THE POSSIBILITY OF SETTING UP REGULAR COSMOLOGICAL SOLUTIONS

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A homogeneous and isotropic cosmological model is considered. The possibility is investigated of setting up regular solutions of the gravitation equations in the presence of a nonlinear increment of the 4-curvature to the Lagrangian density of the gravitational field. An exact solution without singularities is obtained in the case when the Lagrangian density takes the form $L = (R + \alpha R^{4/3})$. With a suitable choice of the integration constants, the solution asymptotically goes over into the Friedmann solution.

1. INTRODUCTION

A characteristic feature of the existing cosmological models is their boundedness in time, due to the influence of a singularity in the space-time metric. The origin of this singularity was initially connected with the assumption of homogeneity and isotropy of the models under consideration. However, the latest papers[1-3] lead to the conclusion that one cannot exclude the presence of a singularity even in the general solution of Einstein's equation, i.e., in the metric of an arbitrary (inhomogeneous and anisotropic) model of the universe.

In this connection, interesting possibilities are uncovered by the paper of Sakharov[4], who advanced physical evidence favoring the existence of increments that are linear in the 4-curvature for the Lagrangian density of the gravitational field. The corresponding gravitation equations are of higher order than Einstein's equations[9], and could contain cosmological solutions with regular transition from contraction to expansion. Such a possibility was noted by Zel'dovich and Novikov[5] and was investigated in detail by the Ruzmaĭkins in[6] for a concrete form of the Lagrangian density of the gravitational field, proposed in[4]:

$$L(R) = L(0) + AR + BR^2 + CR_{ik}R^{ik} + DR_{iklm}R^{iklm} + ER_{iklm}R^{ilkm}. \quad (1)$$

It was shown that in the case of a homogeneous and isotropic model of the universe with a flat co-moving space, a quadratic increment to L makes it possible to eliminate the singularity at the instant of maximum compression (at $t = 0$). However, a solution that is regular at the point $t = 0$ leads to the divergence of R either at $t \to +\infty$ or at $t \to -\infty$. Thus, within the framework of the assumptions made in[6], there is no solution that is regular at all values of t for the gravitational equations corresponding to the Lagrangian density (1). This leaves also open the question of the possibility of constructing such an L, for which the gravitation equations admit of a regular cosmological solution that goes over into the Friedmann solution as $t \to \infty$.

In this paper we present an example of a Lagrangian density with the indicated properties, and obtain exact solutions corresponding to a homogeneous and isotropic model of the universe.

2. FUNDAMENTAL EQUATIONS

We consider the action of a gravitational field in the form

$$S_g = A\int[R + l^{-2}f(l^2R)]\sqrt{-g}\,d\Omega, \quad (2)$$

where $f(l^2R)$ is a certain function of the scalar curvature and l is the characteristic length. Variation of Eq. (2) together with the action for matter yields the following gravitation equations, which go over into the Einstein equations in the case $f \equiv 0$:

$$R_i^k - \frac{1}{2}\delta_i^kR + l^{-2}\left\{\frac{\partial f}{\partial R}R_i^k - \frac{1}{2}f\delta_i^k + (\delta_i^kg^{mn} - \delta_i^mg^{kn})\left(\frac{\partial f}{\partial R}\right)_{;m;n}\right\} = \varkappa T_i^k. \quad (3)$$

Using the Bianchi identity and the rule for interchanging the indices of covariant differentiation, we can easily show that relations (3) contain the equations of motion for matter:

$$T_{i;k}^k = 0. \quad (4)$$

We investigate Eqs. (3) for the case of a homogeneous and isotropic model of the universe with a flat co-moving space:

$$ds^2 = d\tau^2 - b^2(\tau)(dx^2 + dy^2 + dz^2), \quad (5)$$

where $\tau = ct$. We shall describe matter by the equation of state

$$p = \varepsilon/3. \quad (6)$$

At a chosen metric, the system (3) contains only two independent equations ($i = k = 1$, $i = k = 0$). In lieu of the first, it is convenient to use the well known integral of the equations of motion

$$\varkappa\varepsilon = 3b^{-4}l_1^{-2} \quad (7)$$

(l_1 is the integration constant). Then the second equation ($i = k = 0$) defines the $b(\tau)$ dependence:

$$R_0^0 - \frac{1}{2}R + l^{-2}\left\{\frac{\partial f}{\partial R}R_0^0 - \frac{1}{2}f + \frac{3\dot b}{b}\frac{\partial}{\partial\tau}\left(\frac{\partial f}{\partial R}\right)\right\} = 3b^{-4}l_1^{-2}, \quad (8)$$

$$R_0^0 = -3\ddot b/b, \quad R = -6(\ddot b/b + \dot b^2/b^2), \quad \dot b = db/d\tau. \quad (9)$$

