RESONANCE AND OSCILLATORY EFFECTS IN SUPERCONDUCTORS

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A phenomenological theory of superconductivity is developed. It is shown that single-particle excitations in superconductors are quasiparticles with dispersion law (3.3); the quasiparticles move along the phase trajectories shown in Fig. 2 (the notation is given in (4.1)). The motion of the quasiparticles is quantized. This results in a very complex energy spectrum (see Figs. 3–5 and formulas (7.3)–(7.5) and (5.3)–(5.6)), which is characterized by discrete branches that are detached periodically from the continuous spectrum. Case (5.4) is quasiclassical and the quantization conditions are of the Bohr type (5.1); they correspond qualitatively to equidistant changes of the momentum in a classical orbit. The quasiclassical superconductivity equation is derived (formulas (3.6), (3.6a), (3.4), and (3.3)). It is valid in the London case in strong enough magnetic fields when the temperature is not very near the critical one. The periodic nature of the variation of the spectrum results in quantum oscillations of the thermodynamic and kinetic quantities with a relative amplitude (6.2) (analogs of the de Haas-van Alphen effect, the Shubnikov-de Haas effect, etc). Discreteness of the levels results in resonances (electromagnetic and ultrasound) with a relative amplitude of resonance variation $Z'(H)$ (Z is the impedance and H the external stationary magnetic field) of the order of (6.6); the discreteness is also the cause of undamped waves and of natural oscillations in the superconductors.

1. NATURE OF RESONANT AND OSCILLATORY EFFECTS IN SUPERCONDUCTORS

In normal metals there are a number of well known resonant phenomena (of electromagnetic and ultrasonic origin) and oscillatory phenomena (the de Haas–van Alphen or the Shubnikov–de Haas effect). In those cases when the resonance is due to electrons with small momentum spread δp, the resonance is connected with a weakly-damped wave (at a distance on the order of $h/\delta p$) corresponding to excitation of the natural oscillations of the electron plasma of the metal. A common feature of all the foregoing effects is that they are due to periodic motion of the carriers in the constant magnetic field.

In superconductors, the magnetic field attenuates at a depth $\delta_e \ll r$ (r—characteristic Larmor radius; $r \sim \delta_e$ is reached at $H \sim 10^5$ Oe!), and therefore has practically no influence on the motion of most electrons. In quantum mechanics this means[1] a shallow potential well with a single discrete level[2] (surface levels in superconductors were first considered by Pincus [3]; see also[4,5]).

For excitations with $p_y \sim p_F\sqrt{\delta_e/r}$ (see Fig. 1, p—momentum $p_F = mv_F$—Fermi momentum), the magnetic field is quasi-homogeneous, the motion of the charges is localized in a layer of order δ_e and becomes analogous to motion in a plate of thickness δ_e, leading to quantization of the projection of the momentum p_y: $p_y \sim hn/\delta_e$ (n is an integer) and to an associated quantization of the energy. The energy in not too weak a magnetic field $H \parallel z$ differs (see Sec. 3) from the energy at $H = 0$ by a term $-eAv/c \sim -eH\delta_e v_F/c$ ($\mathbf{A} \parallel x$ is the vector potential), so that qualitatively we have

$$\varepsilon_n \sim \left[\Delta^2 + \left(\frac{h^2n^2}{2m\delta_e^2} - \mu_\perp\right)^2\right]^{1/2} - eH\delta_e\frac{v_F}{c}, \qquad \mu_\perp = \mu - \frac{p_x^2 + p_z^2}{2m} \tag{1.1}$$

(μ is the chemical potential).

Discrete levels correspond to bound surface states and should ensure the absence of excitations at infinity. Since there is no magnetic field H there, this means that $H = 0$ such excitations are forbidden, i.e., when $\mu_\perp > 0$ their energy is lower than the gap Δ, and (see (1.1)) when $H < c\Delta\ (eH\delta_e v_F)^{-1}$ we get

$$n_{max} \sim (H/H_1)^{1/4}, \quad H_1 \sim c\hbar a^2\xi_0/e\delta_e^5, \quad a \sim h/p_F, \quad \xi_0 \sim \hbar v_F/\Delta. \tag{1.2}$$

When $H \gg H_1$ we have $n_{max} \gg 1$, i.e., the quasiclassical case. Since $H_1 \sim 10^{-1}$–10^{-6} Oe, this means that the motion of the electrons corresponding to the discrete levels and ensuring the possibility of resonant and oscillatory effects can be regarded classically in the fundamental approximation (in n^{-1}).

The avoidance of different concrete models and the introduction of a physically lucid representation of the carriers as quasiparticles with an arbitrary dispersion law (apart from symmetry) in the case of normal metals have led to appreciable success both in the theory and in experiment. It is therefore natural to introduce such representations also in the case of superconductors, starting not from the microscopic theory (which in the simplest cases leads to the same results by a more complicated path) but semiphenom-

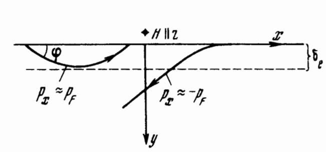

FIG. 1

enologically, all the more since such a scheme admits of different generalizations (to biexcitons, superconductors with impurities, etc.).

2. PHENOMENOLOGICAL THEORY (H = 0)

In normal metals, as is well known, elementary excitations with quasimomentum p are either electrons (e) with energy $\epsilon = \epsilon(p) - \mu \equiv \xi(p) \equiv \xi_p$ if $\epsilon(p) > \mu$, or holes (h) with energy $\epsilon = \mu - \epsilon(p) = -\xi_p$ if $\epsilon(p) < \mu$. This can be written in the form of one equation

$$\begin{pmatrix} \xi_p & 0 \\ 0 & -\xi_p \end{pmatrix} \psi = \varepsilon\psi, \quad \varepsilon > 0. \tag{2.1}$$

We now turn on a weak self-consistent interaction between an electron with charge e and a hole with charge[1] – e. Taking hermiticity into account, we obtain

$$\begin{pmatrix} \xi_p & -\Delta \\ -\Delta^* & -\xi_p \end{pmatrix} \psi \equiv \hat{\mathscr{H}}_0\psi = \varepsilon\psi, \quad \varepsilon > 0. \tag{2.2}$$

Inasmuch as the interaction is significant only at $\epsilon \lesssim |\Delta| \ll \Theta_D$ ($|\Delta|$ is small!), we can set the parameter Δ equal to a constant if $\epsilon < \Theta_D$ and to zero if $\epsilon > \Theta_D$ (Θ_D is the characteristic interval of variation of $\Delta(\epsilon)$). The condition $\epsilon > 0$ always holds—it corresponds to the obvious fact that the excitation energy is positive. We shall consider a representation in which Δ is real. Then it follows from (2.2) that

$$\varepsilon = \varepsilon_0(\mathbf{p}) = \sqrt{\xi_p^2 + \Delta^2}. \tag{2.3}$$

The eigenfunctions of (2.3), normalized to unity ($\psi^2 = 1$), are[2]

$$\psi_s = (u, v) \quad u^2 = \frac{1 + s\alpha}{2}, \quad v^2 = \frac{1 - s\alpha}{2}, \quad \alpha > 0, \quad s = \pm 1, \tag{2.4}$$

$$s\alpha = s\sqrt{\varepsilon^2 - \Delta^2}/\varepsilon = \xi_p/\varepsilon.$$

Thus, excitations in superconductors can be of the "electronic" type (with wave function ψ_+, containing $(1 + \alpha)/2$ electrons and $(1 - \alpha)/2$ holes; we shall henceforth call them "plusons" P), or else of the "hole" type (with wave function ψ_-, containing $(1 - \alpha)/2$ electrons and $(1 + \alpha)/2$ holes—"minusons" M).

