INTERACTION OF HIGH-FREQUENCY AND LOW-FREQUENCY WAVES IN NONLINEAR DISPERSIVE MEDIA. II

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A study is reported of time-independent states consisting of sets of interacting high-frequency and low-frequency waves in nonlinear dispersive media in the presence of an external high-frequency monochromatic force. Cases of ordinary and parametric excitation of waves by the external force in the medium are discussed. The excitation of Langmuir and ion-acoustic plasma oscillations by the external field is considered as an example. The results obtained in this way can be used to describe existing experimental data.

INTRODUCTION

THE dynamics of the interaction of high-frequency and low-frequency waves in nonlinear dispersive media without taking into account dissipation and with no external forces was investigated in[1]. The present paper is concerned with the analysis of interacting high-frequency and low-frequency oscillations in a medium, taking into account dissipation and the presence of external forces.

When a distributed external force is applied to a dissipative medium it sets up time-independent oscillations whose intensity and spectrum depend on the nature and intensity of the external force and on the properties of the medium. It is clear that the frequencies and wave vectors of the steady-state oscillations may not be equal to the frequencies and wave vectors of the applied force although they will, in general, be functions of them.

For given parameters of the applied force there may be a few (not less than one) time-independent states. The problem then is to determine the amplitudes and the dispersion relations for the time-independent states and to establish their stability. In contrast to the well-known problem on the steady-state oscillations of a system with one degree of freedom under the influence of an applied periodic force (see for example[2]), here we have to determine multifrequency time-independent states of a system with an infinite number of degrees of freedom. We shall investigate time-independent states consisting of sets of interacting high-frequency and low-frequency waves (u and v waves) in the presence of an applied high-frequency monochromatic force.

The applied force excites u waves which in turn may generate low-frequency v and u waves at composite frequencies. We shall consider the cases of ordinary and parametric excitation of u waves by the applied force. (In the case of ordinary excitation, the applied force appears on the right-hand side of the equation motion whereas in the case of parametric excitation it appears in the coefficients of these equations.) We shall derive the time-independent states and will investigate their stability. We shall analyze in detail the time-independent states near the instability threshold for the single-frequency time-independent state.

As an example, we shall discuss in the last section the excitation of Langmuir and ion-acoustic plasma oscillations by a monochromatic external field. We shall show that our results are in good agreement with experimental data[3] which until now could not be satisfactorily explained.

1. BASIC EQUATIONS

As in[1], we shall consider a set of interacting u and v waves which are such that the wave vectors and frequencies of the former are much greater than the wave vectors and frequencies of the latter:

$$\omega \gg \Omega, \qquad k \gg \varkappa. \tag{1.1}$$

In the presence of a high-frequency applied force, and ignoring the damping of the waves, the Hamiltonian for the system can be written in the form

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}} u_{\mathbf{k}}^* u_{\mathbf{k}} + \frac{1}{2}\sum_{\varkappa} \Omega_{\varkappa} v_{\varkappa}^* v_{\varkappa} + \frac{1}{2}\sum_{\mathbf{k}} V_H(\mathbf{k}, \mathbf{k}', \varkappa) u_{\mathbf{k}}^* u_{\mathbf{k}'} v_{\varkappa} \times\delta(\mathbf{k}-\mathbf{k}'-\varkappa) + \sum_{\mathbf{k}} f_{\mathbf{k}}(t) u_{\mathbf{k}}^* + \sum_{\mathbf{k},\mathbf{k}',\mathbf{k}''} g_{\mathbf{k}}(t) u_{\mathbf{k}'}^* u_{\mathbf{k}''}^* \delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'') + \text{c.c.} \tag{1.2}$$

In this expression, $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are the Fourier components of the u and v fields, $\mathbf{k}$, κ, $\omega_{\mathbf{k}}$, Ω_{κ} are the wave vectors and frequencies of the u and v waves, $V_H(\mathbf{k}, \mathbf{k}', \kappa)$ are the interaction coefficients for the waves which can be assumed to be real without loss of generality, and $f_{\mathbf{k}}$ and $g_{\mathbf{k}}$ are the spatial Fourier components of the applied forces.

The variables $u_{\mathbf{k}}$ and v_{κ} and the interaction coefficients satisfy the usual symmetry conditions:

$$u_{\mathbf{k}}^* = u_{-\mathbf{k}}, \quad v_{\varkappa}^* = v_{-\varkappa}, \quad V_H(\mathbf{k}, \mathbf{k}', \varkappa) = V_H(\mathbf{k}', \mathbf{k}, -\varkappa). \tag{1.3}$$

The above Hamiltonian takes into account only the three-wave interactions in which two u waves and one v wave participate. We shall assume that the terms in the Hamiltonian which describe this interaction are small in comparison with the terms which are quadratic in u and v. The force $f_{\mathbf{k}}(t)$ describes the ordinary excitation of u waves and $g_{\mathbf{k}}(t)$ describes the parametric excitation.

