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THEORY OF SELF-TRAPPING OF AN ELECTROMAGNETIC FIELD IN A NONLINEAR MEDIUM

V. M. ELEONSKIĬ and V. P. SILIN

P. N. Lebedev Physics Institute, U.S.S.R. Academy of Sciences

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We determine the conditions under which electromagnetic fields specified at the boundary of half-space become self-trapped into fields that are distributed periodically in one dimension in a transparent nonlinear medium. Both weakly-nonlinear fields and fields with a nearly exact one-dimensional distribution with finite amplitude are studied. It is shown that on violation of the conditions of "self-trapping" to a rigorously one-dimensional distribution, proper non-one-dimensional fields are excited in the medium at a sufficient distance from the boundary. In a number of cases these fields are the same as the non-one-dimensional fields investigated previously.[2,3]

1. IN this communication we consider the boundary-value problem for a half-space filled with a medium having a nonlinear dielectric constant. The bulk of the paper deals with a clarification of the conditions under which a distribution of the electromagnetic field specified at the boundary causes some type of steady-state proper distribution of the field in the medium far from the boundary. It is obvious that this boundary-value problem is directly connected with the analysis of the phenomena called "self-trapping" of the electromagnetic field in a nonlinear medium.[1]

We note that it is shown in [2,3] that besides the previously known types of one-dimensional steady-state distributions of the electromagnetic field in a nonlinear nondissipative medium,[4] there exist non-one-dimensional distributions that are close to the one-dimensional ones and possess a periodic structure in one or two spatial dimensions. The properties of such field distributions in unbounded space were considered in detail in the aforementioned papers.[2,3] It will be shown that these very same properties play an important role in the problem of establishment of the self-trapping of the field.

Using as an example a medium that is transparent in the linear approximation, we show that self-trapping of the field specified on the boundary of a half-space into a strictly one-dimensional proper distribution of the field in the medium is an exceptional event. As shown by an analysis of the boundary-value problem, the transition to a strictly one-dimensional proper distribution is possible—in the case of weakly nonlinear field distributions—only if a definite connection exists between the basic parameters characterizing the field on the boundary. On the other hand, in the case when the field specified on the boundary of the half-space has a distribution close to an exact periodic distribution with finite amplitude, the condition of self-trapping into a strictly one-dimensional distribution makes it necessary to satisfy several relations that limit the type of admissible perturbations on the boundary in each order of the asymptotic expansion (in terms of the amplitude) of the solution of the boundary-value problem characterizing the perturbation of the field on the boundary.

It is shown further that proper two-dimensional field distributions are excited in the case when the self-trapping conditions are satisfied for weakly nonlinear distributions far from the boundary. On the other hand, if the self-trapping conditions are not satisfied for boundary fields that are close to the proper one-dimensional periodic distributions with finite amplitude, then there are excited, in some order (in terms of amplitude) of the perturbation, far from the boundary, proper two-dimensional periodic distributions with the amplitude of the fundamental non-one-dimensional mode of the same order of smallness.

Finally, we note that at certain values of the parameters characterizing the field on the boundary of the half-space, far from the boundary, there are excited proper non-one-dimensional distributions with a more complicated structure (compared with those investigated in [2,3]).

2. Let us consider the conditions for self-trapping of the periodic distribution of a field specified on the boundary of the half-space, to strictly one-dimensional and periodic distributions, for the case of a medium that is transparent in the linear approximation.

The system of equations determining the amplitude E and the phase Ψ of the electromagnetic field in the medium with nonlinear dielectric constant is[1-3]

$$\Delta E+[k_\omega^2-\varkappa^2-(\operatorname{grad}\Psi)^2+(\varkappa E/E_c)^2]E=0, \tag{2.1}$$

$$\operatorname{div}(E^2\operatorname{grad}\Psi)=0. \tag{2.2}$$

Let the half-space $z>0$ be filled with a medium, and let the amplitude and the normal derivative of the phase be specified on the boundary:

$$E(x,z=0)=E_0\cos k_\perp x,\quad -\frac{\partial\Psi}{\partial z}\Big|_{z=0}=k_0. \tag{2.3}$$

Relations (2.3) determine the field energy flux density on the boundary of the half-space. We stipulate further that at $z\to\infty$ a one-dimensional periodic distribution of the field is produced in the medium, in the form

$$\lim_{z\to\infty}E(x,z)=E_\infty e_\infty(x),\quad -\lim_{z\to\infty}\frac{\partial\Psi}{\partial z}=k_\infty. \tag{2.4}$$

Here $e_\infty(x)$ is a known[2-4] periodic solution of the equation

$$\frac{d^2e_\infty}{dx^2}+[k_\omega^2-\varkappa^2-k_\infty^2+(\varkappa E_\infty/E_c)^2e_\infty^2]e_\infty=0. \quad (2.5)$$

If $k_\omega^2-\kappa^2-k_\infty^2>0$, Eq. (2.5) admits of a weakly nonlinear solution with known[2,3] expansion of the transverse wave number $k_\perp$ in terms of the small parameter $\mu_\infty=(E_\infty/E_c)^2$:

$$k_\perp^2=k_\omega^2-\varkappa^2-k_\infty^2+\mu_\infty\chi_\perp^{(1)}(\infty)+\mu_\infty^2\chi_\perp^{(2)}(\infty)+\ldots \quad (2.6)$$

Assuming that the problem of self-trapping of the field (2.3), specified on the boundary of the half-space, into the strictly one-dimensional periodic distribution (2.4) has a solution if the transverse wave number $k_\perp$ is conserved, we arrive at the conclusion that it is necessary, when constructing the solution, to determine the connection between $k_\perp$ and the amplitude E_0 at $z=0$, and also the amplitudes of the fundamental modes E_0 and E_∞ at $z=\infty$.

