1})_\omega = \int_0^{\infty} e^{i\omega\tau}\overline{\delta F_{\mathbf{p}}(\tau)\,\delta F_{\mathbf{p}_1}}\,d\tau + \int_0^{\infty} e^{-i\omega\tau}\,\overline{\delta F_{\mathbf{p}_1}(\tau)\,\delta F_{\mathbf{p}}}\,d\tau. \tag{11}$$

However, taking (2) into account, we have

$$\int_0^{\infty} e^{i\omega\tau}\overline{\delta F_{\mathbf{p}}(\tau)\,\delta F_{\mathbf{p}_1}}\,d\tau = \frac{1}{-i\omega + \mathcal{I}_{\mathbf{p}}}\left(\overline{F}_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}_1} + \tilde{\varphi}_{\mathbf{p}\mathbf{p}_1}\right). \tag{12}$$

Hence, using the identity

$$\frac{1}{-i\omega + \mathcal{I}_{\mathbf{p}}} + \frac{1}{i\omega + \mathcal{I}_{\mathbf{p}_1}} = \frac{1}{(-i\omega + \mathcal{I}_{\mathbf{p}})(i\omega + \mathcal{I}_{\mathbf{p}_1})}(\mathcal{I}_{\mathbf{p}} + \mathcal{I}_{\mathbf{p}_1}) \tag{13}$$

and Eq. (7), we obtain ultimately

$$(\delta F_{\mathbf{p}}\delta F_{\mathbf{p}_1})_\omega = \frac{1}{(-i\omega + \mathcal{I}_{\mathbf{p}})(i\omega + \mathcal{I}_{\mathbf{p}_1})}\left[(\mathcal{I}_{\mathbf{p}} + \mathcal{I}_{\mathbf{p}_1})\overline{F}_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}_1} - I_{\mathbf{p}\mathbf{p}_1}^{ee}\{\overline{F}, \overline{F}\}\right]. \tag{14}$$

A formula similar to (14) can be obtained also for spatially inhomogeneous fluctuations. Using Eqs. (3.1) and (3.2) of [2], we obtain

$$(\delta F_{\mathbf{p}}\delta F_{\mathbf{p}_1})_{\omega\mathbf{q}} = \frac{1}{(-i\omega + \hat{B}_{\mathbf{p}}(\mathbf{q}))(i\omega + \hat{B}_{\mathbf{p}_1}^{*}(\mathbf{q}))}\left[(\mathcal{I}_{\mathbf{p}} + \mathcal{I}_{\mathbf{p}_1})F_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}_1} - I_{\mathbf{p}\mathbf{p}_1}^{ee}\{\overline{F}, \overline{F}\}\right]. \tag{15}$$

Here the index $\mathbf{q}$ denotes that one takes the Fourier component with respect to the difference of the spatial coordinates of two points, in which the fluctuations are considered; $-i\omega + \hat{B}_{\mathrm{p}}(\mathbf{q})$ is the operator of the response of a system to an external perturbation that varies like $\exp(-i\omega t + i\mathbf{q}\cdot\mathbf{r})$:

$$\hat{B}_{\mathbf{p}}(\mathbf{q})\psi_{\mathbf{p}} \equiv (i\mathbf{q}\mathbf{v} + \mathcal{I}_{\mathbf{p}})\psi_{\mathbf{p}} - \frac{4\pi e^2}{q^2\mathcal{V}}i\mathbf{q}\frac{\partial\overline{F}}{\partial\mathbf{p}}\sum_{\mathbf{p}'}\psi_{\mathbf{p}'} \tag{16}$$

(ψ_{p} is an arbitrary function). The last term in (16) describes the action of the self-consistent field of electrons that have become redistributed in space under the influence of the external perturbation ($\mathcal{V}$ is the volume of the system).

Formulas (14) and (15) have the same structure as formula (7) of Kadomtsev's paper,[3] but differ from the latter in the presence of the term $I_{\mathrm{pp}_1}^{ee}\{\overline{F}, \overline{F}\}$. This term describes the simultaneous departure of a selected pair of particles from the states p and p_1 (or their arrival at these states) as a result of the mutual collision.3) Connected with these processes is an additional correlation between the occupation numbers of the states p and p_1. This correlation becomes manifest only in the non-equilibrium state, for in the equilibrium state we have $I_{\mathrm{pp}_1}^{ee}\{\overline{F}, \overline{F}\} = 0$, i.e., the arrival and departure terms of (9) cancel each other. On the other hand, in a state far from equilibrium, the contribution from $I_{\mathrm{pp}_1}^{ee}\{\overline{F}, \overline{F}\}$ to the right side of (14) and (15) is of the same order as the contribution from $(\mathcal{I}_{\mathrm{p}} + \mathcal{I}_{\mathrm{p}_1})F_{\mathrm{p}}\delta_{\mathrm{pp}_1}$.