The equations of motion for u_k and v_κ are obtained from Eq. (1.2) in the usual way:

$$i\partial u_k / \partial t = \partial \mathscr{H} / \partial u_k^*, \quad i\partial v_\varkappa / \partial t = \partial \mathscr{H} / \partial v_\varkappa^*. \tag{1.4}$$

In order to take dissipative effects into account we shall introduce into the equation of motion terms which violate invariance under time reversal as follows:

$$i\left(\frac{\partial}{\partial t} + \gamma_k\right) u_k = \omega_k u_k + \sum_{k',\varkappa} [V_H(k, k', \varkappa) + iV_A(k, k', \varkappa)] \times u_{k'} v_\varkappa \delta(k - k' - \varkappa) + f_k(t) + \sum_{k'} g_{k+k'}(t) u_{k'}^*,$$

$$i\left(\frac{\partial}{\partial t} + \Gamma_\varkappa\right) v_\varkappa = \Omega_\varkappa v_\varkappa + \sum_k V_H(k, k - \varkappa, \varkappa) u_{k-\varkappa}^* u_k. \tag{1.5}$$

In these expressions γ_k and Γ_κ are the linear damping coefficients and $V_A(k, k', \kappa)$ are the nonlinear damping coefficients for the u waves. The coefficients V_A satisfy the symmetry relation

$$V_A(k, k', \varkappa) = V_A^*(k', k, -\varkappa).$$

They take into account the slow (in space and time) changes in the damping of the u waves which are due to the presence of the v waves.

The following order of magnitude relations are usually satisfied:

$$|V_A / V_H| \sim \gamma_k / \omega_k \ll 1, \quad \Gamma_\varkappa / \Omega_\varkappa \ll 1.$$

In view of the assumption that the wave interaction is weak, the solutions of Eq. (1.5) can be sought in the form

$$u_k = a_k e^{-i\omega_k t}, \quad v_\varkappa = b_\varkappa e^{-i\Omega_\varkappa t}, \tag{1.6}$$

assuming that a_k and b_κ are slowly-varying functions (over the periods $2\pi/\omega_k$ and $2\pi/\Omega_\kappa$). Substituting Eq. (1.6) in Eq. (1.5), and retaining only the slowly-varying terms on the right-hand sides, we obtain the following sets of equations for a_k and b_κ:

$$i\left(\frac{\partial}{\partial t} + \gamma_k\right) a_k = \sum_{k',\varkappa} V(k, k', \varkappa) \langle a_{k'} b_\varkappa \exp[i\Delta(k, k', \varkappa)t]\rangle \times \delta(k - k' - \varkappa) + \langle f_k(t) \exp(i\omega_k t)\rangle + \sum_{k'} \langle g_{k+k'} a_{k'}^* \exp[i(\omega_k + \omega_{k'})t]\rangle, \tag{1.7a}$$

$$V(k, k', \varkappa) = V_H(k, k', \varkappa) + iV_A(k, k', \varkappa)$$

$$i\left(\frac{\partial}{\partial t} + \Gamma_\varkappa\right) b_\varkappa = \sum_k V_H(k, k - \varkappa, \varkappa) \langle a_{k-\varkappa}^* a_k \exp[-i\Delta(k, k - \varkappa, \varkappa)t]\rangle, \tag{1.7b}$$

where

$$\Delta(k, k', \varkappa) = \omega_k - \omega_{k'} - \Omega_\varkappa,$$

and the angle brackets indicate that we are retaining only slowly-varying terms. Obviously, terms which represent the interaction between the waves are slowly-varying terms if

$$\Omega_\varkappa, \ \omega_k \gg \Delta(k, k - \varkappa, \varkappa). \tag{1.8}$$

The slow variation of the terms describing the effect of external forces is ensured if

$$f_k(t) = f_k e^{-i\omega_{0k} t}, \quad g_k(t) = g_k e^{-i\omega_{0k}' t}, \tag{1.9}$$

where

$$|\Delta_0(k)| \equiv |\omega_k - \omega_{0k}| \ll \omega_k,$$

$$|\Delta_1(k, k')| \equiv |\omega_{0, k+k'}' - \omega_k - \omega_{k'}| \ll \omega_k. \tag{1.10}$$

Condition (1.8) means that we are taking into account only the resonance interaction between the waves, whereas (1.10) shows that external forces excite high-frequency waves in a resonance fashion.

In this paper we shall confine our attention to the excitation of waves in a medium by monochromatic external forces, i.e., we shall consider cases where

$$f_k(t) = f\delta(k - k_0) e^{-i\omega_0 t}, \quad \omega_0 - \omega_{k_0} = \Delta_0, \quad |\Delta_0| \ll \omega_{k_0}$$

$$g_k(t) = g\delta(k - 2k_0) e^{-i\omega_0' t}, \quad \omega_0' - 2\omega_{k_0} = \Delta_1, \quad |\Delta_1| \ll \omega_{k_0}. \tag{1.11}$$

The first case describes ordinary resonance excitation of waves of frequency ω_0 (resonance detuning Δ_0) and the second case represents the parametric excitation (parametric resonance detuning Δ_1).