The substitution $E=E_0\,e(x,z)$ enables us to rewrite the system (2.1) and (2.2) in the form

$$\begin{gathered}\Delta e+[k_\omega^2-\varkappa^2-k_0^2]e=-\mu_0\varkappa^2e^3+[(\operatorname{grad}\Psi)^2-k_0^2]e,\\ \operatorname{div}(e^2\operatorname{grad}\Psi)=0\end{gathered} \quad (2.7)$$

and when $\mu_0\equiv(E_0/E_c)^2\ll 1$ we can seek the solution of the boundary-value problem in the form of the asymptotic expansions

$$\begin{gathered}e=e^{(0)}+\mu_0e^{(1)}+\mu_0^2e^{(2)}+\ldots,\\ \Psi=-k_0z+\mu_0\psi^{(1)}+\mu_0^2\psi^{(2)}+\ldots,\\ k_\perp^2=k_\omega^2-\varkappa^2-k_0^2+\mu_0\chi_\perp^{(1)}+\mu_0^2\chi_\perp^{(2)}+\ldots.\end{gathered} \quad (2.8)$$

The linear approximation admits of the solution

$$e^{(0)}(\varphi_\perp,z)=\cos k_\perp x\equiv\cos\varphi_\perp,\qquad \Psi^{(0)}=-k_0z, \quad (2.9)$$

which satisfies the required boundary conditions. In the next higher approximation, putting

$$\begin{gathered}e^{(1)}=e_1^{(1)}(z)\cos\varphi_\perp+e_3^{(1)}(z)\cos3\varphi_\perp,\\ \psi^{(1)}\cos\varphi_\perp=s_1^{(1)}(z)\cos\varphi_\perp+s_3^{(1)}(z)\cos3\varphi_\perp,\end{gathered} \quad (2.10)$$

we obtain a system of equations for the functions that depend only on the longitudinal variable z:

$$\frac{d^2e_1^{(1)}}{dz^2}+2k_0\frac{ds_1^{(1)}}{dz}=\chi_\perp^{(1)}-\frac{3}{4}\varkappa^2,\quad \frac{d^2s_1^{(1)}}{dz^2}-2k_0\frac{de_1^{(1)}}{dz}=0; \quad (2.11)$$

$$\begin{gathered}\frac{d^2e_3^{(1)}}{dz^2}-8(k_\omega^2-\varkappa^2-k_0^2)e_3^{(1)}+2k_0\frac{ds_3^{(1)}}{dz}=-\frac{1}{4}\varkappa^2,\\ \frac{d^2s_3^{(1)}}{dz^2}-8(k_\omega^2-\varkappa^2-k_0^2)s_3^{(1)}-2k_0\frac{de_3^{(1)}}{dz}=0.\end{gathered} \quad (2.12)$$

The system (2.11), under the condition $\chi_\perp^{(1)}={}^3/_4\kappa^2$, which is connected with the requirement that the solution be bounded at infinity, admits of the trivial solution $e_1^{(1)}\equiv ds_1^{(1)}/dz=0$, and the system (2.12) admits of a solution in the form

$$e_3^{(1)}=\frac{1}{32}\frac{\varkappa^2}{k_\omega^2-\varkappa^2-k_0^2}+Ae^{-\nu z},\qquad s_3^{(1)}=\pm iAe^{-\nu z}, \quad (2.13)$$

where ν is the solution of the equation

$$\nu^2\mp2ik_0\nu-8(k_\omega^2-\varkappa^2-k_0^2)=0. \quad (2.14)$$

Thus, the first terms of the asymptotic expansion of the solution of the weakly nonlinear boundary-value problem for the half-space are of the form

$$e(z,\varphi_\perp)$$

$$=\cos\varphi_\perp+\frac{1}{32}\frac{\mu_0\varkappa^2}{k_\omega^2-\varkappa^2-k_0^2}\left[1-\frac{\cos(k_0z+\delta)}{\cos\delta}e^{-\lambda z}\right]\cos3\varphi_\perp+\ldots, \quad (2.15)$$

$$\Psi(z,\varphi_\perp)=-k_0z-\frac{1}{16}\frac{\mu_0\varkappa^2}{k_\omega^2-\varkappa^2-k_0^2}\times\frac{\sin(k_0z+\delta)}{\cos\delta}e^{-\lambda z}\left[\cos2\varphi_\perp-\frac{1}{2}\right]+\ldots, \quad (2.16)$$

$$k_\perp^2=k_\omega^2-\varkappa^2-k_0^2+{}^3/_4\mu_0\varkappa^2+\ldots. \quad (2.17)$$

We have used here the notation

$$\operatorname{tg}\delta=k_0/\lambda,\qquad \lambda^2=8(k_\omega^2-\varkappa^2)-9k_0^2.$$