The difference between (15) and formula (7) of [3] (the absence of the term $I_{\mathrm{pp}_1}^{ee}$ from the latter) is due to an error made in [3] in the calculation of the correlation function for an extraneous source. This is seen from the fact that the right side of (7) in[3] does not vanish upon integration with respect to $\mathbf{v}$ (or $\mathbf{v}_0$), as is the case for the left side of the same formula. A discussion with Kadomtsev has revealed that a consistent application of the method of [3] lead to the same result as was obtained by us, i.e., to formula (15).

Formulas (14) and (15) exhaust, in principle, the theory of fluctuations in the non-equilibrium stationary state. They can be used to derive expressions for the fluctuations of different physical quantities. Thus, for

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3)Since this term differs from the usual operator of the electron-electron collisions in the absence of one summation with respect to the quasimomentum, it automatically contains a small factor of the type 1/N. For this reason, it contains the main part of the two-particle distribution function (5), equal to $\overline{FF}$, and does not contain the second term, which itself is of the order of 1/N.

example, from (15) we obtain for the fluctuations of the electron concentration

$$(\delta n^2)_{\omega\mathbf{q}} = \frac{2}{|\varepsilon(\omega,\mathbf{q})|^2}\,\mathrm{Re}\left[\sum_{\mathbf{p}}\frac{1}{-i\omega+i\mathbf{q}\mathbf{v}+\mathcal{J}_{\mathbf{p}}}F_{\mathbf{p}} - \sum_{\mathbf{p}\mathbf{p}_1}\frac{1}{(-i\omega+i\mathbf{q}\mathbf{v}+\mathcal{J}_{\mathbf{p}})(i\omega-i\mathbf{q}\mathbf{v}_1+\mathcal{J}_{\mathbf{p}_1})}I^{ee}_{\mathbf{p}\mathbf{p}_1}\{\bar{F},\bar{F}\}\right], \tag{17}$$

where

$$\varepsilon(\omega,\mathbf{q}) = 1 - \frac{4\pi e^2}{q^2\mathcal{V}}\sum_{\mathbf{p}}\frac{i\mathbf{q}}{-i\omega+i\mathbf{q}\mathbf{v}+\mathcal{J}_{\mathbf{p}}}\frac{\partial\bar{F}}{\partial\mathbf{p}}. \tag{18}$$

3. THE LANGEVIN METHOD

The structure of formulas (14) and (15) shows that the fluctuations in a non-equilibrium stationary state can be considered also by the method. To this end it is necessary to add the random force y_p to the linearized kinetic equation for the fluctuating part of the distribution function, so that, for example, for spatially-homogeneous fluctuations this equation takes the form

$$\left(\frac{\partial}{\partial t}+\mathcal{J}_{\mathbf{p}}\right)\delta F_{\mathbf{p}} = y_{\mathbf{p}}. \tag{19}$$

The random forces, on the other hand, can be subjected to the following correlation relation:

$$(y_{\mathbf{p}}y_{\mathbf{p}_1})_\omega = (\mathcal{J}_{\mathbf{p}}+\mathcal{J}_{\mathbf{p}_1})\bar{F}_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}_1} - I^{ee}_{\mathbf{p}\mathbf{p}_1}\{\bar{F},\bar{F}\}. \tag{20}$$

The same relation should be stipulated also for the correlator $(y_p y_{p_1})_{\omega q}$ in the case of spatially-inhomogeneous fluctuations (see formula (15)).4)

Let us stop briefly to discuss the important case of low-frequency long-wave oscillations, i.e., the case of small ω and $\mathbf{q}$ (namely, $\omega\tau_e \ll 1$ and $ql_e \ll 1$, where τ_e is the characteristic electron relaxation time and l_e is their mean free path). In this case the random forces (random extraneous currents) can be introduced directly in the macroscopic (phenomenological) equations. The corresponding system of equations

$$\delta j_i = \sigma_{ik}\mathscr{E}_k + eV_i\delta n - eD_{ik}\frac{\partial}{\partial x_k}\delta n + g_i,$$

$$\mathrm{div}\,\mathscr{E} = 4\pi e\delta n, \qquad e\frac{\partial}{\partial t}\delta n + \mathrm{div}\,\delta\mathbf{j} = 0 \tag{21}$$

was used earlier by one of the authors.[6] Here σ_{ik} is the differential conductivity, V_i is the (differential) drift velocity and D_{ik} is the diffusion tensor (for details see [2]).