2. TIME-INDEPENDENT STATES

A state will be referred to as time-independent if it takes the form of a set of waves with constant amplitudes. For each such state we have a solution of Eq. (1.7) of the form

$$a_k = a_k^0 e^{-i\delta_k t}, \quad b_\varkappa = b_\varkappa^0 e^{-i\varepsilon_\varkappa t}, \tag{2.1}$$

where a_k^0, b_κ^0, δ_k, and ϵ_κ are constants which depend on the parameters of the applied force, the damping, and the nonlinear properties of the medium. The quantities δ_k and ϵ_κ are real; they are in fact the nonlinear additions to the frequencies ω_k and Ω_κ produced by the interaction between the waves. Since the wave interaction is weak, the quantities δ_k and ϵ_κ are small, i.e., $|\delta_k| \ll \omega_k$, $|\epsilon_\kappa| \ll \Omega_\kappa$.

Let us now investigate the time-independent states for different forms of the applied force.

In the absence of applied forces ($f_k = g_k = 0$) there is obviously only one time-independent state: $a_k^0 = b_k^0 = 0$, and this means that there are no waves in the medium. When the forces are applied there will, in general, be nonzero solutions of the form of Eq. (2.1).

Consider the case of ordinary excitation of u waves ($f \neq 0$, $g = 0$). In this case, we have solutions of Eq. (1.7) of the form given by Eq. (2.1):

$$a_k^0 = a_0^0 \delta_{kk_0}, \quad b_\varkappa^0 = 0, \quad a_0^0 = f / (\Delta_0 + i\gamma), \quad \delta_{k_0} = \Delta_0, \tag{2.2}$$

which correspond to a time-independent state consisting of a single u wave of frequency ω_0, wave vector k_0, and amplitude given by Eq. (2.2).

If we investigate the stability of this time-independent state by the standard methods[2] we find that it becomes unstable as soon as the amplitude f assumes the critical value given

$$|f|_c^2 = \min_\varkappa \frac{\Delta_0^2 + \gamma^2}{\rho^2 V_H^0} \{\rho\Delta(\Gamma_\varkappa - \gamma) + (q - \sigma)(\gamma + \Gamma_\varkappa)^2 + (\gamma + \Gamma_\varkappa)\sqrt{[(q - \sigma)(\Gamma_\varkappa - \gamma) + \rho\Delta]^2 + 4\gamma\Gamma_\varkappa[\rho^2 + (q - \sigma)^2]}\}, \tag{2.3}$$

where

$$\rho = \operatorname{Re} V_A(k_0, k_0, \varkappa), \quad \sigma = \operatorname{Im} V_A(k_0, k_0, \varkappa)$$
$$q = V_H(k_0, k_0 - \varkappa, \varkappa) - V_H(k_0 + \varkappa, k_0, \varkappa)$$
$$V_H^0 = V_H(k_0, k_0, \varkappa), \quad \Delta = \Delta(k_0, k_0 - \varkappa, \varkappa).$$

Here we have used the conditions given by (1.1), and all the coefficients of the equations in Eq. (1.7), except for q, are taken in the zero-order approximation in κ/k and Ω/ω.

When $|f| > |f|_c$ we have in addition to the unstable time-independent state given by Eq. (2.2) a further stable state which consists of a discrete set of waves: u waves with wave vectors $\mathbf{k}_n = \mathbf{k}_0 + n\kappa$ and v waves with wave vectors $\kappa_m = m\kappa$ (m and n are arbitrary integers, κ is the wave vector for which the expression given by Eq. (2.3) reaches a minimum; we shall assume for the sake of simplicity that this minimum is reached at a single value of κ). If the v waves with wave vectors κ_m, $m \neq 1$, interact with the u waves in a nonresonant fashion, i.e.,

$$\omega(\mathbf{k}_n) - \omega(\mathbf{k}_{n-m}) \approx \Omega(\varkappa_m)$$

is not satisfied for any $m \neq 1$, then the amplitudes of these v waves are negligible and can be set to zero in Eq. (1.7). In this case, the time-independent state consists of a discrete set of u waves and one v wave. The equations for the time-independent amplitudes are obtained from Eq. (1.7) by substituting Eq. (2.1) into it, taking into account the foregoing discussion:

$$(\delta_n + i\gamma) a_n{}^0 = V(n-1, n) a_{n-1}^0 b^0 \exp[i(\delta_n - \delta_{n-1} - \varepsilon + \Delta)t] + \overline{V}(n, n+1) a_{n+1}^0 (b^0)^* \exp[i(\delta_n - \delta_{n+1} + \varepsilon - \Delta)t] + f\delta_{n0} \exp[i(\Delta_0 - \delta_0)t], \quad (2.4a)$$

$$(\varepsilon + i\Gamma) b^0 = \sum_n V_H(n-1, n)(a_{n-1}^0)^* a_n{}^0 \exp[i(\delta_{n-1} - \delta_n + \varepsilon - \Delta)t],$$

$$V(n-1, n) = V(\mathbf{k}_n, \mathbf{k}_{n-1}, \varkappa), \quad V = V_H + iV_A, \quad \overline{V} = V_H + iV_A^*. \quad (2.4b)$$