It is obvious that the behavior of the solution at infinity required for self-trapping, namely degeneracy of the solution at $z\to\infty$ into a strictly one-dimensional periodic distribution of the field with a plane front, is realized only if relation (2.17) is satisfied together with the inequality

$$k_0^2<{}^8/_9(k_\omega^2-\varkappa^2). \quad (2.18)$$

3. For a more complete elucidation of the structure of the weakly-nonlinear solution of the boundary-value problem for a half-space, as well as of the conditions of self-trapping to strictly one-dimensional periodic distributions of the field in the medium, it is necessary to investigate the higher-order approximations. We find that the solution of the system of equations of the next higher approximation can be written in the form

$$\begin{gathered}e^{(2)}(z,\varphi_\perp)=e_1^{(2)}(z)\cos\varphi_\perp+e_3^{(2)}(z)\cos3\varphi_\perp+e_5^{(2)}(z)\cos5\varphi_\perp,\\ \psi^{(2)}(z,\varphi_\perp)\cos\varphi_\perp=s_1^{(2)}(z)\cos\varphi_\perp+s_3^{(2)}(z)\cos3\varphi_\perp+s_5^{(2)}(z)\cos5\varphi_\perp.\end{gathered} \quad (3.1)$$

In particular, the system of equations for the fundamental mode is given by

$$\begin{gathered}\frac{d^2e_1^{(2)}}{dz^2}+2k_0\frac{ds_1^{(2)}}{dz}=\chi_\perp^{(2)}-\frac{3}{4}\varkappa^2e_3^{(1)}+\left[\frac{ds_3^{(1)}}{dz}\right]^2\\ +8(k_\omega^2-\varkappa^2-k_0^2)[s_3^{(1)}]^2-2k_0e_3^{(1)}\frac{ds_3^{(1)}}{dz},\end{gathered} \quad (3.2)$$

$$\frac{d^2s_1^{(2)}}{dz^2}-2k_0\frac{de_1^{(2)}}{dz}=k_0\frac{d}{dz}[e_3^{(1)}]^2-2\frac{d}{dz}\left[e_3^{(1)}\frac{ds_3^{(1)}}{dz}\right]. \quad (3.3)$$

The solution of the system (3.2), (3.3) must satisfy at $z=0$ the zero boundary conditions for the amplitude and the normal derivative of the phase, and must correspond at $z\to\infty$ to self-trapping to a strictly one-dimensional field distribution. Equation (3.3) has a first integral and leads, when the boundary conditions are taken into account, to the solution

$$\frac{ds_1^{(2)}}{dz}=2k_0e_1^{(2)}+k_0[e_3^{(1)}]^2-2e_3^{(1)}\frac{ds_3^{(1)}}{dz}. \quad (3.4)$$

It can be shown that in all the higher approximations the equations for the fundamental mode, resulting from the divergent form (2.2), also have first integrals similar to (3.4). This circumstance is connected with the conservation of the energy flux through any plane parallel to the boundary of the half-space (see below).

Using relation (3.4), we find that $e_1^{(2)}(z)$ satisfies the equation

$$\begin{gathered}\frac{d^2e_1^{(2)}}{dz^2}+4k_0^2e_1^{(2)}=\chi_\perp^{(2)}-\frac{3}{4}\varkappa^2e_3^{(1)}+\left[\frac{ds_3^{(1)}}{dz}\right]^2\\ +8(k_\omega^2-\varkappa^2-k_0^2)[s_3^{(1)}]^2+2k_0e_3^{(1)}\frac{ds_3^{(1)}}{dz}-2k_0^2[e_3^{(1)}]^2.\end{gathered} \quad (3.5)$$

The latter contains the free parameter $\chi_\perp^{(2)}$, which is determined by the boundary conditions corresponding to the assumed self-trapping. The solution of (3.5) is

$$e_1^{(2)}(z) = e_1^{(2)}(\infty) + \frac{1}{128}\frac{\alpha}{\cos\delta}[A_1\cos(k_0z+\delta) + B_1\sin(k_0z+\delta)]e^{-\lambda z}$$

$$+\frac{1}{1024}\left(\frac{\alpha}{\cos\delta}\right)^2[A_2\cos 2(k_0z+\delta) + B_2\sin 2(k_0z+\delta)]e^{-2\lambda z}, \quad (3.6)$$

$$\chi_\perp^{(2)} = \frac{3}{128}\alpha\varkappa^2 + \frac{1}{512}(\alpha k_0)^2 + 4k_0^2 e_1^{(2)}(\infty). \quad (3.7)$$

