CONDUCTIVITY OF A TWO-DIMENSIONAL TWO-PHASE SYSTEM

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It is shown that the conductivity of a two-phase thin film at equal concentrations of the phases and random distribution of them is equal to the geometric mean of the conductivity of the phases. If one of the phases conducts while the other does not conduct an electric current, the system under consideration experiences a metal-dielectric transition on change of the composition. It is shown that this transition occurs at equal concentrations of the phases, $c_{cr} = 1/2$. The conductivity of a two dimensional polycrystal is determined. These and a number of analogous problems can be solved exactly thanks to a certain symmetry, characteristic of two-dimensional systems.

1. INTRODUCTION

WE shall consider a conducting medium consisting of parts of two types of arbitrary shape and dimensions. The dimensions of the system are assumed to be much larger than the characteristic dimensions of the parts. If such a medium is placed in an electric field currents flow through it and the pattern of the currents will be rather complicated, if the "phase" regions are irregular.[1)] It is of interest to find the average properties of such a medium. The solution of this problem is trivial for a medium with one-dimensional inhomogeneities (a layered medium). There is little interest in this case, however, inasmuch as there are practically no irregularities in the current. The analogous three-dimensional case has been solved approximately in the case when the inhomogeneities are small (see, e.g.,[1]). It will be shown that the corresponding two-dimensional problem admits an exact solution in the case of a 1 : 1 mixture, if both "phases" are in geometrically equivalent conditions (on average). The problem under consideration is of interest principally as an exactly soluble model. Substantially similar problems arise in the calculation of the conductivity of inhomogeneous films, of the surface conductivity for an unevenly covered surface, where the system is inherently two-dimensional, and also in a plasma placed in a magnetic field, where the two-dimensionality can arise as a result of preferential development of instability in the plane perpendicular to the magnetic field (see, e.g.,[2]). In the latter case, however, the conductivity itself can be anisotropic and this requires special treatment.

That it is possible to find an exact solution is due to the fact that the system of equations in the conditions described undergoes a symmetry transformation which does not change the macroscopic properties of the medium. We note that the problem being considered is mathematically equivalent to a number of other physical problems. As an example, we can cite the calculation of various dissipative properties (thermal conductivity, viscosity, etc.) and dynamic properties (dielectric constant, sound velocity, etc.) in randomly non-uniform media.

[1)]For brevity an aggregate of parts of one type, separated from the other parts by an interface, will be called a phase. It is not necessary for these phases to be in a state of thermodynamic equilibrium (but they may be).

2. THE SYMMETRY RELATIONS. CONDUCTIVITY OF A TWO-PHASE SYSTEM

We turn to the formulation and solution of the problem. The system of equations consists of Ohm' law

$$\mathbf{j} = \sigma \mathbf{e} \tag{1}$$

and the equations of a constant current

$$\operatorname{rot} \mathbf{e} = 0, \quad \operatorname{div} \mathbf{j} = 0. \tag{2}$$

The conductivity σ is assumed to be given by a random function of the coordinates (x, y), taking two values; the regions (I, II) with the values $\sigma = \sigma_{1,2}$ are statistically equivalent (in particular, they have equal areas).

We are interested in the relation between the current averaged over the system, $\mathbf{J} = V^{-1} \int \mathbf{j}\, dV$ and the average field $\mathbf{E} = V^{-1} \int \mathbf{e}\, dV$. By virtue of the linearity of Eqs. (1) and (2) this relation will also be linear and, by virtue of the isotropy of the system as a whole, scalar. Thus,

$$\mathbf{J} = \sigma_{\text{eff}} \mathbf{E}, \tag{3}$$

where σ_{eff} is the relevant effective conductivity of the medium. If the medium is uniform on average, σ_{eff} is an automatically averaged macroscopic quantity, the dispersion of which decreases with increase in volume. In place of $\mathbf{j}$ and $\mathbf{e}$ we shall introduce the new unknowns:

$$\mathbf{j}' = (\sigma_1\sigma_2)^{1/2}[\mathbf{ne}], \quad \mathbf{e}' = (\sigma_1\sigma_2)^{-1/2}[\mathbf{nj}], \tag{4*}$$

$\mathbf{n}$ is the unit vector normal to the xy-plane.