It follows from these equations that

$$\delta_n = \Delta_0 + n(\Delta - \varepsilon). \quad (2.5)$$

The solution of Eq. (2.4a), taking Eq. (2.5) into account for $\Delta - \epsilon \neq 0$, in the zero-order approximation in κ/k can be written in the following form:

$$a_{\pm n}{}^0 = f h_{\pm n}(z)$$

$$h_{\pm n}(z) = \begin{cases} (\pm 1)^n (V^0/\overline{V}^0)^{\pm n/2} \dfrac{\pi e^{\pm in\beta}}{2\sin\eta\pi} J_{\pm\eta}(z) J_{n\mp\eta}(z), & \operatorname{Re} z > 0 \\ (\mp 1)^n (V^0/\overline{V}^0)^{\pm n/2} \dfrac{\pi e^{\pm in\beta}}{2\sin\eta\pi} J_{\pm\eta}(-z) J_{n\mp\eta}(-z), & \operatorname{Re} z < 0 \end{cases} \quad (2.6)$$

where

$$z = (V^0\overline{V}^0)^{1/2} |b^0| / (\Delta - \varepsilon), \quad \beta = \arg b^0, \quad \eta = (\Delta_0 + i\gamma)/(\Delta - \varepsilon).$$

In these expressions $J_\nu(z)$ is the Bessel function.

When $|\Delta - \epsilon| \ll |\eta|$ the solution of Eq. (2.4a) can be written in the following form:

$$a_{\pm n}{}^0 = (-i)^{\pm n} (V^0 / \overline{V}^0)^{\pm n/2} \rho^n, \quad (2.7)$$

where ρ is a root of the equation

$$2(\Delta_0 + i\gamma) = (V^0\overline{V}^0)^{1/2} |b^0| (\rho - \rho^{-1}),$$

which satisfies the condition $|\rho| < 1$.

If we substitute the above expressions for a_n into Eq. (2.4b), and separate the real and imaginary parts, we obtain equations for ϵ and $|b^0|$. The phase $\beta = \arg b^0$ is then found to be an indeterminate quantity because the equations in (2.4) are invariant under the transformation

$$b^0 \to b^{0\prime} = b^0 e^{i\alpha}, \quad a_n{}^0 \to a_n{}^{0\prime} = a_n{}^0 e^{in\alpha}. \quad (2.8)$$

Thus if b^0 and a_n^0 are the solutions of this system, then any $b^{0\prime}$ or $a_n^{0\prime}$ defined by Eq. (2.8) are also solutions.

To calculate ϵ and $|b^0|$ we can use the following equations which follow from Eq. (2.4):

$$\sum_n |a_n{}^0|^2 \delta_n = 2\varepsilon |b^0|^2 - \operatorname{Re}\{a_0{}^0 f^*\},$$

$$\gamma \sum_n |a_n{}^0|^2 = |b^0|^2 (\Gamma \operatorname{Im} V_A{}^0 - \varepsilon \operatorname{Re} V_A{}^0)/V_H{}^0 + {}^1/{}_2 \operatorname{Im}\{a_0{}^0 f^*\}. \quad (2.9)$$

Let us now investigate in greater detail the behavior of the time-dependent amplitudes in the case of a stable state as functions of the amplitude of the applied force near the instability threshold when $|f| - |f|_c \ll |f|_c$. In this case, only the four waves with frequencies ω_0, $\omega_0 \pm \Omega$, and Ω are appreciably excited, so that the amplitudes a_n^0, $n = 0, \pm 1$ in Eq. (2.4) can be set equal to zero, and we have

$$a_1{}^0 = \frac{V_+ |b^0|}{\delta_1 + i\gamma} a_0{}^0, \quad a_{-1} = \frac{\overline{V}_- |b^0|}{\delta_{-1} + i\gamma} a_0{}^0;$$

$$\left[1 - \frac{|b^0|^2}{\delta_0 + i\gamma}\left(\frac{V_-\overline{V}_-}{\delta_{-1} + i\gamma} + \frac{V_+\overline{V}_+}{\delta_1 + i\gamma}\right)\right] a_0{}^0 = \frac{f}{\delta_0 + i\gamma}, \quad (2.10)$$

$$(\varepsilon + i\Gamma) = |a_0{}^0|^2 \left[\frac{V_{-H}\overline{V}_-}{\delta_{-1} - i\gamma} + \frac{V_{+H} V_+}{\delta_1 + i\gamma}\right];$$

$$V_+ = V(\mathbf{k}, \mathbf{k} + \varkappa, \varkappa), \quad V_- = V(\mathbf{k} - \varkappa, \mathbf{k}, \varkappa).$$