Since the number of plusons and minusons is not conserved, their chemical potential is equal to zero, and the distribution functions (obviously, of the Fermi type) coincide and equal $[e^{\epsilon/T} + 1]^{-1}$. This means that the "distribution function" for particles remaining in the bound state ("condensate") is

$$n_c(\varepsilon) = 1 - 2[e^{\varepsilon/T} + 1]^{-1} = \operatorname{th}(\varepsilon/2T). \tag{2.5}$$

For a complete description, it remains to find the number of states $d\rho_C$ of the particles in the condensate. It is convenient to carry out the calculation with the number of particles conserved, formally regarding the pair C as consisting of half an electron and half a hole ($\frac{1}{2}$e, $\frac{1}{2}$h). The reaction wherein a pluson P and a minuson M are formally transformed into a pair C results essentially in the fact that the electron, previously unevenly distributed among the two particles (P and M) now belongs to two particles as before, but in equal fractions: P + M $\rightleftarrows$ 2C. Since the number of states is conserved, we equate the phase volumes occupied by the electrons on the left and on the right:

$$\frac{1+\alpha}{2}\frac{1-\alpha}{2} \equiv u^2v^2 = \chi^2$$

(the "effective-mass law" for the concentrations u^2, v^2, and χ of the electrons in P, M, and C). Hence

$$d\rho_c = \chi\frac{2d\mathbf{p}}{h^3} = 2uv\frac{d\mathbf{p}}{h^3} = \frac{2d\mathbf{p}}{h^3}\frac{\Delta}{\varepsilon}. \tag{2.6}$$

We now use the self-consistent character of the interaction. (In essence, this is the only manifestation of the peculiar character of superconductivity, and not of exciton or polaron excitations.) We shall assume that the gap is determined by the pair density N_C. In the case of a weak interaction, the pair density is low and, expanding Δ in terms of N_C, namely $\Delta = \lambda N_C$ (since $\Delta = 0$ when $N_C = 0$), and using (2.5) and (2.6), we get

$$\Delta = \lambda \int_{\varepsilon<\Theta_D} \frac{\Delta}{\varepsilon}\frac{2d\mathbf{p}}{h^3}\operatorname{th}\frac{\varepsilon}{2T}, \tag{2.7}$$

$$\int_0^{\operatorname{Arsh}(\Theta_D/\Delta)} \operatorname{th}\frac{\Delta \operatorname{ch} z}{2T}dz \approx \frac{1}{\lambda\nu(\mu)} \tag{2.8}$$

(since $\Theta_D \ll \mu$), where $\nu(\mu)$ is the density of states of the "normal" electrons.

3. PHENOMENOLOGICAL THEORY IN A MAGNETIC FIELD

The action of the magnetic field, as is well known, reduces to a suitable transformation of the momentum. Since the electron and hole have respective charges e and –e (see Sec. 2), the "electronic" ξ_p must be replaced by $\xi(p - eA/c)$, and the "hole"$-\xi_p$ by$-\xi(p + eA/c)$, where p is the generalized momentum. As a result, (2.2) takes the form

$$\begin{pmatrix} \xi(\mathbf{p} - e\mathbf{A}/c) & -\Delta \\ -\Delta & -\xi(\mathbf{p} + e\mathbf{A}/c) \end{pmatrix} \psi \equiv \hat{\mathscr{H}}\psi = \varepsilon\psi, \quad \varepsilon > 0. \tag{3.1}$$

Since we are interested in $|p_x| \sim p_F$ and certainly $H \lesssim H_c$, we can linearize (3.1) (since $eH\delta_e \ll cp_F$ corresponds to $\delta_e \ll r$, which is satisfied when $H \ll 10^5$–10^6 Oe):

$$\begin{pmatrix} \xi(\mathbf{p}) & -\Delta \\ -\Delta & -\xi(\mathbf{p}) \end{pmatrix} \psi = \left(\varepsilon + \frac{e}{c}\mathbf{A}\mathbf{v}\right)\psi, \quad \mathbf{v} = \frac{\partial\xi}{\partial\mathbf{p}}, \quad \varepsilon > 0. \tag{3.2}$$

From a comparison of (3.2) with (2.2) (see also (2.4)) it follows that

$$\varepsilon = \varepsilon_H^{(\sigma)}(\mathbf{p}) = \sigma\varepsilon_0(\mathbf{p}) - e\mathbf{A}\mathbf{v}/c, \quad \varepsilon_0(\mathbf{p}) = \sqrt{\xi_p^2 + \Delta^2}, \quad \varepsilon > 0, \quad \sigma = \pm 1, \tag{3.3}$$

$$\psi = (u, v) = \left(\sqrt{\tfrac{1}{2}(1 + \sigma\xi_p/\varepsilon_0)}, \sqrt{\tfrac{1}{2}(1 - \sigma\xi_p/\varepsilon_0)}\right). \tag{3.4}$$

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[1]Such an interaction usually takes place between an electron and a *hole having opposite* spins. Since the spin direction does not enter explicitly in the equation, there is no need for noting it specially. A hole with a charge –e and momentum p is another name for the "ordinary" hole with charge e and momentum $p_h = -p$, since in a field with a vector potential A we have $\epsilon_h = -\xi(p-(-e)A/c) = -\xi(p_h + eA/c)$. It is more convenient for us to speak of a hole (–e, p). It is clear that (2.2) corresponds to expansion of the dispersion equation in terms of the weak interaction.

[2]The symbols u and v were not chosen by accident: the plusons and minusons realize in superconductors a Bogolyubov transformation (which diagonalizes the Hamiltonian).

Both signs of $\epsilon_0(p)$ are taken because it is possible to have $\epsilon_H^{(-)} > 0$ when $e\mathbf{A}\cdot\mathbf{v} < -\epsilon_0(p)$.

Let us ascertain when an excitation in a magnetic field is a pluson or a minuson. From (3.3) we obtain $\xi_p = s[(\epsilon + eAv/c)^2 - \Delta^2]^{1/2}$, where $s = \pm 1$. According to (3.4) this means that $s\sigma = 1$ corresponds to a pluson and $s\sigma = -1$ to a minuson. It is important that, inasmuch as ξ_p is not conserved in a magnetic field, it can reverse sign. This means that the ratio of electrons to holes in the excitation changes during the motion, and a pluson—minuson transition takes place at the point $\xi_p = 0$, where the composition of the excitation is ($\frac{1}{2}$e, $\frac{1}{2}$h.

It is seen from (3.3) that the excitation energy in a magnetic field depends not on the combination $p - eA/c$, as in a normal metal, but directly on $\mathbf{A}$ (and, of course, on p). This means that $\mathbf{A}$ acquires a direct physical meaning. Consequently, its choice should be unique, determined by additional physical conditions. In order to find them, let us consider for simplicity the case of a superconducting half-space $y \geq 0$ in a magnetic field $H \parallel z$. In this case any quantity having a physical meaning does not depend on x and z, and if the only preferred directions are those of the axes, this quantity is directed along one of them. At $y \to \infty$, where the magnetic field vanishes, we should have $\mathbf{A}(\infty) = 0$. These conditions, as expected determine $\mathbf{A}$ uniquely (of course, in conjunction with the equation $\mathrm{curl}\,\mathbf{A} = \mathbf{B}$, which follows from $\mathrm{div}\,\mathbf{B} = 0$, $\mathbf{B} \parallel z$)

$$A = A_x(y) = \int_y^\infty B(y')\,dy'.$$

In the general case it is necessary to have[3)] $\mathrm{div}\,\mathbf{A} = 0$ and $\mathbf{A}(\infty) = 0$. (In any conductor with a high charge density, the magnetic field at a given point is automatically self-averaged over the entire set of charge trajectories passing through the point, and therefore the equation contains the magnetic induction B (see[6,2])).

We use the theory developed above to calculate the pair current density j. Inasmuch as the flow of charge plusons and minusons coincide with that in electrons and holes, it suffices to double the charge flux density in plusons and to consider that each pluson with $\psi = (u, v)$ carries $|u|^2$ electrons with velocity $v = \partial\xi(p - eA/c)/\partial p$, and the distribution function of the bound excitations is $\tanh(\epsilon/2T)$ (see (2.5)). We obtain

$$\mathbf{j} = \frac{4e}{h^3}\int \mathbf{v}|u|^2 \,\mathrm{th}\frac{\varepsilon}{2T}\,d\mathbf{p} \qquad (3.5)$$

(the spin factor of 2 is taken into account in the density of states). It is convenient to rewrite formula (3.5), using the invariance of (3.1) against the substitutions $\mathbf{p} \to -\mathbf{p}$ and $\mathbf{A} \to -\mathbf{A}$, in the form

$$\mathbf{j} = \frac{4e}{h^3}\int_{\varepsilon>0}\left\{g\left(\mathbf{p}+\frac{e}{c}\mathbf{A},\mathbf{A}\right) - g\left(\mathbf{p}-\frac{e}{c}\mathbf{A},-\mathbf{A}\right)\right\}\mathbf{v}\,d\mathbf{p}, \qquad (3.6)$$

$$g(\mathbf{p},\mathbf{A}) = |u|^2\,\mathrm{th}\frac{\varepsilon}{2T}.$$

To simplify the obvious derivations, we consider weak fields, when $\sigma = +1$ in formulas (3.3) and (3.4). Recognizing that u depends on p, we find in the approximation linear in A

$$\frac{h^3c}{2e^2}\mathbf{j} = 2\int \mathbf{v}\,d\mathbf{p}\left(\mathbf{A}\frac{\partial}{\partial\mathbf{p}}\right)\{g(\mathbf{p})-1\} - \frac{1}{T}\int \mathbf{v}\,d\mathbf{p}\,(\mathbf{v}\mathbf{A})\,|u|^2\,\mathrm{ch}^{-2}\frac{\varepsilon_0(\mathbf{p})}{2T}. \qquad (3.7)$$

The obtained formula can be easily transformed to the London form.

