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POLARIZATION OF PARTICLES AND QUANTA SCATTERED BY THICK LAYERS OF MATTER

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A method is proposed for calculating the polarization and angular distribution of spin-$\frac{1}{2}$ particles and of quanta scattered by a plane layer of large optical density. The polarization of photons reflected and transmitted by an optically thick medium consisting of freely oriented particles (electrons, atoms, dust particles) is calculated on the assumption that the initial photon beam is incident of an arbitrary angle to the surface of the medium and possesses an arbitrary polarization. Analytic formulas are also obtained for the polarization of neutrons scattered by a plane layer of matter, the polarization being due to spin-orbit interaction with the nuclei. A numerical calculation carried out for the O^{16} nucleus shows that the polarization may be quite large. This permits one to obtain an intense neutron beam with a polarization of several tens of percent.

1. INTRODUCTION

A study of the polarization of particles and quanta emerging from thick layers of matter is of interest many problems in physics and astrophysics, since it yields information on the anisotropy of the scattering medium. Even in those cases when the investigator is interested only in the intensity of the outgoing radiation, its polarization cannot always be neglected. Generally speaking, the polarization and the intensity are determined from a single system of coupled equations, which can be uncoupled only in those cases when the angular distribution of the radiation is close to isotropic. For thin scatterers, both the degree of polarization and the intensity of the radiation can be determined by solving the system of transport equations by successive approximations. We consider here multiple scattering of photons, nucleons, and electrons passing through a thin plane layer of matter, thick enough to cause the radiation to propagate diffusely in the interior of the layer. As noted by the authors elsewhere,[1] in this case the polarization of the emerging radiation is determined completely by the very last collisions prior to leaving the boundary of the layer, since polarization does not arise in the diffusion process. This simplifies greatly the analysis, since it makes it possible to solve the problem of the intensity of the outgoing radiation, without taking into consideration its polarization, and then calculate the polarization.

In the present paper we obtain a system of integral equations for the density matrix of particles and quanta scattered by a plane layer of large optical thickness at an arbitrary law of scattering by an individual force center. By solving this system of equations we calculate the polarization of the photons reflected and transmitted through an optically thick medium consisting of freely-oriented particles (electrons, molecules, dust particles). The initial beam is incident at an arbitrary angle to the surface of the medium and has an arbitrary polarization. We obtain also analytic formulas describing the polarization of neutrons scattered by a plane layer of matter, which arises as a result of spin-orbit interaction with nuclei. A numerical calculation carried out for the nucleus O^{16} at 400 keV shows that the polarization of the neutrons emerging from an optically thick layer can reach an appreciable value (up to 40%). Thus, an optically thick target can serve as an intense source of neutrons with high degree of polarization. The neutrons leaving the target have an energy distribution with a maximum that depends on its thickness. If the thickness is chosen such that this maximum corresponds to an appreciable polarization in single scattering, then the emerging neutrons will be strongly polarized.

2. FORMULATION OF PROBLEM

Let the scattering medium be a plane layer of thickness 2L along the z axis and infinite in the two other directions. The particles of the scatterer are distributed randomly in the layer with a concentration N_0. The intensity and the polarization of the particles with spin $\frac{1}{2}$ and of the photons, as is well known,[2] is described by a two-row density matrix $\rho_{\alpha\beta}(\mathbf{n}_1, z)$ $(\alpha, \beta = -1, 1\,(x, y))$, where $\mathbf{n}_1$ is the direction of the scattered radiation. The sum of the diagonal elements of this matrix $\rho_{\alpha\alpha}(\mathbf{n}_1, z) \equiv I_0(\mathbf{n}_1 z)$ determines the radiation intensity, i.e., the flux of particles through 1 cm^2 per second and per unit solid angle. For photons it is convenient to use energy intensity units I_0 (erg/cm^2 sec-sr). The normalized polarization vector $\mathbf{I}(\mathbf{n}_1 z)$ of particles with spin $\frac{1}{2}$ is determined as the trace of the matrix $\hat{\sigma}\hat{\rho}(\mathbf{n}_1 z)$, where σ is the well known Pauli matrix. The photon polarization is usually determined with the aid of the (non-normalized) Stokes parameters I_1, I_2, I_3, the connection of which with the density matrix elements $\rho_{\alpha\beta}$ depends on the concrete choice of the coordinate system in which $\rho_{\alpha\beta}$ is specified. Thus, if we choose as the polarization unit vectors the usual Cartesian vectors e_x and e_y, then

$$\mathbf{I}(\mathbf{n}_1 z) = \operatorname{Sp}\hat{\sigma}\hat{\rho}(\mathbf{n}_1 z),\ I_i(\mathbf{n}_1 z) = \operatorname{Sp}\hat{\sigma}_i\hat{\rho}(\mathbf{n}_1 z);\ i = 1, 2, 3. \quad (1)$$

The transport equation for the density matrix $(\mathbf{n}_1 z)$ is given by[1,3]

$$\cos\vartheta_1 \frac{d}{dz}\rho_{\alpha\beta}(\mathbf{n}_1 z) = -N_0\sigma_0\rho_{\alpha\beta}(\mathbf{n}_1 z)$$

$$+N_0\int d\mathbf{n}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\,\rho_{\gamma\nu}(\mathbf{n}z)\,t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n})\rangle + R_{\alpha\beta}(\mathbf{n}_1 z). \qquad (2)$$

Here σ_0 is the total cross section for the scattering of particles of quantum with allowance for absorption, $\vartheta_1 = \mathbf{n}_1\mathbf{e}_z$, $t_{\alpha\beta}(\mathbf{n}_1\mathbf{n})$ is the matrix for the scattering of particles (photons) from a direction $\mathbf{n}$ to a direction $\mathbf{n}_1$, $R_{\alpha\beta}(\mathbf{n}_1 z)$ is the source radiation density matrix. The angle brackets $\langle\ \rangle$ denote averaging over the initial states of the target and summation of the final ones. Repeated indices will henceforth imply summation. The quantity

$$B_{\alpha\beta}(\mathbf{n}_1 z) \equiv N_0\int d\mathbf{n}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\,\rho_{\gamma\nu}(\mathbf{n}z)\,t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n})\rangle,$$

which enters in (2) determines the radiation density, i.e., the number of particles per cm^3 per second per sr (erg/cm^2sec-sr for photons). For the matrix $B_{\alpha\beta}$ we can obtain an integral equation[1] by using the Green's function of Eq. (2). It is convenient to use a Fourier transformation in a finite interval

$$\int_0^{2L} dz e^{-iuz} B_{\alpha\beta}(\mathbf{n}_1 z) = 2\exp(-iuL)\,K_{\alpha\beta}(\mathbf{n}_1 u). \qquad (3)$$