Separating the imaginary and real parts of this last equation, we obtain two equations for ϵ and $|a_0^0|^2$ which are independent of f. Consequently, when $f > f_c$ the quantities ϵ and $|a_0^0|$ are constants independent of f. Since for $f < f_c$ it is clear from Eq. (2.2) that $|a_0^0|^2 = |f|^2(\Delta_0^2 + \gamma^2)^{-1}$ and $|a_0^0|$ is a continuous function of f in the neighborhood of $f = f_c$, we have for $f > f_c$

$$|a_0{}^0|^2 = |f|_c^2 (\Delta_0^2 + \gamma^2)^{-1}. \quad (2.11)$$

Using Eqs. (2.10) and (2.11), we find the following expressions for the remaining amplitudes:

$$|a_1{}^0|^2 = |V_+ b^0 f|^2 [(\Delta_0^2 + \gamma^2)(\delta_1^2 + \gamma^2)]^{-1},$$
$$|a_{-1}{}^0|^2 = |\overline{V}_- b^0 f| [(\Delta_0^2 + \gamma^2)(\delta_{-1}^2 + \gamma^2)]^{-1}. \quad (2.12)$$

The quantity $|b^0|^2$ is determined as a real non-negative root of the equation

$$|b^0|^4 |D|^2 + 2A|b^0|^2 + 1 - |f/f_c|^2 = 0,$$

where

$$A = -2\operatorname{Re} D, \quad D = \frac{1}{\Delta_0 + i\gamma}\left[\frac{V_-\overline{V}_-}{\delta_{-1} + i\gamma} + \frac{V_+\overline{V}_+}{\delta_1 + i\gamma}\right]. \quad (2.13)$$

Among the roots of this equation

$$|b^0|_{1,2}^2 = |D|^{-2}[-A \pm \sqrt{A^2 + (|f/f_c|^2 - 1)D^2}] \quad (2.14)$$

there is only one non-negative real root when $A \geq 0$ for $f > |f|_c$. In this region, the time-independent state (2.2) is unstable, and the stable state is defined by Eqs. (2.11)–(2.14). When $f \to f_c$

$$|b^0|^2 = \frac{1}{2A}\left(\left|\frac{f}{f_c}\right|^2 - 1\right),$$

$$|a_1{}^0|^2 = \frac{1}{2A} |V_+|^2 [(\Delta_0^2 + \gamma^2)(\delta_1^2 + \gamma^2)]^{-1} (|f|^2 - |f|_c^2),$$

$$|a_{-1}{}^0|^2 = \frac{1}{2A} |V_-|^2 [(\Delta_0^2 + \gamma^2)(\delta_{-1}^2 + \gamma^2)]^{-1} (|f|^2 - |f|_c^2). \quad (2.15)$$

Therefore, when $A > 0$, $f > f_c$ we have

$$|a_{\pm 1}{}^0|, \; |b^0| \sim \sqrt{|f/f_c|^2 - 1}. \quad (2.16)$$

When $A < 0$ the equation given by (2.13) has one non-negative root for $|f| > |f|_c$, and two such roots in the interval $|f|_{1c} < |f| < |f|_c$, $|f|_{1c}^2 = |f|_c^2(1 - A^2|D^{-2}|)$. The time-independent state defined by Eqs. (2.11) and (2.12) and corresponding to the larger of these roots, $|b^0|_1$, is stable, whereas the corresponding smaller root, $|b^0|_2$, is unstable. Therefore, when $A < 0$ and $f = f_c$ the amplitude b^0 (and also the amplitudes $a_\pm^0$) will discontinuously reach the finite value

$$|b^0(f_c)|^2 = |b^0(f_c)|_1^2 = 2|AD^{-2}|.$$

From Eq. (2.10) we obtain a somewhat unwieldly expression for ϵ, which however simplifies when $\Delta_0 = 0$. In this case,

$$\varepsilon = \frac{1}{\gamma + \Gamma}[-\Gamma\Delta + V_H \operatorname{Re} V_A |a_0^0|^2].$$

Let us now consider the parametric excitation of u waves. The equations for the amplitudes in this case are obtained from Eq. (1.7) if we substitute $f_k = 0$, $g_k \neq 0$ in the first of these. This set of equations has a zero solution $a_k = b_\kappa = 0$ which, however, becomes unstable when the amplitude of the applied force reaches the threshold value:

$$|g^2|_c = \min_{k'} \gamma_{k'}\gamma_{2k_0-k'}\left[1 + \frac{\Delta_1^2(k', 2k_0 - k')}{(\gamma_{k'} + \gamma_{2k_0-k'})^2}\right]. \qquad (2.17)$$

To simplify our analysis we shall assume that the minimum of this expression is reached for $k' = 2k_0 - k' = k_0$. In that case, when $|g| > |g|_c$ the amplitudes of the u waves with wave vectors near to k_0 increase exponentially. This state in turn becomes unstable when the amplitudes of the parametrically excited waves become large enough:

$$\sum_{k \approx k_0} |a_k|^2 \geqslant |a_0^0|_c^2 = |f|_c^2 (\Delta_0^2 + \gamma^2)^{-1} \qquad (2.18)$$

[the quantity $|f|_c$ is defined by Eq. (2.3)].