We note in conclusion one more circumstance. At a fixed chemical potential μ, the magnetic field, by changing the density of states, leads to a change in the charge density ρ'. This means that the electroneutrality of metals leads to a change of the chemical potential in a homogeneous magnetic field. In a superconductor, where the magnetic field is strongly inhomogeneous, the occurrence of an inhomogeneous density is hindered by the resultant electrostatic field E with potential $e\varphi(r)$ (see[6], Sec. 2):

$$\delta\rho' \propto \delta\left[\int |u|^2\,\mathrm{th}\frac{\varepsilon}{2T}\,dp\right]_{\mu\to\mu-e\varphi(r)} = 0.$$

Hence $eE\delta_e \sim \Delta^2/u$.

The condition for the validity of the quasiclassical analysis is $\delta_e \gg h/\delta p$ (where δp is the change of the momentum p_y on moving along the classical trajectory (3.3): $\delta p = p_y(A) - p_y(0)$); the opposite inequality ensures validity of perturbation theory—see also Sec. 5. The condition $\delta_e \gg h/\delta p$ at $T = 0$ for j corresponds to $H \gg H_3$, $H_3 \sim c\hbar\xi_0/e\delta_e^3 \ll H_{c1}$ in the London case and $H_3 \sim c\hbar/e\xi_0\delta_e \ll H_c$ in the Pippard case (for glancing electrons with $\kappa \sim \kappa_F(\delta_e/\xi_0)^2$, where ξ_0 is the pair dimension), and is satisfied already in weak fields. In weaker fields, the quasiclassical analysis is valid for definite p, and also for discrete levels at $H \gg H_1$ (Sec. 5) and the phenomena associated with it (Sec. 6).

4. CLASSICAL MOTION IN A MAGNETIC FIELD

Equation (3.3) determines directly the form of the phase trajectories of the plusons and minusons in a magnetic field in the classical case. For simplicity we consider the one-dimensional case[4)] $A = A_x(y)$ (p) $= p^2/2m - \mu$. Changing over in (3.3) to dimensionless variables:

$$y = \delta_e\xi,\quad p_y = \frac{\hbar\Pi}{\delta_e},\quad \mu_\perp = \varkappa\bar{\varepsilon},\quad \bar{\varepsilon} = \frac{\hbar^2}{2m\delta_e^2},\quad \varkappa_F = \frac{\mu}{\bar{\varepsilon}} \sim \left(\frac{\delta_e}{a}\right)^2,$$

$$\varepsilon_H^{(\sigma)} = z\bar{\varepsilon},\quad \alpha = \frac{\Delta}{\bar{\varepsilon}} \sim \frac{\delta_e^2}{a\xi_0},\quad \xi_0 \sim \frac{\hbar v_F}{\Delta},\quad a \sim \frac{\hbar}{p_F},\quad \rho \equiv \zeta - z = \frac{eH\delta_e v_x}{\bar{\varepsilon}c},$$

$$\rho_F \sim \frac{\delta_e^3}{a^2 r},\quad r \sim \frac{p_F c}{eB_0},\quad A(y) = B_0\delta_e\theta(\xi),\quad B_0 = B(0), \qquad (4.1)$$

$$\theta(0) = 1,\quad \delta_e = B_0^{-1}\int_0^\infty B(y')\,dy'$$

(α plays the role of the constant of the material, since

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3)The need for writing down definite conditons for A is connected with the choice of a real Δ in (2.2). Writing down the equation for ψ in the absence of a magnetic field in the form (2.2) would ensure gauge invariance of A (of course, following a suitable transformation of ψ and Δ); then

$$\Delta^* = \frac{2\lambda}{h^3}\int_{0<\varepsilon<\Theta_D} d\mathbf{p}\,u^*v\,\mathrm{th}\frac{\varepsilon}{2T}.$$

We note that in any case when $\epsilon \gtrsim \Theta_D$, we have $\Delta_p^* = \int\lambda_{pp'}d\rho_p^{(c)}n_c(\epsilon') = 2h^{-3}\int\lambda_{pp'}U_{p'}^*V_{p'}\mathrm{th}(\epsilon_{p'}/2T)dp'$. A convenient model is $\lambda_{pp'} = \Lambda_p\Lambda_{p'}$.

4)We note that the point $\xi_p = 0$ of the pluson—minuson transition is in this case a turning point not only in phase space but also in coodinate space: according to (3.3) we have $\dot{y} = v_y = \partial\epsilon_H^{(\sigma)}/\partial p_y = \sigma\xi_p P_y/m\epsilon_0(P)$.

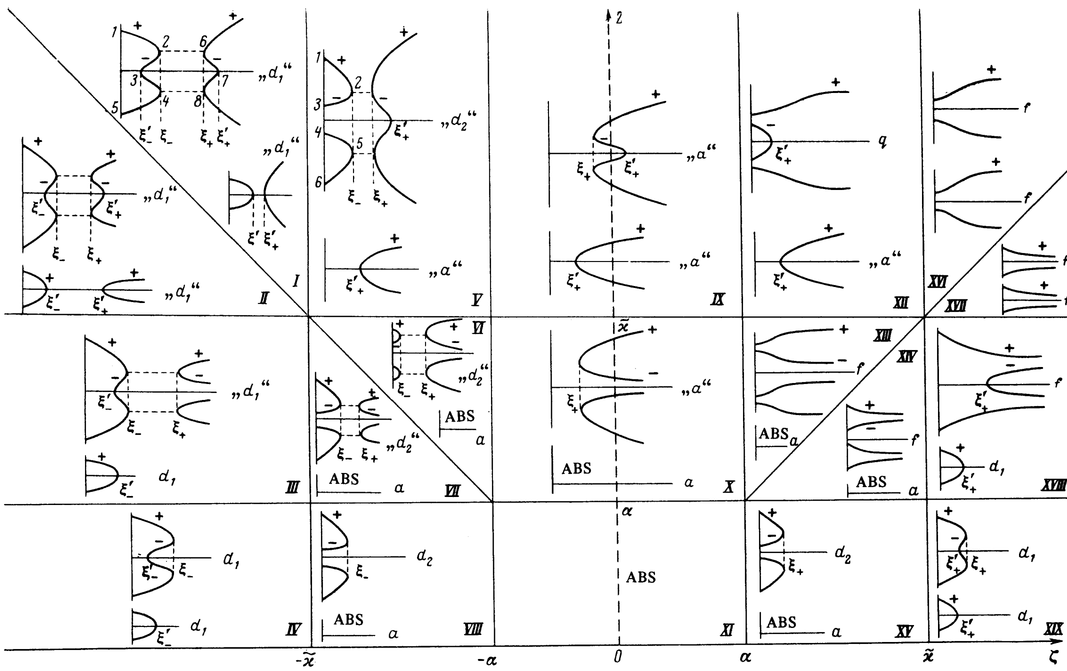

FIG. 2. Plots of $\Pi(\xi)$ for different values of z, κ, and ζ; $\tilde{\kappa} = \sqrt{\kappa^2 + \alpha^2}$.

it depends little on the temperature: on approaching the critical point, when the London case sets in, $\alpha \sim \delta_e^2 \Delta \to$ const), we obtain

$$\Pi(\xi) = \pm(\varkappa + s\{[\zeta\theta(\xi) + z(1-\theta(\xi))]^2 - \alpha^2\}^{1/2})^{1/2}, \quad \zeta - z = \rho, \quad s = \pm 1. \tag{4.2}$$

The form of the function $\Pi = \Pi(\xi)$ for given α and different values of z, κ, and ζ is shown in Fig. 2. In each range of the parameters z and ζ, the upper curve corresponds to $\kappa > 0$ and the lower to $\kappa < 0$; near each branch of the curve is shown the sign of the S that determines it (so that the presence of two signs on the curve denotes the existence of both plusons and minusons in the given region, while only one sign denotes the presence of excitations of only one type; the signs are placed only in the upper quadrant, since the curve is symmetrical in the lower quadrant—see (4.2)). The absence of classical trajectories at given values of the parameters[5] is marked by the symbol ABS.