The "random" extraneous current g_i is collected with the random force y_p as follows:

$$g_i = \frac{e}{\mathcal{V}}\sum_{\mathbf{p}} v_i\mathcal{J}_{\mathbf{p}}^{-1}y_{\mathbf{p}}, \tag{22}$$

so that

$$(g_i g_k)_{\omega\mathbf{q}} = \frac{e^2}{\mathcal{V}^2}\sum_{\mathbf{p}\mathbf{p}'} v_i v_k'\mathcal{J}_{\mathbf{p}}^{-1}\mathcal{J}_{\mathbf{p}'}^{-1}(y_{\mathbf{p}}y_{\mathbf{p}'})_{\omega\mathbf{q}}, \tag{23}$$

where $(y_p y_{p'})_{\omega q}$ is given by formula (20).

If the pair collisions can be neglected,5) then

$$(g_i g_k)_{\omega\mathbf{q}} = \frac{e^2}{\mathcal{V}^2}\sum_{\mathbf{p}\mathbf{p}'} v_i v_k'\mathcal{J}_{\mathbf{p}}^{-1}\mathcal{J}_{\mathbf{p}'}^{-1}(\mathcal{J}_{\mathbf{p}}+\mathcal{J}_{\mathbf{p}'})\bar{F}_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}'}$$

$$= \frac{e^2}{\mathcal{V}^2}\sum_{\mathbf{p}\mathbf{p}'} v_i v_k'(\mathcal{J}_{\mathbf{p}}^{-1}+\mathcal{J}_{\mathbf{p}'}^{-1})\left(\bar{F}_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}'} - \frac{\bar{F}_{\mathbf{p}}\bar{F}_{\mathbf{p}'}}{N}\right) = \frac{e^2 N}{\mathcal{V}^2}(D_{ik}+D_{ki}), \tag{24}$$

since (see [2]

$$D_{ik} = \frac{1}{N}\sum_{\mathbf{p}} v_i\mathcal{J}_{\mathbf{p}}^{-1}(v_k - V_k)\bar{F}_{\mathbf{p}}.$$

When pair collisions are taken into account, $(g_i g_k)_{\omega q}$ does not reduce to the tensor of the diffusion coefficients (see [2]).

4. THE ENTROPY APPROACH

The fluctuations in the state of thermodynamic equilibrium are usually analyzed by the entropy method, described, for example, in the book of Landau and Lifshitz.[5] This method was used successfully also to study fluctuations of the distribution function under equilibrium conditions.[7] The method is based on a combinatory expression for the entropy

$$S = -\sum_{\mathbf{p}}[F_{\mathbf{p}}\ln F_{\mathbf{p}} - F_{\mathbf{p}}], \tag{25}$$

which defines the probability of the distribution F_p for a closed system, and on the fundamental fact that S is extremal in the equilibrium state.

In the situation considered by us, namely an open system in a stationary non-equilibrium state, expression (25) is not extremal and the entropy method, naturally, cannot be employed directly. We deemed it of interest to ascertain whether a generalization of the "entropy" approach to the problem of fluctuations near such a state is possible,6) and if so what are the limits of applicability of such an approach. The generalization turned out to be possible in the case when the correlations due to the electron-electron collisions can be neglected, i.e., in the case of an ideal non-equilibrium gas whose particles interact only with the electric field and with the thermostat (with which energy and particles can be exchanged). We have calculated the probability of such a gas having a given particle distribution F_p different from the mean stationary distribution $\bar{F}_p$. The calculation is given in the Appendix and leads to the following expression for the probability density $w\{F_p\}$ as a functional of F_p:

$$w\{F_{\mathbf{p}}\} = w\{\bar{F}_{\mathbf{p}}\}e^{\mathfrak{S}}, \tag{26}$$

where

$$\mathfrak{S} = -\sum_{\mathbf{p}}[F_{\mathbf{p}}\ln(F_{\mathbf{p}}/\bar{F}_{\mathbf{p}}) - (F_{\mathbf{p}} - \bar{F}_{\mathbf{p}})]. \tag{27}$$

The simplicity of this expression7) should not be surprising, for in our case, when the interelectron collisions can be neglected, there is no correlation between the different electronic states, and the probability of observing an electron in some quantum state does not depend on the occupation numbers of all the other

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4)By eliminating from (20) the terms with E with the aid of (1), we obtain the correlator proposed in [4] (see (2.14) of [4], and also [4a]).