Putting (4) into (1) and (2) we obtain a system of equations for the primed variables:

$$\mathbf{j}' = \sigma' \mathbf{e}', \quad \sigma' = \sigma_1\sigma_2 / \sigma, \tag{5}$$

$$\operatorname{rot} \mathbf{e}' = 0, \quad \operatorname{div} \mathbf{j}' = 0. \tag{6}$$

The system of equations (5) and (6) differs from (1) and (2) by the new conductivity (σ'). The quantity σ' takes the same values σ_1 and σ_2, in the regions II and I respectively. Since, by hypothesis, regions I and II are statistically equivalent, the system (5), (6) generates the averaged Ohm's law

$$\mathbf{J}' = \sigma_{\text{eff}} \mathbf{E}' \tag{7}$$

with the same σ_{eff} as in (3). By means of (4) we find

*$[\mathbf{ne}] \equiv [\mathbf{n} \times \mathbf{e}]$.

$$\mathbf{J}' = (\sigma_1\sigma_2)^{1/2}[\mathbf{nE}], \quad \mathbf{E}' = (\sigma_1\sigma_2)^{-1/2}[\mathbf{nJ}]. \tag{8}$$

Putting (8) into (7) and comparing with (3), we obtain

$$\sigma_{\text{eff}} = (\sigma_1\sigma_2)^{1/2}. \tag{9}$$

Thus the logarithm of the conductivity is found to be additive on mixing.

It is not difficult to show that expression (9) for the effective conductivity is also true for periodic structures, e.g., for a chessboard with all the white (and all the black) squares having equal conductivities.

3. CURRENT AND FIELD DISTRIBUTION CHARACTERISTICS

In the model under consideration it is also possible to calculate other macroscopic characteristics of the distribution of currents and fields over the "phases" and also within an individual phase. We shall calculate the average $\mathbf{A} = \langle(\sigma - \sigma_1)\mathbf{e}\rangle$. Expanding the brackets and averaging term by term, we obtain $\mathbf{A} = \mathbf{J} - \sigma_1\mathbf{e}$. On the other hand, the expression being averaged is nonzero only in the second phase. Taking this into account, we find $\mathbf{A} = \frac{1}{2}(\sigma_2 - \sigma_1)\mathbf{E}_2$, where

$$\mathbf{E}_2 = V_2^{-1}\int_{(V_2)} \mathbf{e}\, dV$$

is the average field in the second phase. Equating the two expressions for $\mathbf{A}$ and using (9), we find

$$\mathbf{E}_2 = \frac{2\sqrt{\sigma_1}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\mathbf{E}$$

and, analogously,

$$\mathbf{E}_1 = \frac{2\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\mathbf{E}.$$

The corresponding expressions for the currents are also easily found.

$$\mathbf{J}_{1,2} = \frac{2\sqrt{\sigma_{1,2}}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\mathbf{J}.$$

To find the distribution over the phases of the energy being dissipated, we calculate the quantity

$$\alpha = \langle j^2\rangle / \langle e^2\rangle = \langle j'^2\rangle / \langle e'^2\rangle. \tag{10}$$

Using (4), we find $\alpha = \sigma_1^2\sigma_2^2\langle e^2\rangle/\langle j^2\rangle = \sigma_1^2\sigma_2^2/\alpha$, whence $\alpha = \sigma_1\sigma_2$. The relation (10) can be rewritten in the form

$$\sigma_1\sigma_2(\langle e^2\rangle_1 + \langle e^2\rangle_2) = \langle j^2\rangle_1 + \langle j^2\rangle_2 = \sigma_1^2\langle e^2\rangle_1 + \sigma_2^2\langle e^2\rangle_2.$$

Hence

$$\sigma_1\langle e^2\rangle_1 = \sigma_2\langle e^2\rangle_2 = (\sigma_1\sigma_2)^{1/2}E^2. \tag{11}$$

The latter equality was obtained by using the relation

$$\langle(\mathbf{je})\rangle = (\mathbf{JE}),$$

which is true if surface effects are neglected. Thus, the energy is dissipated equally in the phases, regardless of the conductivities.

It turns out that this equality applies not only to the average dissipation but also to the distribution function of the dissipated energies. Actually, we shall calculate the quantity

$$\alpha_n = \langle j^{2n}\rangle / \langle e^{2n}\rangle = \langle j'^{2n}\rangle / \langle e'^{2n}\rangle.$$

By means of (4) we shall show that $\alpha_n = (\sigma_1\sigma_2)^n$. Thus

$$(\sigma_1\sigma_2)^n(\langle e^{2n}\rangle_1 + \langle e^{2n}\rangle_2) = \langle j^{2n}\rangle_1 + \langle j^{2n}\rangle_2 = \sigma_1^{2n}\langle e^{2n}\rangle_1 + \sigma_2^{2n}\langle e^{2n}\rangle_2,$$

whence

$$\sigma_1^n\langle e^{2n}\rangle_1 = \sigma_2^n\langle e^{2n}\rangle_2 \text{ or } \langle(je)^n\rangle_1 = \langle(je)^n\rangle_2. \tag{12}$$

Using (12), it is easy to prove the equality

$$\langle\delta((\mathbf{je}) - q)\rangle_1 = \langle\delta((\mathbf{je}) - q)\rangle_2.$$

Thus, the Joule heat distributions are the same in both phases.