The turning points are determined by the formulas

$$\theta(\xi_\pm) = \frac{z \mp \alpha}{z - \zeta}, \quad \theta(\xi_\pm') = \frac{z \mp \tilde{\varkappa}}{z - \zeta}, \quad \tilde{\varkappa} = \sqrt{\varkappa^2 + \alpha^2} \tag{4.3}$$

(so that the excitation can penetrate into the superconductor to an arbitrary depth and return to the surface, in spite of the exponential attenuation of the field; the direction traversed in the x direction is then exponentially larger than in the y direction). The turning points at $\Pi \neq 0$ are the pluson—minuson transition points. Thus, for motion in region I of Fig. 2, at $\kappa > 0$ and $\sigma = +1$, the pluson reflected from the surface of the superconductor at point 1 is transformed into a minuson at point 2, experiences total internal reflection from the field inhomogeneity at points 2, 3, and 4, is again transformed into a pluson at point 4, is specularly reflected from the surface at 5 (in which case Π is replaced by $-\Pi$) and again falls into 1. On the other hand, a pluson moving from the interior is transformed into a minuson at the point 6, is reflected from the field inhomogeneity at 6, 7, and 8, is transformed again into a pluson at 8, and goes off to the interior of the superconductor.

Let us plot the regions of the different characteristic trajectories in "natural" coordinates. We do this in the simplest case of a "cylindrical" dispersion law $\xi(p) = (2m)^{-1}(p_x^2 + p_y^2) - \mu$, when $\kappa = (p_F^2 - p_x^2) \times (2m\bar{\epsilon})^{-1} \sim (\delta_e/a)^2(1 - p_x^2/p_F^2)$, and $(\delta_e/a)^2 \sim 10^5$—10^8 (the value $\kappa \sim \alpha$ is reached at $|p_x/p_F| - 1 \sim a/\xi_0 \sim 10^{-2}$—$10^{-5}$). We recognize that: 1) the values of importance for superconductivity are $|\xi(p)| \sim \Delta$, i.e., $p_x^2 + p_y^2 \approx p_F^2$; 2) the specular reflection of interest to us, from the surface, takes place at $|p_y| \ll p_F$ (ensuring a large de Broglie wavelength), i.e., $p_x \approx \pm p_F$; finally, 3) in superconductors we almost have $r \gg \delta_e$, up to the destruction of the superconductivity. We obtain $\rho \approx \rho_F$ sign n p_x. The forms of the regions of the different trajectories are shown in Figs. 3—5.

The regions marked by the letter a correspond to strict absence of classical orbits (for example, as in region XV of Fig. 2 at $\kappa < 0$). The trajectories marked f and d correspond to infinite and finite motion (d_1—along a singly connected curve such as the curves in region XIX of Fig. 2, d_2—along the doubly connected curve of the type VIII at $\kappa > 0$), q corresponds to infinite motion for the pluson and finite motion for the minuson or vice versa (curve XII at $\kappa > 0$). The quotation marks designate a given type of an orbit near the surface in the presence of infinite motion in the interior (for example, X at $\kappa > 0$ on Fig. 2 corresponds to the region "a"). In each figure, the double line denotes the boundary of the "old" spectrum (in the absence of a magnetic field); there were no orbits below it, and above it the orbits were infinite[6]. In order not to clut-

[5]It is convenient to construct Fig. 2 by extracting the roots graphically in succession and determining in this manner the characteristic regions.

[6]For a concrete determination of the regions of Figs. 3—5 it is convenient to carry out first a classification with respect to ρ_F.

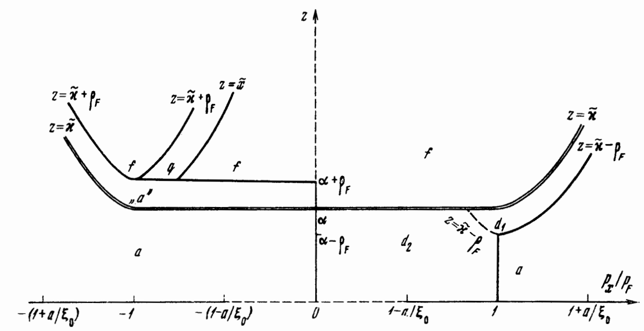

FIG. 3. Regions of characteristic trajectories at $\rho_F < \alpha$.

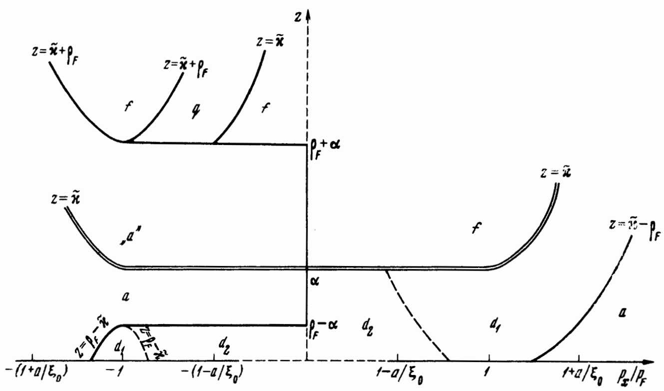

FIG. 4. Regions of characteristic trajectories at $\alpha < \rho_F < 2\alpha$.

ter up the figures, they do not show the boundaries separating the regions where both types of excitation exist (both plusons and minusons), and where there is only one type. These boundaries are clear from Fig. 2; thus, at H = 0 they correspond to $z = \kappa$. The quantitative determination of $y = y(t)$ and $x = x(t)$ reduces to a simple quadrature ($v_x = \partial\xi/\partial p_x$):

$$t = m\sigma\int\frac{\varepsilon_0 dy}{p_y\xi_p}, \quad x(t) = \pm\frac{v_x\xi_p}{\varepsilon_0} - \frac{eA}{mc}.$$

The essential difference between the form of the regions at $p_x > 0$ and $p_x < 0$ is connected with the consideration of $p_x \approx \pm p_F$, when the "surface" trajectories differ strongly already in a normal metal in a homogeneous field—see Fig. 1.

5. QUASICLASSICAL ENERGY LEVELS IN A MAGNETIC FIELD

Figures 2–5 determine not only the character of the classical orbit, but also the energy spectrum in the quantum case.

Let us consider first the quasiclassical spectrum connected with the "depth" excitations in the superconductor. Their reflection near the surface from an inhomogeneous magnetic[7)] (see, for example, region IX in Fig. 2) imitates specular reflection from the surface[8)] in a layer of thickness of the order of δ_e. This

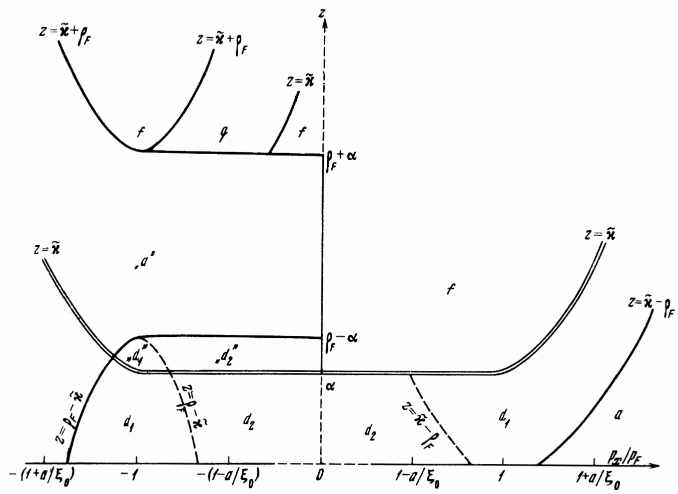

FIG. 5. Regions of characteristic trajectories at $\rho_F > 2\alpha$.

means that it is possible to observe size quantization (and the corresponding ultrasonic resonance and quantum oscillations) in a superconducting plate of thickness d, where $\delta_e \ll d \ll l$ (l—mean free path of the excitations), placed in a two-sided magnetic field (see Fig. 6):

$$p_y = hn/d, \quad 2m\varepsilon_n = [(2m\Delta)^2 + (p_x^2 + p_z^2 + h^2n^2/d^2 - 2m\mu)^2]^{1/2}, \quad n = 1, 2, 3, \ldots. \tag{5.1a}$$

Let us turn now to the quasiclassical surface levels. A strictly discrete spectrum is produced in the case of strictly finite classical motion (regions d_1 and d_2). Classical infinite motion over a trajectory of type f ensures a continuous spectrum, and a combination of finite and infinite motion over the trajectory q ensures a spectrum that is discrete for some excitations and continuous for others; the absence of classical orbits in region a corresponds to forbidden energy regions. The quotation marks correspond to the given type of spectrum, with accuracy to within a tunnel transition through the classically forbidden region (for example X at $\kappa > 0$ in Fig. 2 corresponds to "a"): although the spectrum is strictly speaking continuous (the excitation coming from the interior will always reach the surface), the corresponding density of states is exponentially small, $\sim \exp(-\int|\pi|\,d\xi)$, where the integral is taken over the classically forbidden region of ξ.