5)In this case, of course, $\mathcal{J}_p$ does not contain the operator of the interelectron collisions, so that $\mathcal{J}_p\bar{F}_p = 0$. The addition in (24) of the term $(-F_pF_{p'}/N)$ under the sign of $(\mathcal{J}_p + \mathcal{J}_{p'})$ is essential for the subsequent transformation, since $\mathcal{J}_p$ and $\mathcal{J}_p^{-1}$ commute only with functions whose sum over p is equal to zero (see [2]).

6)We deemed this to be of interest also because a number of authors [8,9] have used the entropy method to calculate fluctuations in the non-equilibrium state.

7)It should be noted that an expression of this type was used by Schlogl [10] in connection with non-equilibrium statistical thermodynamics. Expressions of this type are encountered also in information theory (see [11]).

states.[8] If the probability that the electron will occupy a given state p is $\overline{F}_p$, then the probability of finding N_j electrons in a group of G_j close states (assumed to be equally probable) is

$$\frac{G_j!}{N_j!(G_j-N_j)!}\overline{F}_p^{N_j}(1-\overline{F}_p)^{G_j-N_j}, \tag{28}$$

whence (at N_j, $G_j \gg 1$ and under the condition $N_j/G_j \equiv F_p \ll 1$ and $\overline{F}_p \ll 1$) we immediately obtain for the probability distribution F_p the equations (26) and (27) (see [5], p. 183).

The functional $\mathfrak{S}$ is maximal at $F_p = \overline{F}_p$ (just as (25) is maximal for a closed system in the case of an equilibrium distribution). Thus, the mean (stationary nonequilibrium) distribution $\overline{F}_p$ is also the most probable distribution. The expansion of $\mathfrak{S}$ in terms of $\delta F_p = F_p - \overline{F}_p$ near $\overline{F}_p$ begins with the quadratic term

$$\Delta\mathfrak{S} = -\frac{1}{2}\sum_p \overline{F}_p^{-1}\delta F_p^2, \tag{29}$$

and from then on it is possible to use automatically the scheme of [5] to find the correlator of the random forces.

We introduce the generalized coordinates $x_p \equiv \delta F_p$ and the corresponding forces $X_p \equiv -\partial\Delta\mathfrak{S}/\partial x_p = \overline{F}_p^{-1}\delta F_p$. The connection between x_p and X_p, i.e., the kinetic equation (19), is conveniently represented in the form

$$\delta\dot{F}_p = -\sum_{p'}\mathcal{I}_p\overline{F}_p\delta_{pp'}\overline{F}_{p'}^{-1}\delta F_{p'} + y_p, \tag{30}$$

whence, in accord with [5], we obtain for the correlator of the random forces y_p

$$(y_p y_{p_1})_\omega = (\mathcal{I}_p + \mathcal{I}_{p_1})\overline{F}_p\delta_{pp_1}, \tag{31}$$

which coincides, as it should, with formula (20) in which we discard terms connected with the interelectron collisions.[9]

In order to understand better the physical meaning of the quantity $\mathfrak{S}$, let us find out what happens to it if $\overline{F}_p = \exp[(\mu - \epsilon_p)/T]$, i.e., when we consider fluctuations in the equilibrium state (T is the thermostat temperature). It is easy to see that in this case

$$T\mathfrak{S} = -\Delta(E - TS - \mu N), \tag{32}$$

i.e., at equilibrium $-T\mathfrak{S}$ is none other than the change of the thermodynamic potential Ω of the system (see [5]) in the case of fluctuations. But this is as it should be, for the fluctuation probability in an equilibrium open system is determined by the change of the total entropy of the system and of the thermostat, which, as is well known, equals $\Delta\Omega$ (in the case when the system exchanges both energy and particles with the thermostat).

Thus, $\mathfrak{S}$ is a functional that makes it possible to generalize in natural manner the "entropy" approach to fluctuations about the stationary nonequilibrium state of an ideal gas. On the other hand, if the gas is not ideal, i.e., the interelectron collisions cannot be neglected, then these collisions give rise in the non-equilibrium state to a correlation between the electronic states. It can be shown that in this case the multiparticle equal-time distribution functions (which are needed for the calculation of the probability w) do not reduce to the product of a single-particle function. Thus, to find the probability w we would have to know the correlators $\delta F_p \delta F_{p_1}$, $\delta F_p \delta F_{p_1} \delta F_{p_2}$, etc., for the calculation of which we want precisely to use w. Under these conditions, the entropy method, obviously, becomes meaningless.

We are grateful to A. I. Ansel'm and Yu. N. Obraztsov for a very interesting discussion. We are also most indebted to B. B. Kadomtsev for numerous discussions, which make it possible to compare our method with the method described in [3], and to ascertain the reason for the discrepancy between the corresponding results.