We introduce in place of u the dimensionless quantity $s = u/N_0\sigma_0$. The function $K_{\alpha\beta}(\mathbf{n}_1 s)$ satisfies the equation[1,3]

$$K_{\alpha\beta}(\mathbf{n}_1 s) = K_{\alpha\beta}^{(1)}(\mathbf{n}_1 s) + \frac{1}{\pi\sigma_0}\int d\mathbf{n}\int dq\,\frac{f(s-q)}{1+iq\mathbf{n}\mathbf{e}_z}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\,K_{\gamma\nu}(\mathbf{n}q)\,t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n})\rangle,$$
$$f(x) = x^{-1}\sin(xN_0\sigma_0 L). \qquad (4)$$

The free term $K_{\alpha\beta}^{(1)}(\mathbf{n}_1 s)$ describes the singly-scattered particles. Its explicit form is determined by the integral term (4), in which $K_{\gamma\nu}$ is replaced in accordance with (3) by the Fourier transform of the density matrix of the incident particles. If the matrix $B_{\alpha\beta}(\mathbf{n}_1 z)$ is known, then the density matrix $\rho_{\alpha\beta}(\mathbf{n}_1, 2L)$ of the radiation emerging from the layer is determined in accordance with (3) by simple integration:

$$\rho_{\alpha\beta}(\mathbf{n}_1, 2L) = \int_0^{2L} dz\,\sec\vartheta_1 B_{\alpha\beta}(\mathbf{n}_1 z)\exp[-(2L-z)N_0\sigma_0\sec\vartheta_1]$$
$$= 2\sec\vartheta_1\exp[-N_0\sigma_0 L\sec\vartheta_1]\,K_{\alpha\beta}(\mathbf{n}_1;\ i\sec\vartheta_1). \qquad (5)$$

Analogously, for reflected radiation we have

$$\rho_{\alpha\beta}(\mathbf{n}_1, 0) = 2|\sec\vartheta_1|\exp(-N_0\sigma_0 L|\sec\vartheta_1|)\,K_{\alpha\beta}(\mathbf{n}_1;\ i\sec\vartheta_1). \qquad (6)$$

We put in (4) $s = i\sec\vartheta_1$ and integrate with respect to q, using the analytic properties of the function $K_{\alpha\beta}(\mathbf{n}_1 q)$ with respect to the variable q. Using the connection between the matrices $\hat\rho$ and $\hat K$, we obtain

$$\rho_{\alpha\beta}(\mathbf{n}_1 z) = \frac{x_0}{\sigma_0(x_1+\mu x_0)}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_0)\,I_{\gamma\nu}^{(0)}(\mathbf{n}_0)\,t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n}_0)\rangle\Big[\exp\Big(-\frac{\tau_0 z}{x_1 L}\Big)$$
$$-\exp\Big(-\frac{2\tau_0}{x_0}-\frac{\tau_0}{x_1}\frac{2L-z}{L}\Big)\Big] + 2\exp\Big(-\frac{\tau_0}{x_1}-\frac{\tau_0}{x_0}\Big)\frac{1}{\sigma_0}\int\frac{d\mathbf{n}}{x_1+\mu x}$$
$$\times\Big\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\,K_{\gamma\nu}\Big(\mathbf{n},\frac{i}{x_1}\Big)t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n})\Big\rangle + \frac{1}{\sigma_0}\int_{\Omega^-} d\mathbf{n}\frac{x}{x_1+\mu x}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})$$
$$\times\rho_{\gamma\nu}(\mathbf{n}_1, 0)\,t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n})\rangle\exp\Big(-\frac{N_0\sigma_0 z}{x_1}\Big) - \frac{1}{\sigma_0}\int_{\Omega^+} d\mathbf{n}\frac{x}{x_1+\mu x}$$
$$\times\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\,\rho_{\gamma\nu}(\mathbf{n}, 2L)\,t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n})\rangle\exp\Big(-\frac{\tau_0}{x_1}\frac{2L-z}{L}\Big). \qquad (7)$$

We have introduced here the following notation: $\tau_0 = N_0\sigma_0 L$, z is equal to zero or 2L, $x_0 = \cos\vartheta_0$, $x = \cos\vartheta$, $x_1 = |\cos\vartheta_1|$, $\cos\vartheta = \mathbf{n}\mathbf{e}_z$, μ is equal to 1 or -1 for radiation reflected from or transmitted through the layer, respectively. The integration over the region Ω^+ is with respect to φ from 0 to 2π and integration with respect to ϑ from 0 to $\pi/2$; the integration over the region Ω^- is with respect to ϑ from $\pi/2$ to π and with respect to φ from 0 to 2π; $I_{\alpha\beta}^{(0)}(\mathbf{n}_0)$ is the density matrix of the incident radiation.

In general it is impossible to solve Eq. (7). However, for a medium with large optical thickness ($\tau_0 \gg 1$) we can propose the following approximate solution method. As follows from the results of [1,3], the Stokes parameters of the scattered radiation, describing the polarization, are small compared with the intensity of scattering I_0 at large distances from the source. This effect can be easily explained as being due to diffusion of the radiation, which decreases the anisotropy of the radiation flux at large distances. We can therefore expect the polarization of the radiation to be small inside the plane layer, at distances larger than the mean free path from the boundary of the medium, i.e., where the motion has a diffusion character. The main polarization of the emerging radiation is due to single scattering near the layer boundaries. The foregoing physical considerations allow us to choose as the zeroth approximation for the radiation passing through the layer $\rho_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}I_0$, where I_0 is the scattering intensity, i.e., the solution of Eq. (7), in which we neglect the polarization terms in the single-scattering matrix $t_{\alpha\beta}$. After making the substitution $\rho_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}I_0$ in Eq. (7), we determine the density matrix $\rho_{\alpha\beta}$, and consequently the polarization vector I in the first approximation. Substituting the obtained results again in (7) we obtain the value of the polarization vector I in the second approximation, etc.