Here $\alpha = \kappa^2(k_\omega^2 - \kappa^2 - k_0^2)$ and A_1 and B_1, for example, are the solutions of the system of equations

$$(\lambda^2 + 3k_0^2)A_1 - 2k_0\lambda B_1 = 3\varkappa^2 + {}^1/_4(\alpha k_0)^2, \quad (3.8)$$

$$2k_0\lambda A_1 + (\lambda^2 + 3k_0^2)B_1 = {}^1/_4 k_0\lambda.$$

Thus, if the normal derivative of the phase on the boundary satisfies the inequality (2.18), and the amplitude of the field E_0 on the boundary and the transverse wave number $k_\perp$ are connected by the relation

$$k_\perp^2 = k_\omega^2 - \varkappa^2 - k_0^2 + {}^3/_4\mu_0\varkappa^2$$

$$+\mu_0^2[{}^3/_{128}\alpha\varkappa^2 + {}^1/_{512}(\alpha k_0)^2 + 4k_0^2 e_1^{(2)}(\infty)] + \ldots, \quad (3.9)$$

there is excited in the medium, as $z \to \infty$, a strictly one-dimensional periodic distribution of the field with a plane front and with a fundamental-mode amplitude

$$E_\infty = [1 + \mu_0^2 e_1^{(2)}(\infty) + \ldots]E_0.$$

Consequently, self-trapping to strictly one-dimensional proper distributions of the field in the medium, as shown by an analysis of the asymptotic solution of the weakly nonlinear boundary value problem, is possible only upon satisfaction of a number of strong conditions superimposed on the main parameters of the field at the boundary of the half-space.

In the absence of an energy flux through the boundary of the half-space $k_0 = 0$, and the asymptotic expansion takes the form

$$e(z, \varphi_\perp) = [1 - {}^3/_{1024}(\mu_0\alpha)^2(1 - e^{-Z}) + \ldots]\cos\varphi_\perp$$
$$+ \mu_0\alpha[{}^1/_{32}(1 - {}^1/_4\mu_0\alpha)(1 - e^{-Z}) + {}^3/_{256}\mu_0\alpha Z e^{-Z}]\cos 3\varphi_\perp$$
$$+ {}^1/_{1024}(\mu_0\alpha)^2[1 + {}^1/_2(e^{-\sqrt{3}Z} - 3e^{-Z}) + \ldots]\cos 5\varphi_\perp + \ldots \quad (3.10)$$

Here $\alpha = (\kappa/\Lambda)^2$, $\Lambda^2 = k_\omega^2 - \kappa^2$, $Z = \sqrt{8}\Lambda z$, and finally

$$k_\perp^2 = k_\omega^2 - \varkappa^2 + {}^3/_4\mu_0\varkappa^2 + {}^3/_{128}(\mu_0\varkappa)^2\alpha + \ldots \quad (3.11)$$

Consequently, in this case the expansion coefficients $\chi_\perp^{(1)}$ and $\chi_\perp^{(2)}$ coincide in terms of the parameter μ_0 with the coefficients $\chi_\perp^{(1)}(\infty)$ and $\chi_\perp^{(2)}(\infty)$ of the expansion of the wave number $k_\perp$ in the parameter μ_∞. However, $\chi_\perp^{(3)} \sim (\kappa\alpha)^2$ already differs from $\chi_\perp^{(3)}(\infty)$ by a numerical factor. A similar difference takes place also for the higher-order coefficients. The latter circumstance is connected with the difference between the amplitudes of the fundamental mode $\cos\varphi_\perp$ at the boundary of the half-space and at $z \to \infty$. Indeed,

$$E_\infty = [1 - {}^3/_{1024}(\mu_0\alpha)^2 + \ldots]E_0. \quad (3.12)$$

Thus, in the plane $\{k^2, E_0/E_C\}$ the conditions for self-trapping to a strictly one-dimensional distribution are realized on the curves shown in Fig. 1. The points lying above this curve correspond to the "opacity" region, since the solution of the weakly nonlinear boundary-value problem leads to $\lim e(z, \varphi_\perp) = 0$ as $z \to \infty$.

Before we consider the region lying below the aforementioned curve, we note that the conditions of self-trapping to strictly one-dimensional distributions can be investigated also for the case when the field on the boundary is represented in the form of the asymptotic expansion

$$e(\varphi_\perp, z=0) = \cos\varphi_\perp + \sum_{n>1} e_{2n+1}(0)\mu_0^n\cos(2n+1)\varphi_\perp, \quad (3.13)$$

which degenerates, when $\mu_0 \to 0$, to the boundary condition considered above. Calculations show that when $z \to \infty$ the amplitude of the fundamental mode $\cos\varphi_\perp$ is of the form

$$E_\infty = [1 - {}^3/_{1024}(\mu_0\alpha)^2 + {}^3/_{32}e_3(0)\alpha\mu_0^2 + \ldots]E_0. \quad (3.14)$$

The condition for self-trapping to a one-dimensional distribution is then realized on the surface in the space of the parameters $\{k_\perp^2, E_0/E_C, e_3(0), \ldots\}$, which characterize the field at the boundary.

4. In the investigation of the boundary-value problem for a half-space in the case when the conditions of self-trapping to a strictly one-dimensional distribution are not satisfied, and the point of the plane $\{k_\perp^2, E_0/E_C\}$, which characterizes the field on the boundary, lies below the curve shown in Fig. 1, we confine ourselves to an analysis of flux-free field distributions, $-(\partial\Psi/\partial z)_{z=0} = k_0 = 0$. Let $k_\perp^2 < k_\omega^2 - \kappa^2$, and then the nonlinear approximation has a solution

$$e^{(0)}(\varphi_\perp, z) = \cos\varphi_\perp\cos k_\parallel z \equiv \cos\varphi_\perp\cos\varphi_\parallel \quad (4.1)$$

(where $k_\parallel^2 = k_\omega^2 - \kappa^2 - k_\perp^2$), satisfying the boundary condition and the condition of being bounded at $z \to \infty$. For an unbounded space, such a choice of the fundamental non-one-dimensional mode leads to the two-dimensional periodic field distributions investigated in [2,3]. It is natural to assume that when the condition $k_\perp^2 < k_\omega^2 - \kappa^2$ is satisfied, or more accurately, in the region lying above the curve shown in Fig. 1, proper non-one-dimensional distributions are excited in the medium at $z \to \infty$.

