FINITE AMPLITUDE HELICONS

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We consider inertialess symmetry oscillations, having helical symmetry and an arbitrary amplitude, of the electrons of a conductor or a plasma in a magnetic field. It is shown that ordinary helicons are a particular case of solutions with a frequency that is independent of the amplitude.

1. INTRODUCTION

As is well known, slow electromagnetic waves, called whistlers or helicons, can propagate in a magnetized plasma, in the frequency interval between the cyclotron-ion and the cyclotron-electron frequencies, and also in metals and in semiconductors placed in a strong magnetic field. Usually these waves have a relatively low amplitude and can be described in the linear approximation. More interesting from the general point of view of nonlinear wave processes in continuous media, however, are waves of finite amplitude. They can be also of practical interest. In this paper we consider waves of finite amplitude with helical symmetry. As we shall show, the problem of determining all the possible waves with helical symmetry such as helicons reduces to a solution of one nonlinear differential second-order equation for a scalar function. We derive this equation and present some of its simplest solutions.

2. SLOW ELECTROMAGNETIC WAVES WITH HELICAL SYMMETRY

Let us assume that a slow electromagnetic stationary periodic wave, having helical symmetry, propagates in a magnetized plasma or in a solid conducting body placed in an external magnetic field. The latter means that all the quantities in such a wave change in space in such a way, that they depend only on r and the combination $\zeta = kz - m\vartheta$ of the cylindrical coordinates r, ϑ, and z. Here m is an arbitrary integer, corresponding to the multiplicity of the helix of the wave considered by us.

The slowness of the wave enables us to neglect the displacement current. We assume further that, on the one hand, the oscillation frequency is high enough that we can neglect the displacements of the ions (or of the lattice) in the wave, and on the other hand it is much smaller than the cyclotron frequency of the electrons, i.e., the inertia of the electrons is negligible. For simplicity we neglect also the friction of the electrons against the wave or the lattice, i.e., we assume the conductivity to be infinite. This corresponds to the condition $\Omega_e\tau_e \gg 1$, where $\Omega_e = eB/m_ec$ is the cyclotron frequency of the electrons and τ_e is their average collision time. Under these approximations, we have the following system of equations

$$\operatorname{rot}\mathbf{B} = -\frac{4\pi e}{c}n(\mathbf{v}_e - \mathbf{v}_i), \tag{2.1}$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \quad \mathbf{E} + \frac{1}{c}[\mathbf{v}_e\mathbf{B}] = 0, \tag{2.2*}$$

$$\partial n/\partial t + \operatorname{div} n\mathbf{v}_e = 0, \quad \operatorname{div}\mathbf{B} = 0. \tag{2.3}$$

In these equations, $\mathbf{v}_e$ is the electron velocity, $\mathbf{v}_i$ is the ion velocity, and the remaining symbols are standard. We assume the plasma to be quasineutral, so that the electron and ion densities are equal to n, and the electron temperature is assumed to be zero.

It is convenient to change over to a coordinate system that moves with the phase velocity of the wave $\mathbf{v}_0$. In this coordinate system, the ions move with constant velocity $\mathbf{v}_i = -\mathbf{v}_0$, and the time derivatives vanish.

By virtue of the helical symmetry, the equation div B = 0 takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}rB_r - \frac{1}{r}\frac{\partial}{\partial\zeta}(mB_\vartheta - rkB_z) = 0. \tag{2.4}$$

We see therefore that

$$rB_r = \frac{\partial\psi}{\partial\zeta}, \quad mB_\vartheta - krB_z = \frac{\partial\psi}{\partial r}, \tag{2.5}$$

where ψ is an arbitrary function of r and ζ. We introduce also

$$I = mB_z + krB_\vartheta. \tag{2.6}$$

Then the components of the magnetic field can be expressed in terms of ψ and I:

$$B_r = \frac{1}{r}\frac{\partial\psi}{\partial\zeta}, \quad B_\vartheta = \frac{1}{m^2 + k^2r^2}\left(m\frac{\partial\psi}{\partial r} + krI\right),$$
$$B_z = \frac{1}{m^2 + k^2r^2}\left(-kr\frac{\partial\psi}{\partial r} + mI\right). \tag{2.7}$$