Since our method of calculating the polarization is applicable only to optically thick layers ($2\tau_0 \gg 1$), we can choose as the zeroth approximation for the intensity $\rho_{\alpha\alpha}$ the solution of Eq. (7) with a differential scattering cross section averaged over all angles

$$\frac{1}{8\pi}\int d\mathbf{n}_1\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\,t_{\gamma\alpha}^+(\mathbf{n}_1\mathbf{n})\rangle.$$

The corresponding expression for the intensity of scattering with a plane layer of large optical thickness ($2\tau_0 \gg 1$) is given by[4,5]

$$I_0(\mathbf{n}_1, 2L) \equiv \rho_{\gamma\gamma}(\mathbf{n}_1, 2L) = I_0^{(0)}\frac{x_0}{8\pi}\frac{H(x_0)H(x_1)}{\tau_0+0{,}71},$$
$$I_0(\mathbf{n}_1, 0) \equiv \rho_{\gamma\gamma}(\mathbf{n}_1, 0) = I_0^{(0)}(1-p)\frac{x_0}{4\pi}\frac{H(x_0)H(x_1)}{x_0+x_1}, \qquad (8)$$

$$H(x) = 1 + \frac{1-p}{2}xH(x)\int_0^1 dy\,\frac{H(y)}{x+y}. \qquad (9)$$

Here $I_0^{(0)}$ is the intensity of the incident radiation, H(x) is the tabulated Chandrasekhar-Ambartsumyan function,[4,5] $p = 1-\sigma_S\sigma_0^{-1}$ is the probability of the true absorption of the particle or quantum in the elementary scattering act, σ_S is the total cross section of elastic scattering of a particle or a quantum.

Formulas (8) are valid in the case of weak absorption in the medium and large optical thickness: $p \ll 1$, $\tau_0 \gg 1$, $\sqrt{3p}\,\tau_0 \ll 1$. In practice it turns out, however, that they describe well the scattered radiation starting with $2\tau_0 \approx 3$.[6] In the case of anisotropic scattering, the dependence of $I_0(\mathbf{n}_1, 2L)$ on x_0 and τ_0 will generally speaking be different than in (8), but the dependence on the cosine of the scattering angle x_1 is close to $H(x_1)$.[7,8]

For example, it is shown in [8] that in the case of an anisotropic scattering cross section $1 + \kappa_1 P_1(\cos\vartheta_1)$ formulas (8) remains valid provided τ_0 is replaced by $\tau_0(1-\kappa_1/3)$. Thus, the function $H(x_1)$ always characterizes the angular distribution of diffusely-scattered radiation. Therefore also in the general case of anisotropic scattering, it is possible to choose as the zeroth approximation for the radiation density matrix $\rho_{\alpha\beta}(\mathbf{n}_1, 2L)$

$$\rho_{\alpha\beta}^{(0)}(\mathbf{n}_1, 2L) = \tfrac{1}{2}\delta_{\alpha\beta} A(x_0, \tau_0, p) H(x_1).$$

The quantity $A(x_0, \tau_0, p)$ remains unknown, but when the normalized Stokes parameters

$$\xi_i(\mathbf{n}_1, 2L) = I_i(\mathbf{n}_1, 2L) / I_0(\mathbf{n}_1, 2L)$$

are determined, this quantity cancels out. The uncertainty of $A(x_0, \tau_0, p)$ likewise does not interfere with refinement of the angular dependence of the radiation emerging from the layer:

$$J_0(x_1, 2L) \equiv I_0(x_1, 2L) / I_0(0, 2L),$$

which we shall henceforth call the normalized intensity. Substituting $\rho_{\alpha\beta}^{(0)}(\mathbf{n}_1, 2L)$ in (7), we obtain the density matrix $\rho_{\alpha\beta}(\mathbf{n}_1, 2L)$ in the first approximation:

$$\rho_{\alpha\beta}^{(1)}(\mathbf{n}_1, 2L) = \frac{A}{2\sigma_0}\left[x_1 H(x_1)\int d\mathbf{n}\,\frac{1}{x_1 - x}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\, t^{+}_{\gamma\beta}(\mathbf{n}_1\mathbf{n})\rangle + \int_{\Omega-} d\mathbf{n}\,\frac{xH(x)}{x - x_1}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n})\, t_{\gamma\beta}{}^{+}(\mathbf{n}_1\mathbf{n})\rangle\right] + O(e^{-2\tau_0}). \quad (10)$$

An appreciable contribution to the polarization of the radiation reflected from the layer is made by the singly-scattered particles, the density matrix of which is determined by the first term in expression (7). We therefore take as the zeroth approximation of the matrix $\rho_{\alpha\beta}(\mathbf{n}_1, 0)$ a matrix whose intensity (diagonal elements) coincides with $I_0(\mathbf{n}_1, 0)$ from (8), and whose polarization (nondiagonal) elements coincide with the polarization (nondiagonal) elements of the density matrix of the singly-scattered particles:

$$\hat{\rho}(\mathbf{n}_1, 0) = \frac{1}{2}\left[\hat{I}\,\frac{I_0^{(0)}}{4\pi}\,\frac{(1-p)x_0 H(x_0) H(x_1)}{x_0 + x_1} + \frac{x_0}{x_0 + x_1} I_i^{(1)}(\mathbf{n}_1)\hat{\sigma}_i\right]. \quad (11)$$

Here

$$I_i^{(1)}(\mathbf{n}_1) = \mathrm{Sp}\,\hat{\sigma}_i\langle \hat{t}(\mathbf{n}_1\mathbf{n}_0)\hat{I}^{(0)}(\mathbf{n}_0)\hat{t}^{+}(\mathbf{n}_1\mathbf{n}_0)\rangle\sigma_0^{-1},$$

and $\hat{I}$ is a unit matrix. Since the first term in (7) takes exact account of the singly-scattered particles, and expression (8) for $I_0(\mathbf{n}_1, 0)$ describes with sufficient accuracy the particles scattered twice or more (at not too an anisotropic cross section and in the case of small absorption $p \ll 1$), it follows that our method gives a complete quantitative description of the reflected radiation. Substituting (11) in (7), we obtain the first approximation for $\rho_{\alpha\beta}(\mathbf{n}_1, 0)$:

$$\hat{\rho}^{(1)}(\mathbf{n}_1, 0) = \frac{x_0}{\sigma_0(x_0 + x_1)}\langle \hat{t}(\mathbf{n}_1\mathbf{n}_0)\hat{I}^{(0)}(\mathbf{n}_0)\hat{t}^{+}(\mathbf{n}_1\mathbf{n}_0)\rangle + \frac{1}{2\sigma_0}\int d\mathbf{n}\,\frac{x_1}{x + x_1} \times\left\langle \hat{t}(\mathbf{n}_1\mathbf{n})\left[\hat{I}\frac{I_0^{(0)}}{4\pi}(1-p)H(x_0)H(x_1) + I_i^{(1)}(\mathbf{n})\hat{\sigma}_i\right]\hat{t}^{+}(\mathbf{n}_1\mathbf{n})\right\rangle\frac{x_0}{x_0 + x_1}$$
$$+ \frac{1}{2\sigma_0}\int_{\Omega-} d\mathbf{n}\,\frac{x}{x + x_1}\left\langle \hat{t}(\mathbf{n}_1\mathbf{n})\left[I\frac{I_0^{(0)}}{4\pi}(1-p)H(x_0)H(x_1) + I_i^{(1)}(\mathbf{n})\sigma_i\right]t^{+}(\mathbf{n}_1\mathbf{n})\right\rangle \cdot\frac{x_0}{x_0 + |x|} + O(e^{-\tau_0}). \quad (12)$$

Thus, the problem has been reduced to a calculation of the integrals (10) and (12). After calculating these integrals, we can easily write down with the aid of (7) and (8) expressions for the density matrix $\rho_{\alpha\beta}$ in the second approximation ($\rho_{\alpha\beta}^{(2)}$) etc. Formulas (7)—(12) enable us to calculate the polarization of particles with spin $\frac{1}{2}$ and of phonons when scattered by a thick layer of matter. They can also serve for a refinement of the angular dependence of the radiation emerging from the layer.

In concluding this section, let us discuss briefly the question of determining the polarization of particles in the case of inelastic scattering. If the incident particle has experienced many collisions in the medium, so that its final energy is $E_1 \ll E_0$, then its angular distribution on emerging from the layer will, first, be independent of the energy and, second, have a diffuse character, i.e., it is described by the function $H(x_1)$.[1,9] In this case the scattering intensity is given by

$$I_0(\mathbf{n}_1, 2L) = A_1(E_1)H(x_1), \quad I_0(\mathbf{n}_1, 0) = A_2(E_1)H(x_1).$$

This makes it possible to calculate the polarization by using formulas (7)—(12), in which the matrix $t_{\alpha\beta}(\mathbf{n}_1\mathbf{n})$ is replaced by $t_{\alpha\beta}(\mathbf{p}_1\mathbf{p})$ —the matrix for the scattering of a particle from a state with momentum $\mathbf{p}$ into a state with momentum $\mathbf{p}_1$, and by integrating with respect to $\mathbf{p}$. In the case of elastic scattering, $t_{\alpha\beta}(\mathbf{p}_1\mathbf{p}) = \delta(\mathbf{p}_1 - \mathbf{p})\, t_{\alpha\beta}(\mathbf{n}_1\mathbf{n})$.

3. INTENSITY AND POLARIZATION OF PHOTONS

Let us consider the intensity and polarization of photons scattered by a thick plane layer of matter consisting of freely oriented particles. The brackets in formula (7) denote in this case averaging over the orientation of the scattered particles. The photon density matrix $\rho_{\alpha\beta}$ is conveniently written in terms of cyclic coordinates with unit vectors

$$\varkappa_{-1} = -(\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}, \quad \varkappa_0 = \mathbf{e}_z, \quad \varkappa_{+1} = (\mathbf{e}_x - i\mathbf{e}_y)/\sqrt{2}.$$

In this case the connection of the intensity I_0 and of the Stokes parameters I_1, I_2, I_3 with the elements of $\rho_{\alpha\beta}$ is

$$\rho_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} I_0 + I_2, & -I_3 + iI_1 \\ -I_3 - iI_1, & I_0 - I_2 \end{pmatrix}. \quad (13)$$

The photon scattering matrix $t_{\alpha\beta}(\mathbf{n}_1\mathbf{n}_0)$ is connected with the polarizability tensor of the scattering particle α_{ik}, for which there is a known expression in terms of the matrix elements:[2]

$$t_{\alpha\beta}(\mathbf{n}_1\mathbf{n}_0) = k^2 D_{\alpha n}^{(1)+}(\Omega_{01})\,\alpha_{n\beta}(\beta_{01}|\mathbf{n}_0); \quad \alpha, \beta = \pm 1, \quad n = 0, \pm 1. \quad (14)$$

Here $k = \omega/c$ is the wave number of the photon, $\mathbf{n}_0$ and $\mathbf{n}_1$ are the unit vectors of the photon directions prior to scattering and after scattering, Ω_{01} is the aggregate of the Euler angles $(\alpha_{01}, \beta_{01}, \gamma_{01})$ on going over from a coordinate system with z axis along $\mathbf{n}_0$ to a coordinate system with z′ axis along $\mathbf{n}_1$. The components of the polarizability tensor $\alpha_{ik}(\beta_{01}|\mathbf{n}_0)$ in (14) have been taken in a coordinate system with the z axis along $\mathbf{n}_0$. The Latin indices in this section run through the values 0 and ± 1, while the Greek indices assume only the values ± 1. $D_{mn}^{(l)}(\Omega_{01})$ is an element of the Wigner rotation matrix;[10] its explicit form is given, for example, in [2,11].

Expressions (7) and (8) contain the quantity

$A_{\alpha\gamma\nu\beta}(n_1n) \equiv \langle t_{\alpha\gamma}(n_1n)\, t^{+}_{\nu\beta}(n_1n)\rangle \sigma_0^{-1}$. Substituting (14) in it and averaging over the orientations of the scattering particle, we obtain[3]

$$A_{\alpha\gamma\nu\beta}(\mathbf{n}_1\mathbf{n}_0) = b_1(\beta_{10}) D^{(1)+}_{\alpha\gamma}(\Omega_{01}) D^{(1)}_{\nu\beta}(\Omega_{01}) + b_2(\beta_{10})\delta_{\alpha\beta}\delta_{\gamma\nu}$$
$$\times \delta_{\gamma\nu}\frac{k^4}{\sigma_0}\left[\frac{a^{(s)}}{10} - \frac{a^{(a)}}{6}\right]\left[D^{(1)+}_{\alpha-\nu}(\Omega_{01}) D^{(1)}_{-\nu\beta}(\Omega_{01}) - D^{(1)+}_{\alpha\nu}(\Omega_{01}) D^{(1)}_{\nu\beta}(\Omega_{01})\right], \tag{15}$$