The concrete form of the discrete spectrum is also determined by the corresponding classical trajectories. In fact, according to the quasiclassical correspondence principle, the distance between the levels is equal to $\delta\epsilon = \hbar\Omega$, where $\Omega = 2\pi/T$ is the classical frequency of the periodic motion. (Physically this denotes[6,7] that in quantum mechanics there takes place resonance in an electromagnetic field of frequency ω at $\delta\epsilon = \hbar\omega$, and in classical mechanics at $\omega = \Omega$; in the limit as $\hbar \to 0$, both formulas should give one and the same frequency). But the classical period is

$$T_s = \oint\left(\frac{\partial\varepsilon}{\partial p_y}\right)^{-1} dy = \frac{\partial S}{\partial\varepsilon}, \quad S = \oint p_y dy,$$

so that $\delta S = (\partial S/\partial\epsilon)\delta\epsilon = T\hbar\Omega = h$, hence $S = nh$, where n is an integer and S is the area of the classical trajectory of excitation in the (y, p_y) phase space.

A more detailed analysis shows (see[7]) that in the

[7)]We recall that the gradient of the field in a superconductor may amount to 10^7 Oe/cm.

[8)]In normal metals, an analogous specular behavior and associated phenomena are possible in a specially produced inhomogeneous magnetic field (for example, by varying the frequency). The turning on of such a field can change the resistance of a thin plate by a factor $\sim l/d$, ensure different resonant and oscillatory effects, etc.

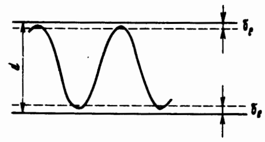

FIG. 6

quasiclassical case (at $S \gg h$), when the trajectories are not close to self-intersection, we have

$$S=\oint p_y dy=(n+1/2)h, \quad \tilde{S}=\oint \Pi(\xi)d\xi=n+1/2 \qquad (5.1)$$

(the generalized Bohr quantization rule), where $\tilde{S}$ is the area of the corresponding orbits in Fig. 2. (Thus, in region I at $\kappa > 0$, we are dealing with the area 123451, and in region V at $\kappa > 0$ with double the area 1231.) The condition for the applicability of (5.1) is a large value of the area $\tilde{S}$ (since n in (5.1) determines the number of zeroes of the wave function, i.e., the ratio of the characteristic dimension to the de Broglie wavelength).

For a concrete calculation of the area it is necessary to use the well known form[8] of the $\theta(\xi)$ dependence, i.e., A(y). When $\xi \lesssim 1$ (this is precisely the region which determines $\tilde{S}$) we have practically always[8] $\theta(\xi) = e^{-\xi}$; the difference between the London and Pippard cases is manifest in the value of δ_e.

The area $\tilde{S}$ can be expressed in terms of elementary functions in three important limiting cases:
1) when κ is the largest parameter of the problem;
2) when $|\zeta| \gg \alpha$ and α^2 can be neglected in (4.2) (this is the case of "normal" electrons with $\epsilon \gg \Delta$ in a superconductor field that attenuates in the interior);
3) when $\rho_F \ll \alpha$ or $\rho_F \gg \alpha$, so that one discards ρ_F^2 or $\alpha\rho_F$.

Since an exact determination of the levels from (5.1) is an elementary but very cumbersome and laborious problem, it is useful to obtain a general representation of the spectrum by writing down simple interpolation formulas for it. It is readily seen that the effective length of all the closed curves in Fig. 2 is of the order of unity. (When $\xi \gg 1$ the "width" of the curve is exponentially small (it is equal to zero at the end point of the curve), so that the region $\xi \gg 1$ has little influence on $\tilde{S}$. On the other hand, if the length of the curve is small ($\xi \ll 1$), then according to (4.3) this smallness is logarithmic.) This means that the quantization condition (5.1) reduces qualitatively to $\delta\Pi \sim n$, where $\delta\Pi$ is the change of the momentum on moving along the classical trajectory (4.2) (in a case of a singly-connected trajectory $\delta\Pi \sim \Pi$, and we arrive at formula (1.1)). It is important that the calculation of $\delta\Pi$ no longer requires any integration. Using Fig. 2, we can verify that in the case of finite motion we have everywhere, except at the boundaries of the regions, $\delta\Pi \sim \zeta(\zeta + |\kappa|)^{-1/2}$, $\tilde{\zeta} = (\zeta^2 - \alpha^2)^{1/2} > 0$. (In region XII, where $\kappa > \tilde{\zeta}$, we have $\delta\pi \sim \sqrt{\kappa} - \tilde{\zeta}\sqrt{\kappa}$, but the renormalization of n reduces this formula to the preceding one. Of course, in writing down the spectrum it is necessary to take into account the limitations imposed by the form of the region on $\tilde{\zeta}$ at a given κ.)

Recognizing that $z_n = \zeta_n - \rho_F \operatorname{sign} p_x$ (see (4.2)) and taking Figs. 3–5 into account, we get

$$z_n \sim \begin{cases} \sqrt{\alpha^2+n^2(n^2+|\kappa|)}-\rho_F \operatorname{sign} p_x, & \rho_F<\alpha \\ [\sqrt{\alpha^2+n^2(n^2+|\kappa|)}-\rho_F]\operatorname{sign} p_x, & \rho_F>\alpha \end{cases}. \qquad (5.2)$$

The analysis of these formulas is so simple, that we shall demonstrate it only for the principal cases.

1) $\rho_F < \alpha$ (i.e. $H < c\hbar/e\xi_0\delta_e < \alpha$. In this case (see Fig. 3; usually $a/\xi_0 \sim 10^{-2}$–10^{-3}) we get by interpolation

$$w_n = [\alpha(z-\alpha+\rho_F)]^{1/4} \sim (n^4+n^2\kappa/\alpha)^{1/4} \sim n+\sqrt{n}(\kappa/\alpha)^{1/4}, \qquad (5.3)$$

and for the considered $z < \alpha$ we get $w_n \leq (\alpha\rho_F)^{1/4}$, $n_{max} \sim (\alpha\rho_F)^{1/4}$. If $\rho_F \sim \alpha$, then $n_{max} \sim \sqrt{\alpha} \sim 10$–100. When $\alpha\rho_F \lesssim 1$, the quasiclassical approach is not valid, and there is[2] a single discrete level (see Sec. 7). In fields $\alpha\rho_F^{(n)} \sim n^4$, when (compare with (1.4))

$$H_n \sim H_1 n^4, \quad H_1 \sim c\hbar\alpha^2\xi_0/e\delta_e^5, \quad H_1 \sim 10^{-1}-10^{-6}\ \text{Oe}. \qquad (5.3a)$$

the n-th level is detached from the continuous spectrum. This means that a new branch of the density of states appears, i.e, singularities are periodically produced in the spectrum at these values of H_n. Their period corresponds to $\delta n = 1$ and in accordance with (5.3a) we have $\delta H^{1/4} \sim H_1^{1/4}$.

With increasing κ, the number of levels decreases, the n-th level vanishes at $\kappa_n \sim \alpha\rho_F/n^2$. Thus, when $1 + \kappa > \alpha\rho_F + \rho_F^2$ there always remains a single level; the condition for applicability of the quasiclassical approach is $\delta\pi \gg 1$, i.e.,

$$1+\kappa \ll \alpha\rho_F+\rho_F^2. \qquad (5.4)$$

(Incidentally, the principal role in many effects connected with the discrete spectrum is played (see Fig. 1) by $\kappa \sim \kappa_F p_y^2/p_F^2 \sim \kappa_F\delta_e/r$, so that $\kappa/\alpha\rho_F \lesssim \alpha\xi_0/\delta_e^2 \ll 1$ and the quasiclassical conditions obtain.) A plot of w_n (see[2]) at $\rho_F \gg \alpha^{-1}$ is shown in Fig. 7A; near $z = \alpha$, the curves come continuously in contact with the boundary of the continuous spectrum (interpolation formula (5.3a) is not applicable in this region).

With increasing magnetic field, at $\rho_F = \rho'$, for the lowest level (at $p_x > 0$) the gap vanishes, and at $\rho_F = \rho''$ with $p_x < 0$ there appear new gapless branches of excitations (see Fig. 4), $\rho' \sim \rho'' \sim \alpha$.