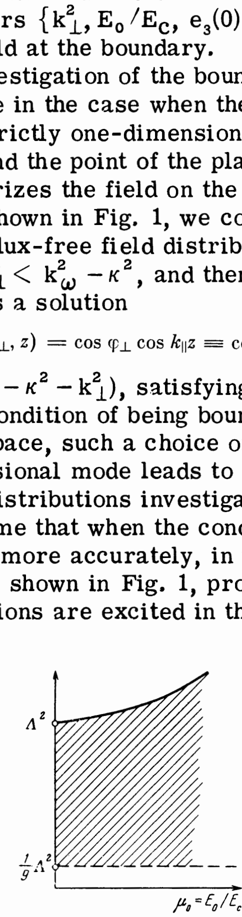

FIG. 1

Assuming that the wave vector $(k_\perp, k_\parallel)$ depends on the amplitude E_0, we find that in the approximation following the linear approximation the solution takes the form

$$e^{(1)}(\varphi_\perp, \varphi_\parallel) = e_1^{(1)}(\varphi_\parallel)\cos\varphi_\perp + e_3^{(1)}(\varphi_\parallel)\cos 3\varphi_\perp. \quad (4.2)$$

In this case the dependence of the functions $e_1^{(1)}$ and $e_3^{(1)}$ on the longitudinal variable is determined by the system of equations

$$\frac{d^2e_1^{(1)}}{dz^2} + (\Lambda^2 - k_\perp^2)e_1^{(1)} = -\frac{3}{16}\varkappa^2\cos 3\varphi_\parallel,$$

$$\frac{d^2e_3^{(1)}}{dz^2} + (\Lambda^2 - 9k_\perp^2)e_3^{(1)} = -\frac{1}{16}\varkappa^2\cos 3\varphi_\parallel - \frac{3}{16}\varkappa^2\cos\varphi_\parallel. \quad (4.3)$$

If the inequalities

$$^{1}/_{9}\Lambda^2 < k_\perp^2 < \Lambda^2 \equiv k_\omega^2 - \varkappa^2 \quad (4.4)$$

are satisfied, the asymptotic expansion of the solution of the boundary-value problem for a half-space takes the form

$$e(\varphi_\perp, z) = \left[1 - \frac{3}{128}\mu_0\left(\frac{\varkappa}{k_\parallel}\right)^2\right]\cos\varphi_\perp\cos\varphi_\parallel + \frac{1}{128}\mu_0\left(\frac{\varkappa}{\Lambda}\right)^2\cos3\varphi_\perp\cos3\varphi_\parallel + \frac{3}{128}\mu_0\left(\frac{\varkappa}{k_\parallel}\right)^2\cos\varphi_\perp\cos3\varphi_\parallel + \frac{3}{128}\left(\frac{\varkappa}{k_\perp}\right)^2\mu_0\cos3\varphi_\perp\cos\varphi_\parallel - \frac{1}{128}\left[\left(\frac{\varkappa}{\Lambda}\right)^2 + 3\left(\frac{\varkappa}{k_\perp}\right)^2\right]\mu_0\cos3\varphi_\perp\exp(-\sqrt{9k_\perp^2 - \Lambda^2}z). \quad (4.5)$$

Elimination of the secular terms leads to the relation

$$k_\perp^2 + k_\parallel^2 = k_\omega^2 - \varkappa^2 + {}^{9}/_{16}\mu_0\varkappa^2 + \ldots. \quad (4.6)$$

The latter shows that, accurate to terms of order μ_0^2, the connection between ($k_\perp$, $k_\parallel$) and E_0 coincides with that established earlier.[2,3] However, by virtue of the fact that the amplitude of the fundamental non-one-dimensional mode at $z \to \infty$

$$E_\infty = \left[1 - \frac{3}{128}\mu_0\left(\frac{\varkappa}{k_\parallel}\right)^2 + \ldots\right]E_0 \quad (4.7)$$

differs from E_0, a difference between the corresponding expansion coefficients in (4.6), compared with the case of non-one-dimensional distributions in all of space, arises already in the next higher approximation. Indeed, analysis shows that

$$k_\perp^2 + k_\parallel^2 = k_\omega^2 - \varkappa^2 + \frac{9}{16}\mu_0\varkappa^2 + \frac{3}{2048}(\mu_0\varkappa^2)^2\left[\frac{1}{k_\omega^2 - \varkappa^2} + \frac{9}{k_\perp^2} + \frac{9}{k_\parallel^2}\right] - \frac{27}{1024}\frac{(\mu_0\varkappa^2)^2}{k_\parallel^2} + \ldots. \quad (4.8)$$

Thus, if the parameters of the fields on the boundary correspond to the region of the plane $\{k_\perp^2, E_0/E_c\}$ bounded from above by the solid line of Fig. 1, and bounded from below by the line $k_\perp^2 = {}^1/_9(k_\omega^2 - \kappa^2)$ shown dashed in Fig. 1, then the solution of the boundary-value problem for the half-space shows that proper non-one-dimensional periodic field distributions are excited in the medium at $z \to \infty$. These distributions are characterized by a wave vector ($k_\perp$, $k_\parallel$) that depends on the amplitude. The properties of the latter were investigated in detail in [2,3].