Finally, using (11) we can calculate the mean square fluctuations characterizing the nonuniformity of the currents and fields in the system. We have

$$\langle e^2\rangle = \frac{1}{2}(\langle e^2\rangle_1 + \langle e^2\rangle_2) = \frac{1}{2}\left(\sqrt{\frac{\sigma_1}{\sigma_2}} + \sqrt{\frac{\sigma_2}{\sigma_1}}\right)E^2,$$

$$\langle j^2\rangle = \frac{1}{2}\left(\sqrt{\frac{\sigma_1}{\sigma_2}} + \sqrt{\frac{\sigma_2}{\sigma_1}}\right)J^2,$$

$$\Delta = \frac{\langle j^2\rangle - J^2}{J^2} = \frac{\langle e^2\rangle - E^2}{E^2} = \frac{1}{2}\left[\left(\frac{\sigma_1}{\sigma_2}\right)^{1/4} - \left(\frac{\sigma_2}{\sigma_1}\right)^{1/4}\right]^2.$$

4. SMOOTH DEPENDENCE OF THE CONDUCTIVITY ON THE COORDINATES

The symmetry we have mentioned also enables us to obtain a solution under less rigid assumptions about the form of the function $\sigma(x, y)$. For convenience we shall introduce the quantity $\chi(x, y) = \ln\sigma - \langle\ln\sigma\rangle$ and consider an ensemble of systems such that the multipoint conductivity distribution function is an even function of the variables χ. As an example of such a distribution, other than that considered above, we could take a Gaussian distribution for the quantities χ. The substitution

$$\mathbf{j}' = \exp\{\langle\ln\sigma\rangle\}[\mathbf{ne}], \quad \mathbf{e}' = \exp\{-\langle\ln\sigma\rangle\}[\mathbf{nj}]$$

transforms Ohm's law

$$\mathbf{j} = \exp(\langle\ln\sigma\rangle + \chi)\mathbf{e}$$

into

$$\mathbf{j}' = \exp(\langle\ln\sigma\rangle - \chi)\mathbf{e}'$$

and does not change Eqs. (2).

Replacing χ by $-\chi$ and using the fact that the distribution functions are even in χ, we again find that the primed system is macroscopically equivalent to the initial one. Hence, repeating the arguments of Sec. 2, we find

$$\sigma_{\text{eff}} = \exp\langle\ln\sigma\rangle = (\langle\sigma\rangle / \langle 1/\sigma\rangle)^{1/2}. \tag{13}$$

For a Gaussian distribution (13) will take the form

$$\sigma_{\text{eff}} = \langle\sigma\rangle\exp(-\Delta^2/2),$$

where $\Delta = \langle\chi^2\rangle^{1/2}$ is the root mean square fluctuation of the logarithm of the conductivity.

5. IMPEDANCE OF AN ELECTRICAL CIRCUIT

A problem analogous to that considered in Sec. 2 can be formulated for a plane electrical circuit. Suppose we have a circuit in the form of a square lattice, the "links" of which have resistances taking the values r_1 and r_2 with equal probability. It is convenient to apply the symmetry transformation to the junction potentials and to the interjunction current functions in the given case, thus generating a transition to another circuit, macroscopically equivalent to the first; this leads to an expression for the effective resistance of the circuit $r_{eff} = (r_1 r_2)^{1/2}$. The same relation is valid for complex resistances in a circuit of quasistationary current.

It is interesting to note that if we take the capacitance and inductance as the two different resistances in such a circuit ($Z_1 = 1/i\omega C$, $Z_2 = i\omega L/c^2$), we obtain the real quantity $Z_{eff} = c^{-1}\sqrt{L/C}$ for the equivalent resistance of circuit. Thus, a circuit comprising imaginary resistances which do not lead to energy dissipation has a real equivalent resistance, i.e., absorbs energy. This apparent contradiction is resolved if we take into account that in a random system, unlike a periodic one, there are localized oscillations, i.e., parts of the circuit, which can be resonant at any frequency. The energy of the source is expended on resonance excitation of localized oscillations; the presence in the system of even the slightest absorption will lead to true (finite) dissipation.

6. THE METAL-DIELECTRIC TRANSITION

If the concentrations of the phases are not equal, the method given does not give a complete solution of the problem. However, it is possible to obtain a relation connecting the conductivities of the "complementary systems" (with concentrations c and 1-c of the high-conductivity phase (the first phase, say)). In fact, for $c \neq 1/2$, the primed system differs from the initial one by the interchange $\sigma_1 \rightleftarrows \sigma_2$ or, which is the same thing, $c \rightleftarrows 1-c$. Then in place of (9) we obtain

$$\sigma_{eff}(c)\,\sigma_{eff}(1-c) = \sigma_1\sigma_2.$$

The greatest theoretical interest attaches to the investigation of a system in which one of the phases does not conduct ($\sigma_2 = 0$). In this case the system experiences a metal-dielectric transition as the composition changes. In fact, for $c_1 \approx 1$, $\sigma_{eff} \approx \sigma_1$, while for $c_1 \ll 1$, $\sigma_{eff} = 0$. There is obviously a critical concentration of the nonconducting phase, above which $\sigma_{eff} = 0$.