2) $\rho_F \gg \alpha$, i.e. $H \gg H_2$, $H_2 \sim \Phi_0/\xi_0\delta_e$, $\Phi_0 = c\hbar/e$. As can be easily seen from (5.2) and Fig. 5,

$$w_n = \sqrt{z_n+\rho_F} = n+\sqrt{n}\,\kappa^{1/4}, \quad \sqrt{\rho_F} < w_n < \sqrt{\alpha+\rho_F}. \qquad (5.5)$$

New branches appear as before at $\kappa = 0$ (i.e., $p_x = p_F$), when $w_n = \sqrt{\alpha + \rho_F} \sim \sqrt{\rho_F}$. This means that $\rho_F^{(n)} \sim n^2$ and

$$H_n \sim H_4 n^2, \quad H_4 \sim c\hbar\alpha/e\delta_e^3, \quad n_{max} \sim \sqrt{\rho_F}, \quad H_4 \sim 1-100\ \text{Oe}. \qquad (5.6)$$

The period of appearance of the levels is $\delta H^{1/2} \sim H_4^{1/2}$, $\delta H/H \sim (H_4/H)^{1/2} \ll 1$.

The singularity of the density of states at the beginning of the spectrum (at $\kappa = 0$) not only appears in the case $\rho_F \gg \alpha$, but also vanishes at $w_n = \sqrt{\rho_F}$ (when $z_n = 0$). The period of its vanishing is the same as of its appearance. The form of the levels at $p_F > p_x > 0$ is shown in Fig. 7B.

In conclusion, we make a few additional remarks.

1. The form of the energy spectrum in the quasiclassical case can be explained in a much more complicated manner than, for example, in[5], but not more

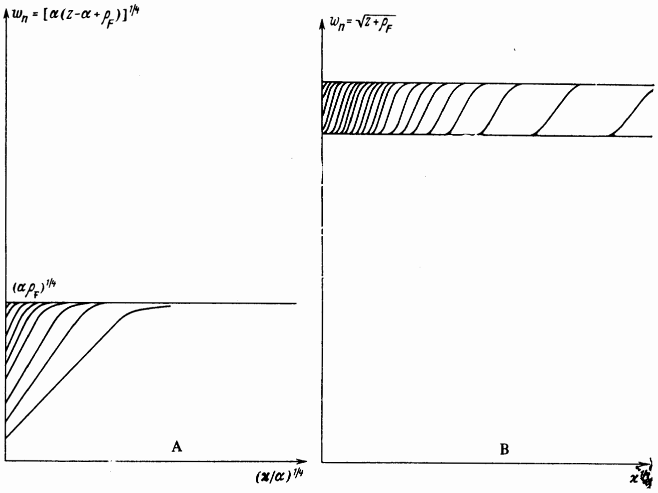

FIG. 7. Plot of w(κ): A—$\rho_F < \alpha$, B—$\rho_F > 2\alpha$.

rigorously. The initial equations in[5] were the quantum equations for the wave functions (which follow from the Gor'kov equations[9]), which were written in the quasiclassical case. The joining together was determined by the transition matrix; its concrete form affected only the next higher approximation (yielding ½ in (5.1)).

2. The obtained levels have a natural width $\delta_i\epsilon$, connected with both the volume and surface collisions:

$$\delta_i\varepsilon \sim \frac{\hbar}{\tau_i}, \quad \frac{1}{\tau_i} \sim \frac{1}{\tau} + \frac{q(\varkappa)}{T}; \tag{5.7}$$

here τ is the volume free path time of the excitations, T is the period of their motion along the surface, and q is the probability of diffuse scattering in the case of surface collision (it decreases with decreasing κ, i.e., with increasing de Broglie wavelength—see[6]).

3. Starting with certain ρ_F, a mixed state arises in the superconductor. This causes the vector potential A to depend not only on y. The entire theory developed above can be readily extended to this case. It is equally easy to take into account the dependence of the gap on the magnetic field and on the coordinates. The point is that in the fundamental approximation all the dependences in the quasiclassical approach should be assumed given (either by the "ordinary" theory, if it yields the fundamental approximation, or by the classical theory, see Item 7).

4. The vanishing of the gap at $\rho_F \sim \alpha$, i.e., at $H \sim c\hbar/e\xi_0\delta_e$, can lead to the appearance of a unique mixed or intermediate surface state. An analysis of this interesting effect is, however, beyond the scope of the present article.

5. The formulas previously written for Δ and j are valid, as is clear from the construction, also in the general quantum case, if the integration is replaced by the sum over the states corresponding to $\epsilon > 0$.

To write down the Schrödinger equation for ψ in the general quantum case it suffices, in accordance with the correspondence principle, to replace p by $-i\hbar\nabla$ in formulas (3.1) and (3.2), and carry out the appropriate symmetrization.

6. For $\epsilon = p^2/2m$, all the equations written out above can be obtained, of course, also from the known formulas[9] for the Green's function by expanding the latter in terms of the eigenfunctions. Just as in the latter, we discard the spin-orbit interaction, the relative contribution of which is of the order of a/δ_e, and the spin paramagnetism.

As usual, $\xi_p \approx v_F(p - p_F)$. However, for a quantum calculation, where $p \to \hat{p}$, it is more convenient, just as in normal metals, to deal with $\hat{p}^2$ and not with $|\hat{p}|$, and to write down (of course, with the same accuracy) $\xi_p = (p^2 - p_F^2)/(2m)$.

7. The condition for quasiclassical behavior in superconductors can be attained also from an equation of the Schrödinger type with a complex potential. To this end, at $\kappa \gg \alpha + |\rho_F|$, it suffices to put in the quantum equation for ψ (see Item 5) $\psi = \exp(i\sqrt{\kappa}\xi)\chi$ and to retain the principal terms in the equations for χ. By the same token, the reasons why the quasiclassical approach holds in a wide range of parameters become clear. In the London case, where $\delta_0 \gg \xi_0$, the reason is natural. In the Pippard case, the effective dimension of the pair decreases as a result of the fact that only the glancing electrons play an important part, since the effective path in the field is increased, and also as a result of the large value of the magnetic field. In very strong fields, the quasiclassical behavior is favored also by the growth of δ_e due to the decrease of the current (with increasing A). As a result, in a wide range of not too weak fields it is possible to write an equation analogous to the Ginzburg-Landau equation. Since in the quasiclassical approach δ_e is the largest characteristic length, a local connection between j and A is ensured, provided the gap is not too small (otherwise it is necessary to have a quasiclassical solution with accuracy to the third essentially nonlocal approximation, and one obtains, in particular, the Ginzburg-Landau equation). Thus, the region of temperatures that are not too close to critical is the simplest for the analysis and leads in the London case to the algebraic equations (3.6), (3.6a), (3.4), and (3.3). In the Pippard case with $\rho \gg \alpha$, perturbation theory in α is valid.

6. OSCILLATORY AND RESONANT EFFECTS

For concreteness, we consider oscillations and resonances of the current density j. The current density connected with the continuous spectrum is well known[8], and there is no need to calculate it anew. We turn to the calculation of the increment due to the quasiclassical discrete levels, which has a singularity. We shall only outline the calculation procedure and present the qualitative results, leaving a detailed discussion of the resonance, as well as of the slowly damped waves and natural oscillations associated with it (analogous, for example, to those in[10]), for a separate communication. We shall also consider separately ultrasonic resonance at discrete levels due to size quantization.

We write j in the physically obvious form[9)]:

[9)]Formula (6.1) can be obtained in the quasiclassical approach, of course, from (3.5); in this case u is determined from (3.4). In the classical case

$$\sum_n \frac{1}{T_s}\cdots \to \int \frac{dn}{T_s}\cdots \infty \int \left(\frac{\partial S}{\partial \varepsilon}\right)^{-1} dS \cdots \to \int d\varepsilon \cdots \to \int \frac{dp_y}{v_y}, \quad \frac{1}{v_y}dt = dy,$$

and (6.1) goes over into (3.5).

$$j = \sum_n \int \frac{1}{T_s} dp_x\, dp_z\, dt\, |u(t)|^2 \delta(y - y(t)) v(t) \operatorname{th} \frac{\varepsilon}{2T}. \quad (6.1)$$