However, when $k_\perp^2 < {}^1/_9(k_\omega^2 - \kappa^2)$, non-one-dimensional fields with a more complicated structure are excited far from the boundary. Indeed, in this case, even in the approximation that follows the linear one, there arises a two-dimensional mode of a new type

$$\cos3\varphi_\perp\cos\nu_3\varphi_\parallel, \quad \nu_3^2 = (k_\omega^2 - \varkappa^2 - 9k_\perp^2)/k_\parallel^2 < 1. \quad (4.9)$$

Moreover, it can be shown that in all the succeeding approximations the presence of this mode leads to the occurrence of secular terms which cannot be eliminated by assuming that the wave vector ($k_\perp$, $k_\parallel$) depends on the amplitude of the fundamental mode $\cos\varphi_\perp\cos\varphi_\parallel$. We note that when $k_\perp^2 < (k_\omega^2 - \kappa^2)/(2n+1)^2$, $n > 1$, there will be excited in the next higher approximations modes of the type

$$\cos(2n+1)\varphi_\perp\cos\nu_{2n+1}\varphi_\parallel, \quad \nu_{2n+1}^2 = \frac{k_\omega^2 - \varkappa^2 - (2n+1)^2k_\perp^2}{k_\parallel^2}. \quad (4.10)$$

Consequently, when $k_\perp \to 0$ the field distribution in the space $z > 0$ will have a more and more complicated structure.

5. We have investigated above the conditions of self-trapping to strictly one-dimensional weakly nonlinear field distributions. It is also possible to investigate in similar fashion the boundary-value problem in the case when the field distributions are close to the exact solutions corresponding to a plane wave of finite amplitude.

We turn, however, to a case when the field distribution specified on the boundary of the half-space is close to an exact one-dimensional periodic distribution with finite amplitude. We confine ourselves to a current-free distribution in a transparent medium ($k^2 - \kappa^2 > 0$). The exact one-dimensional solution of Eq. (2.1) is[2,3]

$$E(x) = E_c e_\infty(x) \equiv E_c A_\infty \,\mathrm{cn}\,[\xi\sqrt{1 + \alpha A_\infty^2};\ \mathscr{K}_\perp]. \quad (5.1)$$

Here $\xi \equiv x\sqrt{k_\omega^2 - \kappa^2}$; cn(y; $\mathscr{K}_\perp$) is the elliptic Jacobi cosine with modulus

$$\mathscr{K}_\perp = \frac{1}{\sqrt{2}}\frac{\sqrt{\alpha}A_\infty}{\sqrt{1 + \alpha A_\infty^2}}.$$

Putting $e(x, z) \Rightarrow e_\infty(x) + a_0 e(x, z)$, where the amplitude $a_0 \ll 1$ characterizes the deviation of the field on the boundary of the half-space from the zero boundary condition, we arrive at the equation

$$\Delta e + [1 + 3\alpha e_\infty^2(x)]e = -3\alpha a_0 e_\infty(x)e^2 - \alpha a_0^2 e^3 \quad (5.2)$$

with the boundary condition $e(x, z)|_{z=0} = e(x, 0)$. In the linear approximation the solution is given by

$$e^{(0)} = \sum_\nu c_\nu^{(0)}\mathscr{E}_\nu(\xi)\cos(\sqrt{-\Gamma_\nu}\zeta) + \sum_n c_n^{(0)}\mathscr{E}_n(\xi)\varepsilon_n(\zeta). \quad (5.3)$$

Here Γ_ν and Γ_n are the negative and positive eigenvalues of the Lame operator, $\mathscr{E}_\nu$ and $\mathscr{E}_n$ are the corresponding eigenfunctions,[3,5] $e_n(\zeta = \exp(-\sqrt{\Gamma_n}\zeta)$, and

$$c_{n,\nu}^{(0)} = \int d\xi\,\mathscr{E}_{n,\nu}(\xi)e(\xi, 0). \quad (5.4)$$

It is known[3,5] that in this case the Lame equation leads only to three non-positive eigenvalues

$$\Gamma_\nu = 0,\ -{}^3/_2\alpha A_\infty^2,\ 1 - [4 + 6\alpha A_\infty^2 + 3\alpha^2 A_\infty^4]^{1/2}.$$

Thus, in the linear approximation the condition for self-trapping of the perturbed field on the boundary to a strictly one-dimensional distribution at $z \to \infty$ require that $c^{(0)} = 0$ for all non-positive eigenvalues of the Lame operator. In the opposite case there are excited in the medium non-one-dimensional modes of the type $\mathscr{E}_\nu(\xi)\cos(\sqrt{-\Gamma_\nu\zeta})$, and at $a_0 \ll 1$, far from the boundary, there are excited proper non-one-dimensional distributions close to the exact one-dimensional distribution (5.1). Several properties of the latter were considered in [2,3].