When Eq. (2.18) is satisfied we have a v wave in the medium with wave vector κ, composite u waves, and a time-independent state which consists of a set of discrete waves. The equations for the time-independent amplitudes are the same as those given by Eq. (2.4) provided we replace f and Δ_0 with $(a_0^0)^* g$ and Δ_1 on the right-hand side of Eq. (2.4a). The solutions for the amplitudes a_n differ from those given by Eq. (2.6) only by a constant factor:

$$a_{\pm n} = C h_{\pm n}(z), \qquad (2.19)$$

and for the amplitude $|b^0|$ we obtain the following equation:

$$|J_\eta(z) J_{-\eta}(z)| = \left|\frac{2 \sin \pi\eta}{\pi g}\right|. \qquad (2.20)$$

In these expressions $h_{\pm n}$, z, and η are the same as in Eq. (2.6).

The constant C in Eq. (2.19), and also ϵ, are determined by Eq. (2.9) with $a_0^0 f^*$ on the right-hand sides replaced with $(a_0^0)^2 g^*$.

Equation (2.20) may have a number of solutions. In fact, when

$$\eta \to 0, \; J_{\pm\eta}(z) \to J_0(z), \; \sin \pi\eta \to \pi\eta$$

we have from Eq. (2.20)

$$|J_\eta(z) J_{-\eta}(z)| = 2|\eta / g|.$$

The solution of this equation is shown graphically in Fig. 1. Each of the roots defines a time-independent amplitude $|b^0|$ which in view of Eq. (2.19) determines the values of the time-independent amplitudes a_n^0. Therefore, each of the roots of Eq. (2.20) defines a time-independent state. It is readily verified that the time-independent states corresponding to the roots indicated in Fig. 1 by the open circles are stable, whereas those indicated by the crosses are unstable against perturbations in the amplitudes a_n and b.

It is clear from Eq. (2.9) that the constant C increases with increasing $|b^0|$. This means that the sum of the squares of the steady-state amplitudes of the u waves increases with increasing $|z|$. Therefore, it may turn out that all the time-independent states, except for the first, which are stable against perturbations of the time-independent amplitudes a_n and b, are unstable against perturbations of the amplitudes of other waves, for example, the v waves with wave vectors $\kappa' \neq \kappa$, and u waves with wave vectors $k_{nm} = k_n + m\kappa'$. It is obvious that the time-independent state corresponding to the smallest value of $|b^0|$ is stable in this sense.

3. EXCITATION OF COUPLED LANGMUIR AND ION-ACOUSTIC PLASMA OSCILLATIONS BY A MONOCHROMATIC EXTERNAL FIELD

As an example, let us consider the excitation of plasma oscillations by an external longitudinal electric field of frequency ω_0 close to the electron plasma frequency $\omega_p = (4\pi n e^2/m)^{1/2}$ (e is the charge, m the mass, and n the mean density of electrons). We note that the parametric excitation of plasma waves by a high-frequency external field was first discussed in[4,5]. Our example is convenient in that the theoretical predictions for it can be compared with experimental results[3] and the theoretical conclusions reported elsewhere[6-8].

The questions for the Langmuir oscillations in an infinite plasma placed in the field of a longitudinal electric wave, taking into account three-wave interactions, can be written in the following form (see, for example,[7]):

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \omega_l^2(\mathbf{k})\right)\varphi_{\mathbf{k}}^l = \frac{i\omega_p^2}{k_0} E_0 e^{i\omega_0 t}\delta(\mathbf{k} - \mathbf{k}_0) + \frac{ie\omega_p^2}{T}\frac{(\mathbf{k}\mathbf{E}_0)}{k^2}\varphi_{\mathbf{k}_0-\mathbf{k}}^s - \frac{e\omega_p^2}{T}\sum_{\varkappa}\frac{(\mathbf{k}, \mathbf{k} - \varkappa)}{k^2}\varphi_{\mathbf{k}-\varkappa}^l\varphi_\varkappa^s + \frac{2i\gamma\omega_l(\mathbf{k})e}{T}\sum_{\varkappa}\varphi_{\mathbf{k}-\varkappa}^l\varphi_\varkappa^s, \qquad (3.1a)$$

$$\left(\frac{\partial^2}{\partial t^2} + 2\Gamma\frac{\partial}{\partial t} + \Omega_s^2(\varkappa)\right)\varphi_\varkappa^s = \frac{e\Omega_s^2(\varkappa)}{m\omega_p^2}\sum_{\mathbf{k}}(\mathbf{k}, \mathbf{k} - \varkappa)(\varphi_{\mathbf{k}-\varkappa}^l)^*\varphi_{\mathbf{k}}^l,$$

where $\varphi_{\mathbf{k}}^l$ and φ_κ^s are the Fourier components of the potentials, and γ and Γ are the damping coefficients of the Langmuir and ion-acoustic oscillations; E_0 is the amplitude, ω_0 the frequency, k_0 the wave vector of the external field, T the electron temperature,

$$\Omega_s^2(\varkappa) = v_i^2\varkappa^2, \quad \omega_l^2(\mathbf{k}) = \omega_p^2 + {}^3/_2 v_e^2 k^2, \qquad (3.2)$$

v_i is the velocity of ion sound, and v_e the thermal velocity of the electrons.