where $b_1(\beta_{01})$ and $b_2(\beta_{01})$ are invariants and are connected with the components of the scalar ($\alpha^{(0)}$), symmetrical $\alpha^{(s)}_{ik}$), and antisymmetrical ($\alpha^{(a)}_{ik}$) parts of the polarizability tensor α_{ik} of the particle in the following manner:

$$\alpha_{ik}(\beta_{01}|\mathbf{n}_0) = {}^1\!/_3\delta_{ik}\alpha^{(0)} + \alpha^{(s)}_{ik} + \alpha^{(a)}_{ik},$$
$$k^{-4}\sigma_0 b_1(\beta_{01}) = a^{(0)}(\beta_{01}) + {}^1\!/_{30}(a^{(s)}(\beta_{01}) - 5a^{(a)}(\beta_{01})), \tag{16}$$
$$k^{-4}\sigma_0 b_2(\beta_{01}) = {}^1\!/_{10}a^{(s)}(\beta_{01}) + {}^1\!/_6 a^{(a)}(\beta_{01});$$
$$a^{(0)} = |\alpha_{ii}|^2, \quad a^{(s)} = \sum_{i,k} |\alpha^{(s)}_{ik}|^2,$$
$$a^{(a)} = \sum_{i,k} |\alpha^{(a)}_{ik}|^2.$$

The last term in (15) makes a contribution only to the intensity of the circularly polarized radiation I_2. As is well known,[3,4], a separate equation, not connected with the other Stokes parameters and with the intensity, is obtained for I_2 from (2) and (14). On the other hand, the parameters I_0, I_1, and I_2 are likewise independent of I_2. Therefore, in determining the intensity I_0 and the linear polarization I_1 and I_3 we can use (15) without the last term. We shall not calculate the circular polarization here. We note that it can occur only in radiation reflected from a thick layer, if the photon beam incident on the layer is already circularly polarized. There can be no circular polarization in radiation emerging from a thick layer of matter, since the initial circular polarization decreases rapidly with depth (like $e^{-0.85\tau}$), and the scattering of unpolarized or linearly polarized radiation does not produce photons with circular polarization.

We find first the density matrix $\rho_{\alpha\beta}(n_1, 2L)$ of the radiation emerging from the layer. To this end we substitute (15) in (10) and calculate the integrals. Expression (10) can be easily calculated in the case when the invariants $b_1(\cos\beta_{01})$ and $b_2(\cos\beta_{01})$ do not depend on the angles. This condition is satisfied in the highly prevalent and important case when the wavelength of the photon is much larger than the dimensions of the scattering particle. Using the known properties of H functions,[4,5] we obtain for $\rho^{(1)}_{\alpha\beta}(n_1, 2L)$ the following expression at p = 0:

$$\rho^{(1)}_{\alpha\beta}(\mathbf{n}_1, 2L) = \frac{A}{2}\left[\delta_{\alpha\beta}H(x_1) + 3\pi b_1\sqrt{\frac{2}{3}}\,C^{2\,\beta-\alpha}_{1\beta\,1-\alpha}D^{(2)}_{0\,\beta-\alpha}(\Omega_{z1})\right.$$
$$\left.\times\left(H_2 + H_1x_1 - \frac{2}{3}H(x_1)\right)\right]. \tag{17}$$

Here $C^{LM}_{l_1m_1l_2m_2}$ is a Clebsch-Gordan coefficient, $H_n = \int_0^1 dx\, x^n H(x)$ is the moment of the H-function, $H_0 = 2$, $H_1 = 1.155$, $H_2 = 0.822$; $H_3 = 0.639$, $H_4 = 0.524$. In the calculations we have used the well known relation:[3]

$$1 - p = (8\pi/3)(b_1 + 3b_2).$$

For purely scalar (Rayleigh, Thomson) type of scattering ($b_2 = 0$, $b_1 = 3(1-p)/8\pi$) and for p = 0, results of a numerical solution of the transport equation for

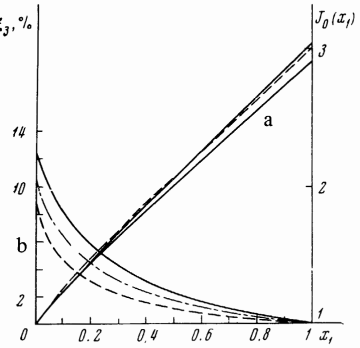

FIG. 1. Polarization and angular distribution of photons passing through a layer, in scalar type of scattering. The solid curves were obtained by numerically solving the problem, and the dashed curves were obtained from formula (17), while the dash-dot curves from (18). a – Ambartsumyan-Chandrasekhar function at p = 0.

this problem are given in [5,12] A comparison of the results given by formula (17) in this particular case with the results of numerical calculation is shown in Fig. 1. We see that the angular dependence of the radiation practically coincides with the exact angular dependence, and the degree of polarization in the entire interval of angles is smaller than the true value by approximately one-half.

Let us find the density matrix $\rho_{\alpha\beta}(n_1, 2L)$ in the second approximation. To this end we substitute (17) in (7) and carry out elementary integration. As a result we obtain

$$\rho^{(2)}_{\alpha\beta}(\mathbf{n}_1, 2L) = \frac{A}{2}\left\{\delta_{\alpha\beta}\left[H(x_1) + \frac{1}{3}\pi b_1 N_2(x_1)\right]\right.$$
$$+ \pi b_1\sqrt{\frac{2}{3}}\,C^{2\,\beta-\alpha}_{1\beta\,1-\alpha}D^{(2)}_{0\,\beta-\alpha}(\Omega_{z1})\cdot[3(H_2 + H_1x_1) - H(x_1)$$
$$\left.+ 4\pi b_1({}^7\!/_{15}N_0(x_1) - {}^{10}\!/_{21}N_2(x_1) + {}^{12}\!/_{35}N_4(x_1))]\right\}. \tag{18}$$