The appearance of a new branch of levels gives in (6.1) a periodically appearing new term. The period of these oscillations is determined in Sec. 5. Their amplitude can be roughly estimated from the following considerations. The "total" current density is determined by an interval of the order of μ, whereas the discrete levels are "located" in an interval of the order of ρ_F (this is easiest to see at $T = 0$ by integrating (3.7) by parts), and each of them "ensures" an interval of the order of ρ_F/n_{max}. Therefore the relative amplitude f of the oscillations is of the order of

$$f \sim \rho_F (\mu n_{max})^{-1}. \quad (6.2)$$

The oscillations of the specific heat and of the gap are of the same order. To obtain the resonant increment of the current density (corresponding to a transition between discrete levels), it is necessary to write j in a constant magnetic field H_0 and an alternating electromagnetic field of frequency ω (the vector potential of the latter is $A_1 e^{i\omega t}$, with $A_1 \to 0$). In this case[10)]

$$\psi_n = \psi_n^{(0)} \exp\left(\frac{-i\varepsilon_n t}{\hbar}\right) + \exp\left(\frac{-i\varepsilon_n t}{\hbar} - i\omega t\right) \sum_m \frac{1}{\omega - \omega_{mn}} a_{mn} \psi_m^{(0)}, \quad (6.3)$$

$$a_{mn} = \frac{1}{T_s} \int_0^{T_s} a_1(t) \exp(i\omega_{mn} t)\, dt, \quad a_1 = \frac{e}{c} v_x A_1, \quad \omega_{mn} = \varepsilon_n - \varepsilon_m. \quad (6.4)$$

Substituting (6.3) in (6.1) and assuming $\hbar\omega_{mn} \ll \Delta$, so that $m \approx n$ and $u_m \approx u_n$, we obtain

$$j = e^{-i\omega t} \sum_{nm} \int \frac{dp_x\, dp_z\, dt}{(\omega - \omega_{mn}) T_s} v(t) \delta(y - y(t)) |u(t)|^2 \exp(-i\omega_{mn} t) \operatorname{th} \frac{\varepsilon}{2T}. \quad (6.5)$$

The case most favorable for observation of resonance is that of a crylindrical equal-energy surface. The resonance, as is usual in the case of the presence of one continuous parameter, takes place at $\omega = \omega_{mn}$; ω_{mn} corresponds to the edge of the spectrum or to the maximum density of states: $\partial\omega_{mn}/\partial p_x = 0$. The amplitude of the resonant increment to the current density is small—proportional to $\Delta\omega\tau_i/\mu$. However, its contribution to the derivative of the surface impedance with respect to the magnetic field is of the order of

$$\delta Z'(H) / Z'(H) \sim (\omega\tau_i)^2 \Delta / \mu, \quad (6.6)$$

where τ_i is determined by formula (5.7) and can apparently be observed experimentally in sufficiently pure samples with good surfaces. The effect is easiest to observe in Pippard superconductors, where only electrons glancing at an angle on the order of δ_e/ξ_0 to the surface are significant; this decreases greatly the nonresonant part of the impedance (and increases the relative contribution of the resonant part, for which only glancing electrons are always important).

7. QUANTIZATION IN RELATIVELY WEAK FIELDS

So far we have considered quantization of levels in the quasiclassical case, when the wave function in the classically inaccessible region attenuates over distances that are small compared with the characteristic distances of the problem.

Let us consider the opposite limiting case of slow damping of a wave function[2], where it is necessary to have the quantum equation for the wave function. To simplify the derivations, we write it in the case $\epsilon = p^2/2m$. In the dimensionless coordinates (4.1) we have

$$\left(\frac{d^2}{d\xi^2} + \varkappa\right)\psi + \begin{pmatrix} z & \alpha \\ -\alpha & -z \end{pmatrix}\psi = -\rho\theta(\xi)\hat{\sigma}_z\psi \equiv \sigma_z f(\xi), \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.1)$$

We confine ourselves in the analysis to the most interesting region of a strictly discrete spectrum (for given κ and ρ), when the wave function attenuates at infinity. In order to ensure immediately the satisfaction of the boundary condition corresponding to specular reflection, $\psi(0) = 0$, we continue all the functions to the negative semi-axis: $\psi(-\xi) = -\psi(\xi)$, and take the Fourier transform of (7.1). We obtain

$$\psi(\xi) = -\int_0^\infty \frac{\sin(k\xi)\, dk}{(k^2 - \varkappa)^2 + (\alpha^2 - z^2)} \begin{pmatrix} -k^2 + \varkappa - z & -\alpha \\ \alpha & -k^2 + \varkappa + z \end{pmatrix} \tilde{f}(k),$$
$$\tilde{f}(k) = \int_0^\infty \sin(k\xi) f(\xi)\, d\xi. \quad (7.2)$$

We are interested in a case opposite to the quasiclassical one, when there is a single level close to the boundary of the "old" spectrum[11)] (i.e., the spectrum at $H = 0$), where $z = \alpha$ at $\kappa > 0$ and $z = \tilde{\kappa}$ at $\kappa < 0$. Let us consider, for concreteness, the former case.

When $z = \alpha$ (and $\kappa > 0$) the denominator is equal to $(k - \sqrt{\kappa})^2 (k + \sqrt{\kappa})^2$, so that there is a divergence at $k = \sqrt{\kappa}$. This means that the main contribution to the integral (7.2) is made by $k \approx \sqrt{\kappa}$, and we can put $k = \sqrt{\kappa}$ everywhere, with the exception of the denominator and $\sin(k\xi)$. We obtain

$$\psi(\xi) = \begin{pmatrix} z & \alpha \\ -\alpha & -z \end{pmatrix} \tilde{f}(\sqrt{\varkappa}) I(\xi), \quad I(\xi) = \frac{\pi \operatorname{Im}[\lambda^{-1} e^{i\lambda\xi}]}{2\sqrt{\alpha^2 - z^2}}, \quad (7.3)$$

Multiplying (7.3) by $-\rho\theta(\xi)$ and taking the Fourier component, we obtain a homogeneous equation for $f(\sqrt{\kappa})$, which determines (accurate to a normalization factor) both $\tilde{f}(\sqrt{\kappa})$ (i.e., according to (7.3), $\psi(\xi)$) and the discrete level:

$$\frac{1}{2}\rho(\alpha + z) \int_0^\infty \theta(\xi) I(\xi) \sin(\sqrt{\varkappa}\,\xi)\, d\xi = 1. \quad (7.4)$$

---

10)Formulas (6.3) and (6.4) follow from perturbation theory ($A_1 \to 0$) for the equations $i\hbar\dot{\psi} = \mathscr{H}\psi = \mathscr{H}_0\psi + a_1\psi e^{i\omega t}$. Writing ψ in the form (6.3), we obtain a_{mn}, which coincides in the quasiclassical approach with the Fourier component with respect to the classical trajectory (see [1], Sec. 48; the proof is repeated verbatim for the vector wave function). We note immediately that when $m \approx n$

$$\int u_m^* h u_n\, dy \approx |u_m|^2 \frac{1}{T} \int \exp(i\omega_{mn} t)\, h(t)\, dt,$$

where u_m is the "classical value" of u, obtained in Sec. 3. For the proof, it is necessary to obtain a formula for the quasiclassical vector wave function. Replacing in (3.2) $\mathbf{p} \to -i\hbar\nabla$ and putting

$$\psi = (F_c + F_1) \exp\left\{\frac{i}{\hbar} \int (p_c + ip_1)\, dy\right\},$$

where $F_1 \ll F_c$ and $p_1 \ll p_c$, we obtain (determining p_1 from the condition for the solvability of the inhomogeneous equation) the natural quasiclassical formula

$$\psi = \frac{F_c}{\sqrt{v_H}} \exp\left(\frac{i}{\hbar} \int p_c dy\right), \quad v_H = \frac{\partial\varepsilon_H}{\partial p_y},$$

where F_c and p_c are determined by (3.3) and (3.4).

11)For the case of slow attenuation of ψ, which is of interest to us, the poles must be "almost real," and real poles, i.e., purely oscillating ψ, correspond, as can be readily seen, to the boundary of the spectrum in the absence of an external field.

From this we see immediately that the local level exists only when $\rho > 0$, i.e., when $p_x > 0$, and it is determined qualitatively by the formula

$$\alpha - z \sim \begin{cases} \alpha\rho^2/\varkappa, & \varkappa \gtrsim 1 \\ \alpha\rho^2(\varkappa + \alpha^2\rho^2), & \varkappa \lesssim 1 \end{cases}. \tag{7.5}$$

The condition for the applicability of the theory, according to (7.5), is $\alpha\rho \ll \kappa + 1$ (compare with (5.4)).

For the existence of the obtained level, it is very important, as always for a shallow potential well, that the problem be one-dimensional. Essential violation of one-dimensionality (poor surface, mixed or intermediate state) makes the level exponentially close to α in the two-dimensional case and causes it to disappear completely in the three-dimensional case.