Analogously, from the equation div $n\mathbf{v}_e = 0$ it follows that

$$rnv_r = \frac{c}{4\pi e}\frac{\partial\chi_e}{\partial\zeta}, \quad mnv_{\vartheta e} - krnv_{ze} = \frac{c}{4\pi e}\frac{\partial\chi_e}{\partial r}, \tag{2.8}$$

where χ is an arbitrary function of the variables r and ζ. If we again introduce a quantity similar to (2.6),

$$\mu_e = \frac{4\pi e}{c}(mnv_{ze} + krnv_{\vartheta e}), \tag{2.9}$$

then the components of the electron density of the current can be expressed in terms of χ_e and μ_e with the aid of relations analogous to (2.7).

For ions it suffices to consider the particular case $\mathbf{v}_i = -\mathbf{v}_0$. It is simpler, however, to consider the more general case of helical flow of ions, which can also be described by the functions χ_i and μ_i, and then change over to the particular case $\partial\chi_i/\partial\zeta = 0$ and $\partial\mu_i/\partial\zeta = 0$.

We now proceed to Eqs. (2.2). The first of them shows that in the stationary case the electric field is potential, i.e., $E = \nabla\varphi$. Consequently, the second equation takes the form

$$[\mathbf{v}_e\mathbf{B}] = c\nabla\varphi, \tag{2.10}$$

*$[\mathbf{v}_e\mathbf{B}] \equiv \mathbf{v}_e \times \mathbf{B}$.

from which follows $\mathbf{B}\cdot\nabla\varphi = 0$ and $en\mathbf{v}_e\cdot\nabla\varphi = 0$, i.e., the force lines and the electron-current lines lie on the equipotentials φ = const. Written out in terms of the components, these conditions lead to relations of the type of the vanishing of Jacobians, from which it follows that

$$\varphi=\varphi(\psi)\quad \chi_e=\chi_e(\varphi), \tag{2.11}$$

i.e., χ_e and φ are functions of ψ. In other words, χ_e and φ are constant on the magnetic surfaces ψ = const. The third component of (2.10), which we choose conveniently to be the projection on $\nabla\psi$, leads to the relation

$$\mu_e=I\frac{\partial\chi_e}{d\psi}-4\pi e(m^2+k^2r^2)n\frac{\partial\varphi}{d\psi}. \tag{2.12}$$

We now consider Eq. (2.1). Its r-component and the difference between the ϑ-component and the z-component, multiplied by kr/m, lead to the condition:

$$I=\chi_e-\chi_i, \tag{2.13}$$

and the sum of the ϑ-component of Eq. (2.1) and of the z-component, multiplied by m/kr, gives the relation

$$\beta\Delta^*\psi+2mkI/\beta=\mu_i-\mu_e, \tag{2.14}$$

where

$$\Delta^*\psi=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\beta}\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\zeta^2},\qquad \beta=m^2+k^2r^2. \tag{2.15}$$

If we now substitute here I from (2.13) and μ_e from (2.12), we obtain one nonlinear differential equation of second order for ψ:

$$\Delta^*\psi+\frac{2mk}{\beta^2}(\chi_e-\chi_i)+\frac{1}{\beta}(\chi_e-\chi_i)\frac{d\chi_e}{d\psi}-4\pi en\frac{d\varphi}{d\psi}=\frac{\mu_i}{\beta}. \tag{2.16}$$

In this equation, χ_e and φ are arbitrary functions of ψ. As to the density n and the ion functions χ_i and μ_i, they are certain specified functions of r and ζ.

For the most interesting case of a cylindrical conductor or a plasma cylinder, they should be regarded as independent of ζ, and if the ions are at rest in the laboratory system of coordinates, then we must put

$$\mu_i=-\frac{4\pi e}{c}mnv_0,\qquad \chi_i=\frac{4\pi e}{c}kv_0\int_0^r nr\,dr. \tag{2.17}$$

Equation (2.16) describes the most general helical flows of magnetized electrons, neglecting their inertia and collisions. These flows, although complicated by the unusual configuration of the helical magnetic fields, have precisely the same physical nature as simple helicons or whistlers, and it is therefore advantageous to call them here helicons in a certain generalized sense.