The explicit form of the functions $N_i(x_1)$ is

$$N_0(x) = \frac{3}{2}(H_2 - H_1x)\left[1 + x\ln\left(1 + \frac{1}{x}\right)\right] + \frac{\sqrt{3}}{2} - 2H(x),$$
$$N_2(x) = H(x) + \frac{3}{2}(H_2 + H_1x)[\varphi_2(x) - 1] + \frac{\sqrt{3}}{8},$$
$$N_4(x) = H(x)\left(\frac{35}{12}x^2 - \frac{3}{4}\right) + \frac{3}{2}(H_2 + H_1x)\left[\varphi_4(x) + \frac{5}{2}\right]\frac{\sqrt{3}}{48}$$
$$- \frac{35}{8}(H_4 + H_3x + H_2x^2 + H_1x^3);$$
$$\varphi_n(x) = x\int_0^1 dy\,\frac{P_n(y)}{x+y}, \quad \varphi_2(x) = \frac{3}{4}x(1-2x) + xP_2(x)\ln\left(1 + \frac{1}{x}\right),$$
$$\varphi_4(x) = \frac{x}{8}\left(-35x^3 + \frac{35}{2}x^2 + \frac{55}{3}x - \frac{25}{4}\right) + xP_4(x)\ln\left(1 + \frac{1}{x}\right). \tag{19}$$

Using (13) and (18), we can easily obtain expressions for the degree of linear polarization $\xi^{(2)}_3(n_1, 2L)$ and the normalized radiation intensity $J^{(2)}_0(n_1, 2L)$ in the second approximation. A plot of $\xi^{(2)}_3(n_1, 2L)$ for the particular case of purely scalar scattering is shown in Fig. 1. Comparing it with the results of the numerical solution, we see that the second-approximation formulas describe well the polarization of the radiation emerging from the layer. The formulas for $\xi^{(2)}_3(n_1, 2L)$ and $J^{(2)}_0(n_1, 2L)$ do not depend on the concentration of the scatterers N_0, since the polarization in the angular distribution are determined by the last scattering acts.

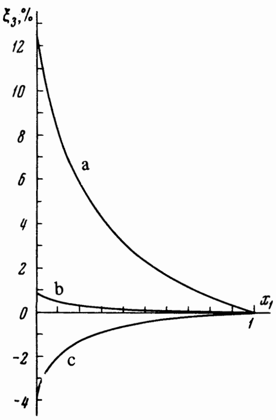

FIG. 2. Degree of linear polarization $\xi_3(x_1 2L)$ in the case of scalar (a), symmetrical (b), and antisymmetrical (c) types of scattering.

Figure 2 shows also the degree of linear polarization $\xi_3(n_1, 2L)$ in the general case when all three types of scattering are present in the medium (scalar, symmetrical, and antisymmetrical). We see that in this case the dependence of the degree of polarization in the scattering angle is the same as in pure scalar (Rayleigh) scattering, but the maximum value of the polarization is much smaller and depends only on b_1. This factor characterizes the decrease of the degree of polarization of the radiation under the influence of the depolarizing effect of the scattering by the anisotropy fluctuations. It can assume different values, depending on the relative contribution of the different types of scattering (scalar, symmetrical, and antisymmetrical) to the cross section. Thus, in scalar Rayleigh scattering ($a^{(s)} = a^{(a)} = 0$), we have $b_1 = 3/8\pi$ and $b_2 = 0$. For purely symmetrical types of scattering ($a^{(0)} = a^{(e)} = 0$) we have $b_1 = 3/80\pi$ and $b_2 = 3b_1$, and for antisymmetrical scattering ($a^{(a)} = a^{(s)} = 0$) we have $b_1 = -b_2 = 3/16\pi$.

Thus, measurement of ξ_3 can serve as a means of obtaining information concerning the presence of any particular type of scattering in a medium.

Formula (18) also yields the polarization and the angular distribution for Compton scattering of photons by free electrons, if the photon energy is $\lesssim$ 100 keV. This scattering is pure scalar: $b_1 = 3(1-p)/8\pi$, $b_2 = 0$. In isotropic scattering ($b_1 = 0$, $b_2 = 1/8\pi$), as expected, the density matrices $\rho^{(1)}_{\alpha\beta}(n_1, 2L)$ and $\rho^{(2)}_{\alpha\beta}(n_1, 2L)$ are identically equal to the matrix $\rho^{(0)}_{\alpha\beta}(n_1, 2L) = \frac{1}{2}\delta_{\alpha\beta}H(x_1)$.

The density matrix $\rho^{(1)}_{\alpha\beta}(n_1, 0)$ of the radiation reflected from the layer is determined from (12). Substituting (15) in (12) and carrying out simple integration, we obtain

$$\rho^{(1)}_{\alpha\beta}(\mathbf{n}_1\mathbf{n}_0 0) = b_1\frac{x_0}{x_0+x_1}\{D^{(1)+}_{\alpha\gamma}(\Omega_{01})I^{(0)}_{\gamma\nu}(\mathbf{n}_0)D^{(1)}_{\nu\beta}(\Omega_{01})$$

$$+2\pi b_1(-1)^n C^{2\beta-\alpha}_{1\beta\,1-\alpha}D^{(2)}_{n\,\beta-\alpha}(\Omega_{z1})\Big[\sqrt{\frac{2}{3}}I^{(0)}_{11}(\mathbf{n}_0)D^{(2)+}_{0n}(\Omega_{z0})$$

$$+I^{(0)}_{-11}(\mathbf{n}_0)D^{(2)}_{2n}(\Omega_{z0})+I^{(0)}_{1-1}(\mathbf{n}_0)D^{(2)}_{-2n}(\Omega_{z0})\Big]C^{j0}_{222-2}C^{j0}_{2n2-n}$$

$$\times[\varphi_j(x_0)+\varphi_j(x_1)]+\frac{1}{3}\delta_{\alpha\beta}I^{(0)}_{\gamma\gamma}(\mathbf{n}_0)[H(x_0)H(x_1)-1]$$

$$+\frac{1}{4}\sqrt{\frac{2}{3}}I^{(0)}_{\gamma\gamma}(\mathbf{n}_0)C^{2\,\beta-\alpha}_{1\beta\,1-\alpha}D^{(2)}_{0\,\beta-\alpha}(\Omega_{z1})\Big[\frac{3}{2}H_1(1-p)(x_0+x_1)H(x_0)$$

$$+(3px_1^2-1)H(x_0)H(x_1)-2P_2(x_0)+2\sqrt{p}H(x_0)P_2(x_0)$$

$$-2\sqrt{p}H(x_0)P_2(x_1)\Big]\Big\}+b_2\delta_{\alpha\beta}I^{(0)}_{\gamma\gamma}(\mathbf{n}_0)\frac{x_0}{x_0+x_1}H(x_0)H(x_1). \quad (20)$$

Equation (20) implies summation over j = 0, 2, 4 and n = 0, ± 1, ± 2.