APPENDIX

Let us derive formulas (26) and (27) for the probability distribution of different macroscopic states of an ideal non-equilibrium electron gas, whose particles interact only with an external electric field and with a thermostat (phonons and impurities), but not with one another. As is usually done (see, for example [5], p. 143), we subdivide the region of p-space occupied by the electrons into cells. The number of states G_j, as well as the number of particles N_j in each cell j, is assumed to be large. Then the set of numbers N_j characterizes the macroscopic state of the gas.

The probability of appearance of a given set of numbers N_j is given by the following expression:

$$w\}N_j\{ = \mathrm{Sp}\left[\rho\prod_j \Delta(\hat{N}_j - N_j)\right]. \tag{A.1}$$

Here ρ is the density matrix of the electron system under consideration, Δ is the Kronecker symbol ($\Delta(x) = 0$ for integer $x \neq 0$ and $\Delta(x) = 1$ for $x = 0$); $\hat{N}_j = \sum_{p_j}\hat{n}_p$ is the operator of the number of particles in the cell j; $\hat{n}_p$ is the operator of the number of particles in the state p; the summation is over all values of p belonging to a given cell j.

The factors in the product of (A.1) can be represented in the form

$$\begin{aligned}\Delta(\hat{N}_j - N_j) &= \frac{1}{2\pi i}\oint\frac{dz}{z}\exp\{(\hat{N}_j - N_j)\ln z\}\\ &= \frac{1}{2\pi i}\oint\frac{dz}{z}\exp\{-N_j\ln z\}\prod_{p_j}\exp\{\hat{n}_p\ln z\}.\end{aligned} \tag{A.2}$$

but $(\hat{n}_p)^k = \hat{n}_p$ when $k \neq 0$, and therefore

$$\exp\{\hat{n}_p\ln z\} = 1 - \hat{n}_p + \hat{n}_p z, \tag{A.3}$$

so that

$$\Delta(\hat{N}_j - N_j) = \frac{1}{2\pi i}\oint\frac{dz}{z^{N_j+1}}\prod_{p_j}(1 - \hat{n}_p + \hat{n}_p z). \tag{A.4}$$

Substituting such expressions in (A.1), we see that the problem reduces to the calculation of the mean value of a certain number of operators $\hat{n}_p$ with different indices p. The answer is almost obvious and consists in the fact

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[8] As is well known (see, for example, [1]), the very requirement that the number of particles be constant leads to a correlation between the occupation numbers of the different states. However, the probability distribution (26), (27) can be used also for a system with a specified number of particles, and the requirement that the number of particles be constant can be imposed in the form of an additional condition (see [1]).

[9] In some papers, a simplified relation (31) is used, in which the operators $\mathcal{I}_p$ and $\mathcal{I}_{p^1}$ are replaced by a certain number having the meaning of the collision frequency. The possibility of such a substitution must be verified in each concrete case.

that for an ideal gas the mean product is equal to the product of the means. Such a statement follows rigorously from our earlier paper[1] [10]). The mean value of n_p within the confines of one cell must be regarded as equal to one another, so that the mean number of particles in the cell is $\overline{N}_j = G_j \overline{F}_p$. Thus the probability $w\{N_j\}$ breaks up into a product of individual factors, equal to

$$w_j\{N_j\} = \frac{1}{2\pi i} \oint \frac{dz}{z^{N_j+1}} \exp[G_j \ln(1 - \overline{F}_p + \overline{F}_p z)]. \qquad (A.5)$$

Calculating this integral by the saddle-point method, we get

$$\tilde{w}_j(F_p) = w_j(\overline{F}_p) \exp\left\{ -G_j \left[F_p \ln \frac{F_p}{\overline{F}_p} + (1 - F_p) \ln \frac{1 - F_p}{1 - \overline{F}_p} \right] \right\}. \qquad (A.6)$$

We have introduced here the symbol $F_p = N_j / G_j$ for the number of particles averaged over the cell (but not over the time!) in each of the quantum states of the group j. For the total probability of the distribution F_p we obtain

$$w\{F_p\} = w\{\overline{F}_p\} e^{\mathfrak{S}}, \qquad (A.7)$$

where

$$\mathfrak{S} = -\sum_p \left[F_p \ln \frac{F_p}{\overline{F}_p} + (1 - F_p) \ln \frac{1 - F_p}{1 - \overline{F}_p} \right]. \qquad (A.8)$$

When F_p, $\overline{F}_p \ll 1$, Eq. (A.8) goes over into formula (27) of the main text.

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[10]) The analysis given in [1] for the product of two such factors can be extended directly to the case of the product of an arbitrary number of these factors.