The last term on the right-hand side of Eq. (3.1a) describes the nonlinear damping of the Langmuir waves,

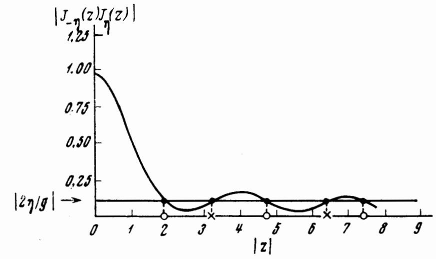

FIG. 1

whereas the second term represents the interaction between the external field and the Langmuir and ion-acoustic waves. However, in the case of resonance excitation of Langmuir waves by the external field $[\omega_0 \approx \omega_l(k_0)]$ this term can be omitted because the field strength of the Langmuir wave is much greater than the strength of the external field: $k_0\varphi_{k_0}^l \approx E_0\omega_p/\gamma \gg E_0$.

We shall seek the solutions of Eq. (3.1) in the form

$$\varphi_{\mathbf{k}}{}^l = u_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}t}, \quad \varphi_\varkappa{}^s = v_\varkappa e^{-i\Omega_\varkappa t},$$

where $u_{\mathbf{k}}$ and $v_\varkappa$ are slowly-varying functions. If we normalize these functions so that

$$u_{\mathbf{k}} = a_{\mathbf{k}}\frac{1}{k}\left(\frac{e\omega_p}{T}\right)^{1/2}, \quad v_\varkappa = b_\varkappa\left(\frac{e\Omega_\varkappa}{m\omega_p{}^2}\right)^{1/2},$$

then Eq. (3.1) will assume the form of Eq. (1.7) in which

$$V_H(\mathbf{k}, \mathbf{k}', \varkappa) = \frac{(\mathbf{kk}')}{2kk'}\frac{e}{T}\left(\frac{e\Omega_\varkappa}{m}\right)^{1/2},$$

$$V_A(\mathbf{k}, \mathbf{k}', \varkappa) = \frac{\gamma k}{\omega k'}\frac{e}{T}\left(\frac{e\Omega_\varkappa}{m}\right)^{1/2},$$

$$f = \frac{1}{2}\left(\frac{T\omega_p}{e}\right)^{1/2}E_0.$$

Consequently, the threshold amplitude of the external field is determined by Eq. (2.3) in which $q = 2(k_0\varkappa)/k_0$, $V_H^0 = (e\Omega_\varkappa/m)^{1/2}e/2T$. In particular, $\gamma \gg \Gamma$ and $q \gg V_A$, which is usually the case, we have

$$|f|_c{}^2 = \frac{2\Gamma}{qV_H{}^0\gamma}(\Delta_0{}^2 + \gamma^2)(\Delta^2 + \gamma^2). \tag{3.3}$$

The amplitudes of waves excited in the plasma by the external field before the threshold, near the threshold, and after the threshold are given by Eqs. (2.2), (2.5), and (2.9)–(2.12).

In the experiment described in[3] the Langmuir oscillations were excited in mercury plasma filling a glass tube of external diameter 0.8 cm and placed in a waveguide at right angles to the waveguide axis. A transverse wave with frequency $\omega_0 = 2.75 \times 10^{10}$ sec^{-1} close to the plasma electron frequency $(\omega_0 \approx \omega_p)$ propagated through the waveguide. High-intensity oscillations were excited in the plasma and the absolute amplitudes of these oscillations were determined. For microwave amplitudes $E_0 < 12$ V/cm, the plasma was found to execute oscillations of frequency ω_0 and amplitude a_0 proportional to E_0 (Fig. 2a). For $E_0 = E_{0c} \approx 12$ V/cm there were ion-acoustic oscillations of frequency $\Omega = 7.5 \times 10^5$ sec^{-1} and composite oscillations of frequencies $\omega_0 \pm \Omega$ (Fig. 2b). As the amplitude E_0 increases there is an increase in the amplitude of

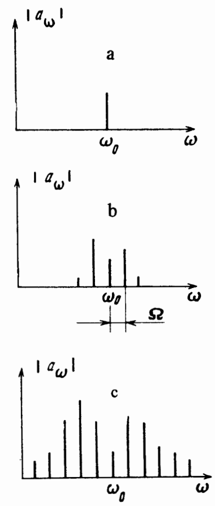

FIG. 2

the ion-acoustic oscillations and, at the same time, in the number of waves at the composite harmonics. The spectrum of the high-frequency oscillations for $E_0 = 15$ V/cm is shown in Fig. 2c.