Of course, Eq. (2.16) in its general form cannot be solved analytically, so that it is of interest to consider some of the simpler particular cases. These include the helicons proper in a longitudinal magnetic field, and also problems involving the flow of current in an ideal conductor with allowance for the Hall effect.

We begin with consideration of helicons proper.

3. HELICONS

We consider a simple but, as will be shown below, a most interesting case, when the functions χ_e and $d\varphi/d\psi$ depend linearly on ψ:

$$\chi_e=A+Ck\psi,\quad 4\pi ed\varphi/d\psi=D-L\psi, \tag{3.1}$$

where A, C, D, and L are constants.

In this case Eq. (2.16) becomes linear, and its solution can be represented in the form $\psi=\psi_0+\psi'$, where ψ_0 depends only on the coordinate, and the dependence of ψ' on ζ for each individual Fourier harmonic can be chosen in the form $\exp i\zeta$. For ψ' we obtain a homogeneous second-order equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\beta}\frac{\partial\psi'}{\partial r}\right)-\frac{1}{r^2}\psi'+\frac{2mk^2C}{\beta^2}\psi'+\frac{C^2k^2}{\beta}\psi'+Ln\psi'=0. \tag{3.2}$$

We see therefore that at sufficiently large C and L the solutions have the form of functions that oscillate with r. In the particular case L = 0 and C = 1, Eq. (3.2) has a solution ψ' = r corresponding to ordinary helicons propagating in an unbounded medium along the magnetic field.

The equation for ψ_0, which we shall not present here (it can be readily obtained from (2.16) by making the substitition (3.1)) enables us, given n, v_0, and the constants A, C, D, and L, to find ψ_0 and then, in accordance with (3.1), also χ_e, i.e., I_0. By the same token, we determine the most general form of the fields $B_z^0(r)$ and $B_\vartheta^0(r)$, a feature of which is that helicon-type waves of arbitrary amplitude can propagate in them, while the values of B_r^0 and B_ϑ^0 averaged over the azimuth remain the same as before. In other words, the frequency of the helicons in such fields does not depend on their amplitude. Equation (2.16) possesses this property only in the case of the linear dependence (3.1). Accordingly, the frequency of the waves does not depend on the amplitude only in the class of solutions (3.1).

We now consider an even more particular case, when the density, the longitudinal magnetic field, and the longitudinal current density are constant, i.e., n = const, $B_z = B_0$ = const, and B_ϑ^0/r = const. In this case

$$\psi_0=-\frac{k_\parallel B_0}{2}r^2,\quad I_0=mB_0+kr^2\left(\frac{B_\vartheta{}^0}{r}\right), \tag{3.3}$$

where $k_\parallel = k\pm mB_\vartheta^0/B_0 r$.

If the second relation of (3.3) is substituted in (2.13) and account is taken of (2.17) and (3.1), then we obtain after equating the coefficients of equal powers of r^2

$$v_0=-\frac{k_\parallel cB_0}{4\pi en}C-\frac{c}{2\pi en}\frac{B_\vartheta{}^0}{r}, \tag{3.4}$$

$$A=mB_0. \tag{3.5}$$

Relation (3.4) is none other than the dispersion equation for the phase velocity of the helicons, and the second term in (3.4) corresponds simply to the drift of the helicons at the electron drift velocity.

If we substitute (3.1), (3.3), and (2.17) in (2.16), then we readily obtain by equating the coefficients of equal powers of β

$$L=0,\quad D=CB_\vartheta{}^0/rn. \tag{3.6}$$

It is easy to consider also the case $B_\vartheta\sim 1/r$, when an external current flows along the axis of the pinch. In this case L = D = 0, $A = mB_0 - \tfrac{1}{2}CkmrB_\vartheta^0$ and $v_0 = -(ckE_0/4\pi en)C$, i.e., the frequency of the helicon does not depend on B_ϑ^0.

We are thus left with only one arbitrary constant C. Equation (3.2) with L = 0 is, as can be readily verified, the usual equation for helicons, more accurately, an equation for the rB_r' component of the unperturbed field $\mathbf{B}'$ satisfying the equation[1]

$$\operatorname{rot}\mathbf{B}'=Ck\mathbf{B}'. \tag{3.7}$$

This equation can be easily obtained in the linear approximation from (2.1) and (2.2), and the natural frequency of the oscillations $\omega = kv_0$ is connected with C by relation (3.4).