The order of Eq. (8) can be lowered by introducing in place of $b(\tau)$ a new unknown function

$$y = l_1^2b^2\dot b^2, \quad (10)$$

and by choosing b as the independent variable. Knowing that in terms of the new variables we have

$$R = l_1^{-2}\rho = l_1^{-2}(-3b^{-3}y'), \tag{11}$$

where $y' = dy/db$, we get from (8)

$$y + \left(\frac{l_1}{l}\right)^2 \left\{ \frac{\partial f}{\partial \rho}\left(y - \frac{b}{2}y'\right) - \frac{b^4}{6}f + yb\frac{\partial}{\partial b}\left(\frac{\partial f}{\partial \rho}\right)\right\} = 1. \tag{12}$$

The last relation can be written also in the form of the Lagrange equation

$$\frac{d}{db}\frac{\partial \mathscr{L}}{\partial y'} - \frac{\partial \mathscr{L}}{\partial y} = 0 \tag{13}$$

with a Lagrangian function

$$\mathscr{L} = (l_1/l)^2 b^4 y^{-1/2} f + 6(y^{1/2} + y^{-1/2}), \tag{14}$$

where

$$f = f(-3b^{-3}y') = f(\rho).$$

In accordance with the statements made in the introduction, we shall seek solutions of the system (10) and (12), describing a continuous transition from compression to expansion at the instant $\tau = 0$. In other words, the function $b(\tau)$ should have at the indicated point a regular minimum:

$$b(\tau) = b_0 + \ddot{b}_0\tau^2/2 + \dddot{b}_0\tau^3/6 + \ldots \tag{15}$$

This makes it possible to formulate boundary conditions for Eq. (12) at the point $b = b_0$. We note beforehand that owing to the non-monotonicity of the function $b(\tau)$, a single-valued $y(\tau)$ dependence corresponds to a double-valued $y(b)$ dependence:

$$y[b(\tau)] = \begin{cases} y_+(b), & \tau > 0 \\ y_-(b), & \tau < 0 \end{cases}. \tag{16}$$

The boundary conditions for y_+ and y_- at the point $b = b_0$ follow from Eqs. (10) and (15), and are given by

$$\begin{gathered} y_+(b_0) = y_-(b_0) = 0, \\ (y_+')_{b=b_0} = (y_-')_{b=b_0} = 2l_1^2 b_0^2 \dddot{b}_0, \\ \lim_{b\to b_0}(y_+^{1/2}y_+'') = -\lim_{b\to b_0}(y_-^{1/2}y_-'') = 2l_1^3 b_0^3 \bar{b}_0. \end{gathered} \tag{17}$$

According to (17), the term containing the highest-order derivative in (12) vanishes at the point $b = z_0$, i.e., there is a unique connection between $y'_\pm$ and b_0 or, what is the same, between $\ddot{b}_0$ and b_0:

$$\left(\frac{1}{2}by_\pm'\frac{\partial f}{\partial \rho} + \frac{b^4}{6}f + l^2 l_1^{-2}\right)_{b=b_0} = 0. \tag{18}$$

This is precisely the reason for the need for specifying the third derivative $\dddot{b}_0$ at the point $\tau = 0$ for the third-order equation (8).

The problem now consists of determining the form of the function f, at which there exists a solution of (12) with boundary conditions (17), without leading to a singularity in the physical quantities at all values of τ (including $\tau = \pm\infty$).

3. REGULAR COSMOLOGICAL SOLUTION

In this section we shall stop to investigate in greater detail Eq. (12) with a power-law $f(R)$ dependence[1)]:

$$f(R) = \alpha R^n, \quad \alpha = l^{2n}. \tag{19}$$

We choose here $\alpha > 0$, for otherwise it is impossible

1) Such a nonlinear increment to the Lagrangian density of the gravitational field in vacuum was considered earlier in [7] in order to ascertain the possibility of a limited transition to Newton's law.

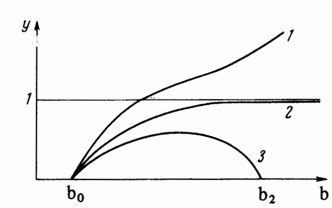


FIG. 1

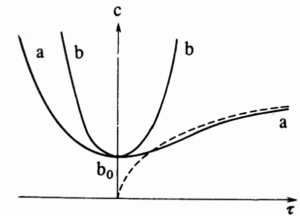


FIG. 2

FIG. 1. Types of solutions $y_+(b)$ as functions of the constant C_+: 1–$C_+ > -1$, 2–$C_+ = -1$, 3–$C_+ < -1$.