Moreover, the following approximate values were obtained in the experiment:

$$T \sim 1 \text{ eV} \quad \varkappa = 6 \text{ cm}^{-1} \quad v_i \approx 1.2\cdot 10^5 \text{cm/sec} \quad v_e \approx 7.5\cdot 10^7 \text{ cm/sec}$$

$$\gamma/\omega_p \sim 10^{-2} - 10^{-3}, \quad \Gamma/\Omega \lesssim 1. \tag{3.4}$$

Before we compare the above theoretical results with experimental data, we note that, in contrast to the case of the homogeneous infinite plasma considered above, the experiment was performed with a bounded inhomogeneous plasma whose proper oscillations are not plane waves. This fact cannot, however, modify the form of the equations for the amplitudes or the order of magnitudes of their coefficients whose exact calculation is not easy.

We shall use the above expressions, and in the formulas for the coefficients we shall replace the wave vectors by quantities which are reciprocals of the characteristic inhomogeneity scales for the proper oscillations. As regards the longitudinal component of the external field, whose strength and distribution in the plasma are unknown, we shall assume for the sake of simplicity that the spatial distribution is close to one of the Langmuir modes.

In order to be able to use Eq. (3.3) to estimate the threshold amplitude of the external field, we must first estimate the quantity k which was not measured in the experiment. This can be done by using experimental data on near-threshold values of the amplitudes of the composite waves $a_{\pm 1}$ (Fig. 2b). In fact, in view of Eq. (2.10), near the threshold

$$|a_1/a_{-1}| \approx 1 + q/V_H{}^0 \approx 1 + \varkappa/k_0.$$

It is clear from Fig. 2b that $|a_1/a_{-1}| \sim 1.1$ and, consequently,

$$k \approx 10\varkappa \approx 60 \text{ cm}^{-1} \tag{3.5}$$

and

$$q \sim 10^{-1} V_H{}^0. \tag{3.6}$$

Since resonance excitation was carried out in the experiments, it follows that the detuning Δ_0 is of the same order as γ, i.e., $\Delta_0 \sim \gamma$. We then have

$$\Delta = \omega_l(\mathbf{k}+\varkappa) - \omega_l(\mathbf{k}) - \Omega_s(\varkappa) \approx \frac{3v_e^2}{2\omega_p} k\varkappa \sim 10^8 \text{ sec}^{-1} \quad (3.7)$$

By substituting Eqs. (3.4)—(3.7) in Eq. (3.3), we obtain the following estimate for the threshold amplitude of the external field: $E_{0C} \sim 1$ V/cm. This value for the critical amplitude of the external field is in satisfactory agreement with the experimental result $E_{0C} \sim 10$ V/cm.

We note that the estimates of E_{0C} given in[6,7] are higher by several orders of magnitude than the experimental result. This discrepancy is due to the fact that the possibility of resonance excitation of proper oscillations in plasma by the external field was not taken into account in[6,7]. This type of excitation ensures that the amplitude of the Langmuir oscillations is much greater than the amplitude of the external field, and the excitation of the ion-acoustic oscillations occurs as a result of the presence of high-intensity Langmuir oscillations.

The time-independent amplitudes of the Langmuir oscillations behind the threshold in the zero-order approximation in κ/k are given in accordance with Eq. (2.6) by the formulas

$$|a_n{}^0| \sim |J_{n+\eta}(z)|, \quad \eta = (\Delta_0 + i\gamma)\Delta^{-1}, \quad z = V_H{}^0\Delta^{-1}|b^0|. \quad (3.8)$$

Unfortunately, there are no experimental data which could be used to obtain an independent determination of the parameters η and z.

However, if we carry out a comparison of Eq. (3.8) with the relative amplitudes $|a_n|$ shown in Fig. 2c, and assume that z = 2.7 and $\eta \ll 1$ (which is not inconsistent with the above estimates), we obtain

$$|a_n{}^0| \sim |J_n(2.7)|$$

which is in adequate agreement with experimental data. The results of the comparison are shown in the following table:

n	−4	−3	−2	−1	0	1	2	3	4
$\|J_n(2.7)\|$	0.11	0.25	0.45	0.44	0.14	0.44	0.45	0.25	0.11
a_n^0 experiment, rel. units	0.16	0.35	0.48	0.35	0.14	0.42	0.35	0.14	0.16

If we take terms $\sim\kappa/k$ into account in the equations for the time-independent amplitudes, we can find the expressions for a_n^0 which may be in better agreement with experimental data.

The value of z found from the comparison with experimental data enables us to estimate the amplitude of the ion-acoustic oscillations for $E_0 = 15$ V/cm:

$$E_s = \varkappa\varphi^s = \varkappa|b|\left(\frac{e\Omega_\varkappa}{m\omega_p{}^2}\right)^{1/2} = \varkappa\frac{2.7\Delta}{V_H{}^0}\left(\frac{e\Omega_\varkappa}{m\omega_p{}^2}\right)^{1/2} \approx 10^{-1} \text{ V/cm}.$$

Finally, the amplitudes of the Langmuir oscillations can readily be estimated if we know the estimated value for the threshold amplitude of the external field (3.8). In fact, $E_l \sim E_{0l}\,\omega_p/\gamma \sim 100$—1000 V/cm.

We thus see that the above theory is in good agreement with existing experimental data and can be used to estimate quantities which have not as yet been measured.

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