We have thus shown that if the density, the magnetic field B_z, and the current density among z are homogeneous, then helicons of arbitrary amplitude can propagate in the medium, and their frequency or phase velocity does not depend on the amplitude. This conclusion has been obtained for cylindrical waves, but pertains also of course to plane waves. We note in this connection that the result obtained in [2], which differs from ours, namely that the frequency of a plane wave of the helicon type depends on its amplitude, is connected with allowance for very small additions due to the perturbation of the electron density in an oblique wave.

The constant C is determined by the boundary conditions. For example, we can stipulate vanishing of the B'_r component, and consequently also of ψ', on the boundary of the cylindrical conductor, say at $r = R$. The condition $\psi'_{r=R} = 0$ also follows from the vanishing of the radial component of the electron current, and as a result, the function χ_e, and together with it also ψ (according to (3.1)) should not depend on ζ at $r = R$. In such a solution, however, B'_z and B'_ϑ differ from zero on the boundary of the conductor, so that it should be interpreted as oscillations in a conducting body with allowance for the Hall effect, surrounded by conductor with infinite conductivity, along which surface current could flow. If the conductor borders on vacuum then, as shown in [1], the presence of surface currents leads to a finite damping decrement of the waves even when $\tau_e \to \infty$. Then the condition $\psi'_{r=R} = 0$ determines the wave with the smallest damping decrement.[3]

It is possible to get rid of the damping connected with the surface currents by making n a smoothly decreasing function of r.[3] In fact, for a small-amplitude wave it is easy to obtain from the system (2.1)–(2.3), in the particular case $B_z = B_0 = \text{const}$, $B_\vartheta = 0$, and $m \neq 0$, a linearized equation for the perturbation ψ':

$$\Delta^*\psi' + Q(r)\frac{2mk^2}{\beta^2}\psi' + Q^2(r)\frac{k^2}{\beta}\psi' = \frac{m}{\beta}Q(r)\frac{1}{r}\frac{d\ln n}{dr}\psi', \quad (3.8)$$

where $Q = -(4\pi ev_0/ckB_0)n(r)$. Let us assume that $n \to 0$ as $r \to \infty$. Then, as can be readily seen, $Q \to 0$ as $r \to \infty$, and Eq. (3.8) has exponentially decreasing solutions: $\psi' \sim \exp -kr$. These solutions will hardly be affected by the boundary of the conductor with the vacuum if this boundary is located at sufficiently large kr.

We shall show that Eq. (3.8) actually has such solutions satisfying the boundary condition $\psi' = 0$ at $r = 0$. In fact, we choose v_0 such that Q is much larger than unity in the main region. Then in this region we can neglect in (3.8) the terms linear in Q and retain only the quadratic term. But the remaining equation, as can be readily seen, for example, in the quasiclassical approximation, has solutions that oscillate with r. By choosing Q it is possible to make one of the zeroes of this solution fall at the point $r = 0$, i.e., to satisfy the boundary condition $r = 0$ and cause thereby ψ' to decrease at infinity.

Thus, if the density profile decreases with r, there exist in a homogeneous magnetic field helicons which are linear in the amplitude and localized in r. However, their nonlinear analog does not belong to the class of nonlinear helicons considered by us above, if for no other reason than the fact that the coefficients of the terms in (3.8), having the same structure as in (3.2), depend on the coordinates. Therefore the waves under consideration are not included in the class of helicons whose frequency and phase velocity are independent of the amplitude. This can be demonstrated also in a more direct manner.

Indeed, it is easy to see that in the class of solutions (3.1), in the case under consideration when $B_\vartheta = 0$ and $\psi_0 = -kB_0r^2/2$, we have

$$I_0 = mB_0 = \chi_e - \chi_i = A - \frac{k^2CB_0}{2}r^2 - \frac{4\pi e}{c}k\int_0^r nv_0 r\,dr = \text{const.} \quad (3.9)$$

where χ_e is substituted from (3.1) and χ_i is expressed in accordance with (2.17). Differentiating (3.9) with respect to r we get $nv_0 = -(ckB_0/4\pi e)T = \text{const}$, which leads in the case of $v_0 = \text{const}$ to homogeneity, $n = \text{const}$. If it is assumed that the profile of the ion velocities v_0 is not homogeneous in r, and is such that $v_0 \sim 1/n(r)$, then it can be readily seen from ψ_0 that $L = 0$. But then the term with the density drops out of Eq. (3.2), and we again arrive at helicons that are not localized in r.