FIG. 2. Plot of $b(\tau)$: a–solution that goes over into the Friedmann solution as $\tau \to +\infty$, b–time-symmetrical cosmological solution (the dashed line shows the Friedmann solution).

to satisfy the boundary conditions (17) on the function $y_\pm$ at the point $b = b_0$. We write out the Lagrangian (14) and the equation (13) that follows from it, using formula (19) for f:

$$\mathscr{L} = \gamma b^{4-3n} y^{-1/2}(y')^n - 2(y^{1/2} + y^{-1/2}), \tag{20}$$

$$y + \frac{\gamma}{b^2}\left(\frac{y'}{b^3}\right)^{n-2}\left\{n(n-1)yy'' + \frac{(1-n)}{2}y'^2 + n(4-3n)\frac{yy'}{b}\right\} = 1 \tag{21}$$

$$\gamma = (-3)^{n-1}(l/l_1)^{2n-2}. \tag{22}$$

When $n = 4/3$, it is possible to obtain an exact solution of this equation and to construct a cosmological model with the properties indicated in the introduction. Indeed, if $n = 4/3$ the Lagrangian (20) does not depend explicitly on the "time" b, and consequently Eq. (21) possesses an "energy" integral, which we write in the form

$$(y_+')^{4/3} = (8\beta)^{4/3}(y_+ + 2C_+ y_+^{1/2} + 1), \quad \beta^2 = \frac{3\sqrt{2}}{32}\frac{l_1}{l}. \tag{23}$$

The constant C_+ which enters here is determined, in accordance with the boundary conditions (17), by the values of b_0 and $\dddot{b}_0$:

$$C_+ = \frac{1}{24\beta^2}(l_1 b_0)^3 \dddot{b}_0. \tag{24}$$

The solutions of (23) which correspond to different values of the constant C_+ are shown in Fig. 1. A plot of the function y_- at the same "initial" conditions is obtained by replacing C_+ by $C_- = -C_+$ in formula (23). As can be seen from the figure, depending on the value of C_+, there are three qualitatively different forms of the solution y_+ emerging from the point $b = b_0$, $y = 0$, with identical derivatives.

When $C_+ > -1$, the solutions increase without limit with increasing b in accordance with the asymptotic equation

$$y_+ = (2\beta b)^4\{1 - O(b^{-2})\}, \quad b \to \infty. \tag{25}$$

This corresponds (see (11)) to the scalar curvature R tending to a constant negative value:

$$R = -(3/2)^3 l^{-2}. \tag{26}$$

2. When $C_+ = -1$, the function approaches unity asymptotically, i.e., the solution goes over, with increasing b, into the Friedmann solution ($y = l_1^2 \dot{b}^2 b^2 = 1$, $b = \sqrt{2l_1^{-1}\tau}$[8]).

3. If $C_+ < -1$, then the solution, after going through a maximum ($y_{max} < 1$), again vanishes at a certain $b = b_2$.

4. Finally, in the case when the roots of the quadratic trinomial in (23) are real, Eq. (23) has the following singular solutions:

$$y_+ = (-C_+ \pm \sqrt{C_+^2 - 1})^2, \quad |C_\pm| > 1. \tag{27}$$

Let us consider first a cosmological model that goes over into the Friedmann model as $\tau \to \infty$. Such a solution exists (see Sec. 2) at $C_+ = -1$. Let us trace qualitatively the behavior of this model in the entire range of variation of τ from $-\infty$ to $+\infty$. At $\tau < 0$ the solution is determined by the constant $C_- = -C_+ = 1$, i.e., it is described by a curve of type 1. As $\tau \to -\infty$ ($b \to \infty$) this solution, in accordance with formula (26), is equivalent to the Friedmann solution with a cosmological constant $\Lambda = -2(3/4)^3 l^{-2}$. When τ increases from $-\infty$ to $\tau = 0$, the solution y_- approaches the point of maximum compression (see Fig. 1). Then, when τ varies from 0 to $+\infty$, the solution y_+ emerges from this point along a curve of type 2, approaching unity asymptotically:

$$y_+ = 1 - (2\beta^2 b^2)^{-1}, \quad b \to \infty. \tag{28}$$

It is likewise easy to obtain from (23) and (27) the exact form of the plots of $y_\pm(b)$ and $b(\tau)$:

$$(1 \mp y_\pm^{1/2})^{1/2} + (1 \mp y_\pm^{1/2})^{-1/2} = 2\beta(b - b_0) + 2, \tag{29}$$

$$\pm(b^2 - b_0^2) + \beta^{-2}\{[\beta(b + b_0) - 1] \times [\beta^2(b - b_0)^2 + 2\beta(b - b_0)]^{1/2} + \operatorname{arch}[\beta(b - b_0) + 1]\} = 4\tau l_1^{-1}. \tag{30}$$

For positive and negative τ we choose here, the upper and lower signs respectively. A plot of $b(\tau)$ is shown in Fig. 2, curve a. The constants b_0 and l_1 in formulas (29) and (30) are connected with the energy density at the point of maximum compression ϵ_0 by the relation (7). Thus, at a given value of ϵ_0 the solution $b(\tau)$ is defined accurate to an arbitrary constant which, as in the Friedmann solution, corresponds to the leeway in the choice of the scale of b, and therefore does not enter in the relations between the observed quantities[2)].