8. ENERGY SPECTRUM IN THE PINCUS MODEL

In Secs. 5 and 7 we obtained the energy levels in superconductors in weak ($\alpha\rho \ll \kappa + 1$) and strong ($\alpha\rho \gg \kappa + 1$) magnetic fields. It is of interest to show in conclusion how to obtain the spectrum in arbitrary fields in the idealized model of Pincus[3], where $\theta(\xi) = 1$ when $\xi < 1$ and $\theta(\xi) = 0$ when $\xi > 1$. We confine ourselves, for simplicity, to the discrete spectrum.

In the region $\xi > 1$ the attenuating solution of (7.1) is

$$\psi = (u, v) = \sum_\sigma d_\sigma(-1, \varphi_\sigma)\exp(-\xi\delta_\sigma),$$
$$\varphi_\sigma = (z - \sigma\tilde{v})/\alpha, \quad \tilde{v} = \sqrt{z^2 - \alpha^2}, \quad \delta_\sigma = \sqrt{-\varkappa - \sigma\tilde{v}}, \quad \sigma = \pm 1. \tag{8.1}$$

When $\xi = 1$, the wave function is continuous together with its first derivative. To write this condition in a convenient form, it is meaningful to find from (8.1) the quantities $d_\sigma \exp(-\xi\delta_\sigma)$ and to write their logarithmic derivatives at $\xi = 1$. As a result we get

$$(\varphi_{-\sigma}u' + v') + \delta_\sigma(\varphi_{-\sigma}u + v) = 0. \tag{8.2}$$

When $\xi < 1$ a solution of (7.1), satisfying $\psi(0) = 0$, is

$$\psi = \sum_\sigma h_\sigma(-1, \tilde{\varphi}_\sigma)\sin(\xi\tilde{\delta}_\sigma), \quad \tilde{\varphi}_\sigma = \alpha^{-1}(z + \rho - \sigma\sqrt{(z+\rho)^2 - \alpha^2}),$$
$$\tilde{\delta}_\sigma = (\tilde{\varkappa} + \sigma\sqrt{(z+\rho)^2 - \alpha^2})^{1/2}. \tag{8.3}$$

Substituting (8.3) in (8.2), we obtain the dispersion equation

$$\left[1 + \frac{\delta_+}{\tilde{\delta}_+}\operatorname{tg}\tilde{\delta}_+\right]\left[1 + \frac{\delta_-}{\tilde{\delta}_-}\operatorname{tg}\tilde{\delta}_-\right]$$
$$= \left[1 + \frac{\delta_+}{\tilde{\delta}_-}\operatorname{tg}\tilde{\delta}_-\right]\left[1 + \frac{\delta_-}{\tilde{\delta}_+}\operatorname{tg}\tilde{\delta}_+\right]\frac{z - \sqrt{z^2 - \alpha^2}}{z + \sqrt{z^2 - \alpha^2}}, \tag{8.4}$$

which leads in limiting cases qualitatively to the results of Secs. 5 and 7. The Pincus model is very convenient also for the exact solution of the Gor'kov equation[9] and for obtaining the results of Sec. 6 from the Green's function.

I. A phenomenological theory of superconductivity was constructed.

II. The energy spectrum of a superconducting half-space $y \geq 0$ in a constant magnetic field $\mathbf{H} \parallel z$ was obtained.

1. In extremely weak magnetic fields $H < H_1$, $H_1 \sim \Phi_0 a^2\xi_0/\delta_e^5 \sim 10^{-1}$–$10^{-6}$ Oe at $p_x > 0$, a branch $\epsilon = \epsilon_1(p_x, p_z)$ of discrete levels (for given p_x and p_z) is detached from the boundary of the continuous spectrum. Its distance $\delta\epsilon$ from the boundary is maximal at $|1 - p_\perp/p_F| \sim (a/\delta_e)^2$ and $\delta\epsilon_{max} \sim (\tilde{\epsilon}^2/\Delta)(H/H_1)^2$, $\tilde{\epsilon} = \hbar^2/m\delta_e^2$ and decreases rapidly with increasing $|p_F - p_\perp|$—see formulas (7.4), (7.5), (7.3), and (4.1). (Notation: $\Phi_0 = \hbar c/e$, $a = \hbar/p_F$, $\xi_0 = \hbar v_F/\Delta$, $p_\perp = \sqrt{p_x^2 + p_z^2}$, p_F and v_F—limiting Fermi momentum and velocity, Δ—gap, δ_e—characteristic depth of attenuation of the field.)

2. With increasing magnetic field, the branch of the discrete levels moves away from the continuous spectrum. When $H_1 < H < H_2$, $H_2 \sim \Phi_0/\xi_0\delta_e$ and $p_\perp = p_F$, new branches of levels are detached from the continuous spectrum periodically, at $H = H^{(n)} \sim H_1 n^4$, $n = 1, 2, \ldots,$. These levels correspond, as before, to $p_x > 0$ and come in contact with the continuous spectrum when $|1 - p_\perp/p_F| \sim (aH/n\delta_e H_1)^2$. The number of levels n_0 at $p_\perp = p_F$ is of the order of $n_0 \sim (H/H_1)^{1/4}$, and the distance $\delta\epsilon_n$ between them is $\delta\epsilon_n \sim (n/n_0)^3\sqrt{\Delta\tilde{\epsilon}}(H/H_2)^{3/4}$. The form of the spectrum is determined by formulas (5.3), (5.1), (4.2), and Fig. 7A, 3.

3. When $H > H_2$, the periodic splitting of the branches occurs at $H = H^{(n)} \sim H_4 n^2$, $H_4 \sim \Phi_0 a/\delta_e^3$. The number of branches is of the order of $(H/H_4)^{1/2}$, and the maximum distance between them can be of the order of Δ. The branches not only appear, but also vanish, coalescing with the ground state ($\epsilon = 0$). When $p_x < 0$, branches of discrete levels are also detached from the ground state. The form of the spectrum is determined by formulas (5.5), (5.6), (5.1) and by Fig. 7B, 4, 5.

The appearance of gapless excitations in fields $H > H_2$ can lead to a mixed or intermediate surface state.

4. For large numbers of levels, and also in the continuous spectrum at $\delta_e \gg h/\delta p_y$ (δp_y—change of p_y on moving along a classical trajectory), classical excitations have been introduced with a dispersion law $\epsilon = \pm(\xi_p^2 + \Delta^2)^{1/2} - ec^{-1}A\partial\xi/\partial p$, $\epsilon > 0$. The corresponding orbits in phase space, for different values of the parameters, are shown in Fig. 2. Superconductors are characterized by the presence of a great variety of trajectory types (particularly, finite and infinite), which are accompanied in many cases by total internal reflection of the excitations from a large magnetic-field gradient, with a transition of an electronic type of excitation (pluson) into a hole type (minuson). For excitations going from the interior of the superconductor, this is equivalent to specular reflection of the excitation from the surface.

5. Quasiclassical quantization is determined by formula $S = (n + 1/2)h$, where S is the area of the classical phase trajectory of the excitations, and $n \gg 1$ is an integer. Qualitatively, quantization corresponds to the quantization of δp_y, viz., $\delta p_y \sim hn/\delta_e$.

Discrete levels arise also as a result of size quantization (formula (5.1a)) of excitations that are specularly reflected from the surface (see Item 4).

6. The penetration of a magnetic field into a superconductor is accompanied by the occurrence of an electrostatic potential difference $\delta\varphi$ between the surfaces of the superconductor: $\delta\varphi \sim E\delta_e \sim \Delta a/e\xi_0$.

7. In fields $H \gg H_3$, it is possible to construct a quasiclassical theory of superconductivity and write down equations that are generalizations of the equations of Ginzburg-Landau to not too weak magnetic fields at

all temperatures, and make it possible to study the surface mixed or intermediate state (see Item 3). For Pippard superconductors, $H_3 \sim H_2 \ll H_c$. For London superconductors, in calculating the current, $H_3 \sim \Phi\xi_0/\delta_e^3$ if the temperature is $T < ec^{-1}v_F H\delta_e$, and $H_3 \sim (\Phi_0/\delta_e^2)\sqrt{T/\Delta} \sim H_{c1}\sqrt{T/\Delta}$ if $T > ec^{-1}v_F H\delta_e$. In calculating the gap, $H_3 \sim \Phi_0/\delta_e^2$. In the simplest London case, the quasiclassical equations of superconductivity are (3.7), (3.5a), (3.4), and (3.3).

III. The discreteness of the levels and the nonperiodicity of their appearance lead to quantum oscillations of the thermodynamic and kinetic quantities (of the gap, specific heat, magnetic moment, surface impedance, etc.), to resonances (electromagnetic and ultrasonic), the natural oscillations, and to waves that attenuate weakly in the superconductor.

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