The results obtained above allow us to calculate the critical concentration c_{cr}. The question of the nature of the singularity in $c_{eff}(c)$ at $c = c_{cr}$ remains open for the present. To find the critical concentration we proceed as follows. We shall consider a system consisting of parts of three types, with conductivities σ_1, σ_2 and $\sigma_3 = (\sigma_1\sigma_2)^{1/2}$; σ_{eff} will be equal to $(\sigma_1\sigma_2)^{1/2}$ if the regions I and II are statistically equivalent, and region III is arbitrary. We pass now to the limit $\sigma_1 \to \infty$, $\sigma_2 \to 0$, $\sigma_1\sigma \to \infty$. The limiting system becomes a two-phase system with a conducting-phase (regions I and III) concentration $c \gg 1/2$, and $\sigma_{eff} = \infty$. Passing to the limit with $\sigma_1 \to \infty$, $\sigma_2 \to 0$ and $\sigma_1\sigma_2 \to 0$, we obtain that for $c < 1/2$, $\sigma_{eff} = 0$. Thus for the system considered ($\sigma_1 = \infty$, $\sigma_2 = 0$)

$$\sigma_{eff} = \begin{cases} \infty, & c_1 > 1/2 \\ 0, & c_1 < 1/2, \end{cases} \tag{14}$$

and $c_{cr} = 1/2$. We shall show that the same value of c_{cr} is obtained for arbitrary σ_1. In fact, for $\sigma_2 = 0$ we have the relation $\sigma_{eff} = \sigma_1 f(c)$, with $f(c_{cr} + 0) = 0$. Passing here to the limit $\sigma_1 \to \infty$ and comparing with (14), we find that f(c) is finite for $c < 1/2$ and $f(c) = 0$ for $c > 1/2$. Hence $c_{cr} = 1/2$.[2)]

7. CONDUCTIVITY OF A TWO-DIMENSIONAL POLYCRYSTAL

In the case of a polycrystal the local Ohm's law can be written in the form

$$\mathbf{j} = \hat{P}_\varphi \hat{\sigma} \hat{P}_{-\varphi} \mathbf{e}. \tag{15}$$

Here $\hat{\sigma}$ is the conductivity tensor of the monocrystal, which is independent of the coordinates. One of its principal axes is assumed to be directed along the z-axis; this guarantees the two-dimensionality of the currents and fields. $\hat{P}_\varphi$ is the rotation matrix for a rotation through an angle φ in the xy plane

$$\hat{P}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

The angle φ, which defines the orientation of the given crystallite, is a random function of the coordinates. The transformation

$$\mathbf{j} = (\det\hat{\sigma})^{1/2}\hat{P}_{\pi/2}\mathbf{e}', \quad \mathbf{e} = (\det\hat{\sigma})^{-1/2}\hat{P}_{\pi/2}\mathbf{j}'$$

leads to the relation

$$\mathbf{j}' = (\det\hat{\sigma})\hat{P}_{-\pi/2}\hat{P}_\varphi\hat{\sigma}^{-1}\hat{P}_{-\varphi}\hat{P}_{\pi/2}\mathbf{e}'. \tag{16}$$

By making use of the commutativity of rotations in a plane and the easily verifiable identity

$$\hat{\sigma} = (\det\hat{\sigma})\hat{P}_{-\pi/2}\hat{\sigma}^{-1}\hat{P}_{\pi/2},$$

we rewrite (16) in the form

$$\mathbf{j}' = \hat{P}_\varphi\hat{\sigma}\hat{P}_{-\varphi}\mathbf{e}'. \tag{17}$$

Comparing (15) and (17) and again repeating the arguments of Sec. 2, we find

$$\sigma_{eff} = (\det\hat{\sigma})^{1/2}.$$

Thus, the conductivity of a two-dimensional polycrystal is equal to the geometric mean of the principal values of the conductivity tensor.

2)This result, for the discrete (lattice) model, was obtained by another method in [3].

[1] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Fizmatgiz, M., 1959 (English translation published by Pergamon Press, Oxford, 1960).

[2] E. P. Velikhov, A. M. Dykhne and I. Ya. Shipuk, in "Ionization Phenomena in Gases" (Proceedings of the VIth International Conference, Belgrade, 1965).

[3] M. F. Sykes and J. W. Essam, J. Math. Phys. 5, 1117 (1964).

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