We note that the solution that goes over into the Friedmann solution as $\tau \to \infty$, as seen from the figure, is not symmetrical with respect to the reversal of the sign of the time. A cosmological model having such a symmetry is obtained by choosing $C_+ = -C_- = 0$ and accordingly $\dddot{b}_0 = 0$:

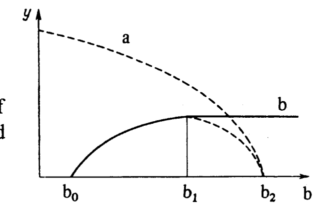
$$y = [2\beta(b - b_0) + 1]^4 - 1. \tag{31}$$

The corresponding form of the $b(\tau)$ dependence is shown in Fig. 2, curve b. When $|\tau| \to \infty$ this solution goes over into the Friedmann solution with a Λ term that coincides in magnitude with that calculated above.

Both solutions investigated above are continuous together with their derivatives with respect to τ. In addition to them there is also a number of regular cosmological solutions with $C_+ > -1$. All these solutions, in spite of the very complicated dependence of $y_\pm(b)$, are qualitatively similar to those considered above, and are therefore not presented here.

Assume now that the initial conditions are such that

FIG. 3. The solution $y_+(b)$ obtained by joining together the solutions of type 3 and 4 at the point b_1 (the dashed line shows the solution of type 3, continued analytically beyond the point b_2).



the function y_+ is of type 3. Then there exists no solution that is regular for all values of τ, since analytic continuation of the solution beyond the point $b = b_2$ (see Fig. 3, curve a) at a certain $\tau = \tau_0$ leads inevitably to a singularity ($b(\tau) \sim \sqrt{\tau - \tau_0}$). However, in the case considered here it is possible to construct a physically reasonable solution by foregoing the continuity of the higher derivatives of $b(\tau)$. We use to this end a function y_+ "pieced together" from solutions of type 3 and 4 at the maximum of curve 3, as shown in Fig. 3, curve b. At the joining point, in accordance with formula (23), the three first derivatives with respect to b are equal to zero, and d^4y_+/db^4 experiences a finite jump. As follows from the definition (10) of y, this is equivalent to a jump of only the fifth derivative $b_\tau^{(V)}$. Since the initial system for $b(\tau)$ is of fourth order, we are dealing here with an exact solution of the gravitational equations. Such solutions exist for all cases of type 3 and denote that the Friedmann solution is reached at a finite τ. When $\tau < 0$ these solutions are described by solutions of type 1.[3)]

Thus, the nonlinear increment $f = \alpha R^{4/3}$ to the Lagrangian density of the gravitational field makes it possible to construct cosmological models without singularities for all values of τ. We shall show that the expression given above for f is in this sense not unique. Indeed, even in the case of a power-law form of the nonlinear increment $f = \alpha R^n$ there exists an interval of values of n, for which Eq. (21) admits of solutions that are qualitatively similar to those analyzed above.

To determine this interval we stipulate that Eq. (21) admit of solutions that increase as $b \to \infty$ no faster than the asymptotic form (25) for $n = 4/3$. (In the opposite case, in accordance with formula (11), the scalar curvature R will diverge.) This puts an upper limit on the degree of n in (19): $n < 2$. The lower limit can be obtained from the condition that the equation admit of solutions that tend asymptotically to the Friedmann solution in accordance with a power law, analogous to the case of $n = 4/3$:

$$y = 1 - Ab^{-a}. \tag{32}$$

Here a and A are of the form

$$a = 4\frac{n-1}{2-n}, \quad A = \left[\frac{2-n}{(5n-6)na^{n-1}|\gamma|}\right]^{1/(n-2)}, \tag{33}$$

which yields the necessary conditions for the existence of the indicated asymptotic forms:

$$6/5 < n < 2. \tag{34}$$

Computer calculations have shown a qualitative

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2) Unlike the Friedmann solution, where the energy density is determined only by the world constant and by the time $\epsilon = 3(4\kappa\tau)^{-1}$, in the solution considered here it depends also on ϵ_0, the influence of which on $\epsilon(\tau)$ vanishes only asymptotically as $\tau \to \infty$.

3) We point out that the construction of the indicated solutions is possible because the joining point is a branch point of the solutions of the equation. The latter results from the non-analytic character of the nonlinear increment $f = \alpha R^{4/3}$ at $R = 0$.

similarity between the solutions in the interval (34) and the solutions considered above, and the upper limit for n is confirmed by the result of[6], where a solution that diverges as $b \to \infty$ increases more rapidly than b^4 and leads to a divergence of R.

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