4. FLOW OF CURRENT THROUGH A CONDUCTOR WITH HELICAL SYMMETRY

If we put in (2.6) $v_0 = 0$, i.e., $\mu_i = \chi_i = 0$, and assume n to be a function of r and ζ, then we arrive at the problem of the flow of electric current, with allowance for the Hall effect, in a resting ideal conductor with helical symmetry (at $\Omega_e\tau_e \gg 1$). Equation (2.16) itself then takes the form

$$\Delta^*\psi + \frac{2mkI}{\beta^2} + \frac{I}{\beta}\frac{dI}{d\psi} = 4\pi en\frac{d\varphi}{d\psi}, \quad (4.1)$$

where $I = \chi_e$ and φ are arbitrary functions of ψ.

Equation (4.1) is similar to the equation of helical equilibrium in magnetohydrodynamics:[4,5]

$$\Delta^*\psi + \frac{2mkI}{\beta^2} + \frac{I}{\beta}\frac{dI}{d\psi} = -4\pi\frac{dp}{d\psi}, \quad (4.2)$$

where p is the pressure. This of course is no accident, since (4.2) is, in some sense, a particular case of (4.1). In fact, in the case of hydrodynamic equilibrium of the ions, $\mathbf{v}_i = 0$, we have

$$\nabla p_i = en\mathbf{E}, \quad (4.3)$$

where $p_i = nT_i$, T_i is the ion temperature, and it is necessary to add in the second equation (2.2) the additional term $e^{-1}n\nabla p_e$, where $p_e = nT_e$ is the electron pressure. This means that in the stationary equation of equilibrium for the electrons (the second equation of (2.2)) it is necessary to substitute in place of $\nabla\varphi$ the quantity $-(en)^{-1}\nabla p$, and this leads to the replacement of (4.1) by (4.2).

The analogy between (4.1) and (4.2) shows that when $n = \text{const}$ or, in the more general case $n = n(\varphi)$, the equation for the electron current reduces to an equilibrium equation. It can be stated that the electrons should form an equilibrium configuration in the magnetic and electric fields. Accordingly, the only electron currents that can be realized are those having the analog of a hydrodymagnetic equilibrium.[6]

In the case of azimuthal symmetry ($m = 0$) and $B_z = 0$, as is well known from the theory of magnetohydrodynamic equilibrium,[7] equilibrium is possible only with cylindrical symmetry. A similar conclusion holds also in this case. Namely, as seen from (4.1), when $m = 0$ and $B_z \to 0$, i.e., $\psi \to 0$, condition $dI^2/Id\varphi = 8\pi enk^2r^2$ should be satisfied, i.e.,

$$I = krR_\vartheta = F(r^2n), \tag{4.4}$$

where F is an arbitrary function.

At $n = \text{const}$ it follows therefore that I and B_ϑ are functions of r only, i.e., the current can flow only along cylindrical surfaces. This conclusion agrees with the results of [6,8], where it was shown that when $\Omega_e\tau_e \gg 1$ the electric current does not flow into the convex parts of a corrugated conductor. As we see from (4.4), only in the particular case when $n \sim r^{-2}$ can the current I depend on ζ, and furthermore in arbitrary manner.

5. CONCLUSION

We have thus shown that the problem of finding all the possible waves of finite amplitude with helical symmetry reduces to the solution of one nonlinear second-order differential equation. This equation has a class of formally linear solutions. These solutions have the feature that their frequency does not depend on the (finite) amplitude. Ordinary helicons, including those propagating at an angle to the magnetic field, are contained just in this class of solutions. Unfortunately, in a homogeneous sample the helicons have appreciable damping that does not depend on the collision frequency, owing to dissipation of the surface current. In the case of a density profile $n(r)$ that decreases with r, it is possible to obtain a situation wherein helicons of small amplitudes are localized along the radius and have no strong damping as a result of surface currents, but such solutions do not enter in the class of waves with a frequency independent of the amplitude.

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