Let us discuss now the accuracy of expression (20). In the case of purely isotropic scattering ($b_1 = 0$, $b_2 + (1-p)/8\pi$) the formula gives an exact solution of the problem. This is natural, since (9) is, by definition, the solution of (12) in the case of isotropic scattering. For this reason (20) will be a good approximation for $(n_1, 0)$ if $b_1 \ll b_2$. The accuracy of formula (20) in the opposite limiting case of pure scalar scattering ($b_1 = 3(1-p)/8\pi$, $b_2 = 0$) can be estimated from a comparison with some numerical solution of the problem. In [5,12] are given the results of a numerical solution of the problem for the case of unpolarized radiation perpendicularly incident on the layer. In Fig. 3 are shown the intensity and the polarization, calculated from formula (20), as well as their exact values taken from [5]. The error of our approximation for the intensity of the reflected radiation is maximal in a direction parallel to the surface of the layer ($\vartheta_1 = \pi/2$), and is equal to 8%, while for polarization the error amounts to 18%, inasmuch as the polarization decreases with decreasing angle ϑ_1. Since the contribution of single-scattered photons to the intensity of the reflected radiation increases in the case of oblique incidence of the beam, one should expect formula (20) to be more accurate in these cases, since formula (20) takes exact account of the singly-scattered photons. For an extensive class of problems, particularly in astrophysics, such an accuracy is perfectly adequate. We therefore do not present formulas for $\rho^{(2)}_{\alpha\beta}(n_1, 0)$.

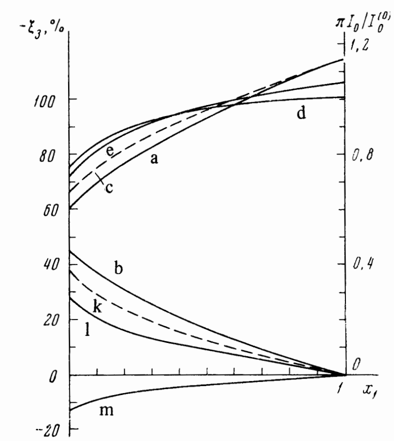

FIG. 3. Polarization and intensity of radiation reflected from a layer for the case when unpolarized photons are perpendicularly incident on the layer: a and b – exact intensity of polarization in scalar scattering, c, d, e, and k, l, m – intensity and polarization in scalar, symmetrical, and antisymmetrical types of scattering, calculated from formula (20). The values of curve l have been magnified 10 times.

Figure 3 shows also a plot of the reflected radiation in the case of a general type of scattering in a medium following perpendicular incidence of an unpolarized beam of protons on the surface of a plane layer.

4. INTENSITY AND POLARIZATION OF PARTICLES WITH SPIN $\frac{1}{2}$

Let us consider the problem of determining the polarization and the angular distribution of particles with spin $\frac{1}{2}$, emerging from a plane layer of large optical thickness $2\tau_0 \gg 1$. The nuclei of the target are assumed to be unpolarized. In this case the scattering matrix of particles with spin $\frac{1}{2}$ is given by[2,13]

$$t(\mathbf{n}_1\mathbf{n}_0) = \hat{I}f(\beta_{01}) - ig(\beta_{01})\hat{\sigma}\tau_{01}, \quad \tau_{01} = [\mathbf{n}_0\mathbf{n}_1](\sin\beta_{01})^{-1}. \qquad (21)$$

The functions $f(\beta_{01})$ and $g(\beta_{01})$ can be expanded in partial waves:

$$f(\beta) = \frac{1}{k}\sum_{l=0}^{\infty}[(l+1)e^{i\delta_l^+}\sin\delta_l^+ + le^{i\delta_l^-}\sin\delta_l^-]P_l(\cos\beta),$$

$$g(\beta) = \frac{\sin\beta}{k}\sum_{l=1}^{\infty}[e^{i\delta_l^+}\sin\delta_l^+ - e^{i\delta_l^-}\sin\delta_l^-]\frac{d}{d\cos\beta}P_l(\cos\beta). \quad (22)$$

Here k is the wave vector of the incoming particle; $\delta_l^+ = \delta_{l+1/2}$ and $\delta_l^- = \delta_{l-1/2}$ are the scattering phases.

The quantity $\langle \hat{t}(\mathbf{n}_1\mathbf{n}_0)\hat{t}^+(\mathbf{n}_1\mathbf{n}_0)\rangle$, in the coordinate system connected with the mass center, is given by[14]

$$k^2\langle\hat{t}(\mathbf{n}_1\mathbf{n}_0)\hat{t}^+(\mathbf{n}_1\mathbf{n}_0)\rangle = \hat{I}\sum_{l=0}^{\infty}B_lP_l(\cos\beta_{01}) \qquad (23)$$

$$+ \hat{\sigma}[\mathbf{n}_0\mathbf{n}_1]\sum_{l=1}^{\infty}C_l dP_l(\cos\beta_{01})/d\cos\beta_{01},$$

$$B_l = \sum_{J_1=0}^{\infty}\sum_{J_2=0}^{\infty}\sum_{l_1=J_1-1/2}^{J_1+1/2}\sum_{l_2=J_2-1/2}^{J_2+1/2} Z^2(l_1J_1l_2J_2;\,{}^1/_2 l)$$

$$\cdot\sin\delta_{l_1J_1}\sin\delta_{l_2J_2}\cos(\delta_{l_1J_1}-\delta_{l_2J_2}), \qquad (24)$$

$$C_l = \left(\frac{3(2l+1)}{2l(l+1)}\right)^{1/2}\sum_{J_1}\sum_{J_2}\sum_{l_1}\sum_{l_2}(-1)^{J_1+l_2-l+1/2}Z^2(l_1J_1l_2J_2;\,{}^1/_2 l)$$
$$\times X(J_1l_1{}^1/_2;\,J_2l_2{}^1/_2;\,ll1)\,W^{-1}(l_1J_1l_2J_2;\,{}^1/_2 l)\sin\delta_{l_2J_2}\sin\delta_{l_1J_1}\sin(\delta_{l_1J_1}-\delta_{l_2J_2}).$$

Here $W(l_1J_1l_2J_2;\,\frac{1}{2}l)$ is the usual Racah coefficient, $X(J_1l_1\frac{1}{2};\,J_2l_2\frac{1}{2};\,ll1)$ is the Fano function; the coefficient Z is expressed in terms of the Racah coefficients and the Clebsch-Gordan coefficients $C^{LM}_{lml'm'}$:

$$Z(l_1J_1l_2J_2;\,sl) = i^{l-l_1+l_2}\sqrt{(2l_1+1)(2J_1+1)(2J_2+1)}\,W(l_1J_1l_2J_2;\,sl)\,C^{l0}_{l_10l_20},$$

and δ_{lJ} are the corresponding scattering phase shifts.