Let us assume that the field on the boundary satisfies the self-trapping conditions in the linear approximation. The next higher approximation in the parameter a_0 leads to the equation

$$\Delta e^{(1)} + [1 + 3\alpha e_\infty^2(\xi)]e^{(1)} = -3\alpha e_\infty(\xi)\left[\sum c_n^{(0)}\mathscr{E}_n(\xi)\varepsilon_n(\zeta)\right]^2 \quad (5.5)$$

with zero boundary condition or, after expanding $e^{(1)}$ in terms of the system of eigenfunctions of the Lame operator, to the system of equations

$$\left(\frac{d^2}{d\zeta^2}-\Gamma_m\right)e_m^{(1)}=-3\alpha\sum_{n',n''}\mathfrak{M}_m(n',n'')c_{n'}^{(0)}c_{n''}^{(0)}\varepsilon_{n'}(\zeta)\varepsilon_{n''}(\zeta). \quad (5.6)$$

where

$$\mathfrak{M}_m(n'\,n'')=\int d\xi\, e_\infty\mathscr{E}_m\mathscr{E}_{n'}\mathscr{E}_{n''},\qquad e_m^{(1)}=\int d\xi\,\mathscr{E}_m e^{(1)}(\xi,\zeta).$$

For positive eigenvalues Γ_n the solution of (5.6) takes the form

$$e_n^{(1)}=-3\alpha\sum\frac{\mathfrak{M}_n(n',n'')}{[\sqrt{\Gamma_{n'}}+\sqrt{\Gamma_{n''}}]^2-\Gamma_n}c_{n'}^{(0)}c_{n''}^{(0)}[\varepsilon_{n'}(\zeta)\varepsilon_{n''}(\zeta)-\varepsilon_n(\zeta)], \quad (5.7)$$

provided only $[\sqrt{\Gamma_{n'}}+\sqrt{\Gamma_{n''}}]^2\neq\Gamma_n$. However, no essential complications arise also in the opposite case. For negative eigenvalues Γ_ν we obtain

$$e_\nu^{(1)}=c_\nu^{(1)}\cos(\sqrt{-\Gamma_\nu}\,\zeta)-3\alpha\sum_{n',n''}\frac{\mathfrak{M}_\nu(n',n'')}{[\sqrt{\Gamma_{n'}}+\sqrt{\Gamma_{n''}}]^2-\Gamma_\nu}c_{n'}^{(0)}c_{n''}^{(0)}\varepsilon_{n'}(\zeta)\varepsilon_{n''}(\zeta). \quad (5.8)$$

Here

$$c_\nu^{(1)}=3\alpha\sum_{n',n''}\frac{\mathfrak{M}_\nu(n',n'')}{[\sqrt{\Gamma_{n'}}+\sqrt{\Gamma_{n''}}]^2-\Gamma_\nu}c_{n'}^{(0)}c_{n''}^{(0)}. \quad (5.9)$$

Thus, in order for the conditions for self-trapping to be satisfied not only in the linear approximation but also in the next higher approximation it is necessary to satisfy the conditions $c_\nu^{(0)}=c_\nu^{(1)}=0$ for all the non-positive eigenvalues of the Lame operator. Otherwise there are excited far from the boundary non-one-dimensional distributions with the fundamental mode $\mathscr{E}_\nu(\xi)\times\cos(\sqrt{-\Gamma_\nu}\,\zeta)$ and with amplitude $a_0^2c_\nu^{(1)}$, which at $a_0\ll 1$ are close to the exact one-dimensional fields (5.1).

Consequently it is possible to distinguish among types of boundary fields satisfying the self-trapping conditions with accuracy up to n-th order of smallness in the amplitude of the perturbation on the boundary. Far from the boundary, there are excited in this case non-one-dimensional fields that are close to strictly one-dimensional fields with the amplitude of the fundamental non-one-dimensional mode of (n + 1)-st order of smallness (in terms of the characteristic amplitudes of the perturbation on the boundary).

6. In conclusion we note that a number of properties of the boundary value for a half space can be revealed in the analysis of the exact integral relations that result from the local conservation laws (divergent forms) of the nonlinear nondissipative electrodynamics. Indeed, in the case of a plane geometry the system (2.1), (2.2) leads to the following divergent forms:

$$\frac{\partial T_{zz}}{\partial z}+\frac{\partial T_{zx}}{\partial x}=0,\qquad \frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xz}}{\partial z}=0. \quad (6.1)$$

Here

$$T_{zz}=\frac{1}{2}\left(\frac{\partial E}{\partial z}\right)^2+\frac{1}{2}E^2\left(\frac{\partial\Psi}{\partial z}\right)^2-\frac{1}{2}\left(\frac{\partial E}{\partial x}\right)^2-\frac{1}{2}E^2\left(\frac{\partial\Psi}{\partial x}\right)^2+\frac{1}{2}(k_\omega^2-\varkappa^2)E^2+\frac{1}{4}\left(\frac{\varkappa}{E_c}\right)^2E^4, \quad (6.2)$$

$$T_{xx}=-\frac{1}{2}\left(\frac{\partial E}{\partial z}\right)^2-\frac{1}{2}E^2\left(\frac{\partial\Psi}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial E}{\partial x}\right)^2+\frac{1}{2}E^2\left(\frac{\partial\Psi}{\partial x}\right)^2+\frac{1}{2}(k_\omega^2-\varkappa^2)E^2+\frac{1}{4}\left(\frac{\varkappa}{E_c}\right)^2E^4,$$

$$T_{zx}=T_{xz}=\frac{\partial E}{\partial z}\frac{\partial E}{\partial x}+E^2\frac{\partial\Psi}{\partial z}\frac{\partial\Psi}{\partial x}.$$

In addition, Eq. (2.2) is a divergent form. Integrating (2.2) and (6.1) with respect to the half-space and assuming that when $z\to\infty$ there is realized a strictly one-dimensional distribution with a plane front (for example, a plane waveguide layer is excited[1-4]), we obtain the exact integral relations

$$-\int_{-\infty}^{\infty}dx\,E^2(x,0)\left(\frac{\partial\Psi}{\partial z}\right)_0=k_\infty\int dx\,E^2(x,\infty)\equiv S, \quad (6.3)$$

$$\int_{-\infty}^{\infty}dx\left[\frac{\partial E(x,0)}{\partial x}\left(\frac{\partial E}{\partial z}\right)_0+E^2(x,0)\left(\frac{\partial\Psi}{\partial x}\right)_0\left(\frac{\partial\Psi}{\partial z}\right)_0\right]=0, \quad (6.4)$$

$$\int_{-\infty}^{\infty}dx\left[\frac{1}{1}\left(\frac{\partial E}{\partial z}\right)_0^2+\frac{1}{2}E^2(x,0)\left(\frac{\partial\Psi}{\partial z}\right)_0^2-\frac{1}{2}\left(\frac{\partial E(x,0)}{\partial x}\right)_0^2-\frac{1}{2}E^2(x,0)\left(\frac{\partial\Psi}{\partial x}\right)_0^2+\frac{1}{2}(k_\omega^2-\varkappa^2)E^2(x,0)+\frac{1}{4}\left(\frac{\varkappa}{E_c}\right)^2E^4(x,0)\right]$$
$$=2\int_{-\infty}^{\infty}dx\left[\frac{1}{2}(k_\omega^2-\varkappa^2)E^2(x,\infty)+\frac{1}{4}\left(\frac{\varkappa}{E_c}\right)^2E^4(x,\infty)\right]. \quad (6.5)$$

In the derivation of (6.3)–(6.5) we used the assumptions

$$\lim_{x\to\pm\infty}E(x,z)=0,\quad \lim_{z\to\infty}\frac{\partial\Psi}{\partial x}=0,\quad -\lim_{z\to\infty}\frac{\partial\Psi}{\partial z}=k_\infty,$$

and also the explicit form of the first integral of the one-dimensional equation for $E(x,\infty)$.

For the case when the amplitude $E(x,0)$ and the normal derivative of the phase $(\partial\Psi/\partial z)_0$ are specified on the boundary, the integral relation (6.3) determines the normal derivative of the phase at $z\to\infty$. The unknown quantities in this case are $(\partial E/\partial z)_0$ and $\Psi(x,0)$. We can, however, consider a different formulation of the boundary-value problem, in which the amplitude $E(x,y)$ and the phase $\Psi(x,y)$ are specified on the boundary. In this case the known quantities are the normal derivatives $(\partial E/\partial z)_0$ and $(\partial\Psi/\partial z)_0$. Assuming that the field at the boundary is characterized by a plane front, we find that (6.4) leads to orthogonality of the tangential and normal derivatives of the field amplitude, and (6.5) leads to the obvious relation

$$\int_{-\infty}^{\infty}dx\left[-\left(\frac{dE}{dx}\right)_0^2+\frac{1}{2}(k_\omega^2-\varkappa^2)E^2(x,0)+\frac{1}{4}\left(\frac{\varkappa}{E_c}\right)^2E^4(x,0)\right]$$
$$<2\int_{-\infty}dx\left[\frac{1}{2}(k_\omega^2-\varkappa^2)E^2(x,\infty)+\frac{1}{4}\left(\frac{\varkappa}{E_c}\right)^2E^4(x,\infty)\right]. \quad (6.6)$$

The latter, at a specified value of the energy flux through the boundary S, determines in the space of the parameters characterizing the field at the boundary $E(x,0)$ a region outside of which the self-trapping to a plane waveguide layer is impossible. For example, if the field on the boundary is given by

$$E(x,0)=E_0\exp[-(x/x_\perp)^2]$$

then the allowed region in the plane $\{h,\ l_\perp\}$, where

$$h=\frac{\varkappa^2}{\varkappa^2-k_\omega^2}\left(\frac{E_0}{E_c}\right)^2,\quad l_\perp=x_\perp\sqrt{\varkappa^2-k_\omega^2}$$

is designated by the shaded area in Fig. 2.

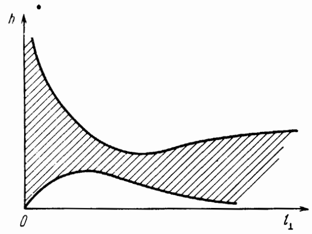

FIG. 2

It should be noted that an analysis of the exact integral relations, and also of inequalities similar to (6.6), is, in our view, of undisputed interest in connection with the fact that the results of the investigation are not connected with any particular form of the asymptotic expansions in terms of a small parameter.

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