An analytic expression for the density matrix of the transmitted radiation can be obtained by substituting (23) in (10). However, since the transport equation is written in the laboratory coordinate system, the expression for the density matrix (10), following substitution of (23), is valid only for scattering of neutrons by heavy nuclei, when the recoil of the nucleus in scattering can be neglected. For light and medium nuclei, the recoil of the nucleus plays an important role in the scattering process and must be taken into account. To this end it is necessary first of all to write down the expression for the transition probability (23) $\langle \hat{t}(\mathbf{n}_1\mathbf{n}_0)\hat{t}^+(\mathbf{n}_1\mathbf{n}_0)\rangle$ in the laboratory system, which can be readily done by using the well known formulas for the conversion from the c.m.s. to the laboratory system (see, for example [13]). The formulas (8) for the scattering intensity are themselves valid only in the case of pure elastic scattering in the laboratory frame, i.e., only for heavy nuclei. However, as stated in Sec. 2, in the case of inelastic scattering the intensity of the particles passing through a plane layer of large thickness is of the form $I_0(\mathbf{n}_1, 2L) = A_1(E_1)H(x_1)$. The coefficient A_1 is determined by the cross section of the elementary act of inelastic interaction. It was calculated in [9] for the case when it is possible to neglect the anisotropy of the neutron scattering and the dependence of the scattering cross section on the energy in the c.m.s.

As follows from the results of that paper, the neutrons emerging from an optically thick layer have a broad energy spectrum, the maximum of which takes place at an energy E_1 given by the expression

$$E_1 \approx E_0\exp[-6\tau_0\gamma(1-{}^4/_3\gamma)], \qquad (25)$$

where γ is the reciprocal mass number of the nucleus. Formula (25) can be used for estimates in the general case, when the neutron scattering cross section in the c.m.s. is anisotropic and depends on the energy. Thus, if a beam of neutrons with energy E_0 is incident on a plane layer of thickness $2\tau_0$, then the scattered neutrons will have, in the main, an energy E_1 determined by the expression (25), and will be polarized. The polarization of the neutrons is determined by formula (20), if we substitute in it expression (23), transformed to the laboratory coordinate system. The normalized polarization does not depend on the concrete form of the coefficient $A_1(E_1)$.

We shall not present the explicit form of (10) in the general case, and confine ourselves to the case of practical interest, when it is possible to retain in the expansion (22) only the S and P waves. This corresponds to a scattered-particle energy not exceeding 10 MeV. In this case the calculation of the polarization by means of formula (10) yields

$$\mathbf{I}^{(1)}(\mathbf{n}_1;2L) = \frac{\pi A_1}{\sigma_0}[\mathbf{e}_z\mathbf{n}_1]P(x_1),$$

$$P(x_1) = 2H_1(C_1-3\gamma C_2) + 3(C_2+\gamma C_1)x_1[3(H_2+H_1x_1)-2H(x_1)]$$
$$+ 12\gamma C_2[(5x_1^2-1)(H_3+H_2x_1+H_1x_1^2-{}^2/_3x_1H(x_1))$$
$$+(1-3x_1^2)H_1] + O(2\gamma^2). \qquad (26)$$

We get accordingly for the scattering intensity

$$I_0^{(1)}(\mathbf{n}_1, 2L) = \frac{4\pi A_1}{\sigma_0}\Big\{B_0H(x_1) + \left(\frac{1}{2}B_1+\gamma B_0+\frac{1}{5}\gamma B_2\right)H_1P_1(x)$$
$$+{}^3/_4(B_2+2\gamma B_1)[H_2+H_1x_1-{}^2/_3H(x_1)]P_2(x_1) + 3\gamma B_2[H_3+H_2x_1$$
$$+H_1x_1^2-{}^3/_5H_1-{}^2/_3x_1H(x_1)]P_3(x_1)\Big\} + O(2\gamma^2). \qquad (27)$$

It is easy to see that the normalized polarization vector

$$\xi^{(1)}(\mathbf{n}_1, 2L) \equiv \mathbf{I}^{(1)}(\mathbf{n}_1, 2L)\,/\,I_0'(\mathbf{n}_1, 2L)$$

and the normalized angular distribution

$$J_0^{(1)}(\mathbf{n}_1, 2L) \equiv I_0^{(1)}(\mathbf{n}_1, 2L)/I_0^{(1)}\ (x_1=0, 2L)$$

do not depend on the unknown constant A_1 or on the total cross section σ_0.

By way of an example of the application of the obtained formulas (26) and (27), let us calculate the angular dependence and the polarization of the neutrons emerging from an oxygen target. At an energy of approximately 435 keV, there is in oxygen O^{16} a resonance for neutron scattering, connected with the state $\frac{3}{2}^+$. At this energy, the only scattering phases that can be regarded as significant are S and $P_{3/2}$ (see [15]).

Figure 4 shows the values of $|\xi^{(1)}(\mathbf{n}_1, 2L)|$ and

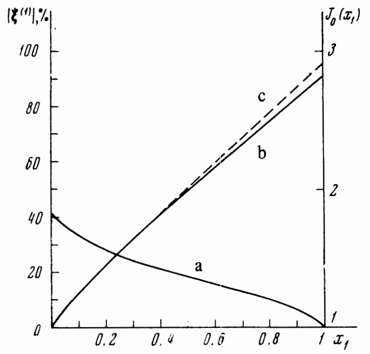

FIG. 4. Absolute value of the normalized polarization vector of 400-keV neutrons emerging from a thick layer of oxygen (a), normalized neutron intensity $J_0^{(1)}(x_1, 2L)$ (b), and the Ambartsumyan-Chandrasekhar function at p = 0 (c).

$J^{(1)}(x_1, 2L)$ at 400 keV. As seen from the figure, the neutron polarization is quite appreciable. Its maximum value reaches 41%. The angular distribution (27) is hardly different from $H(x_1)$ in this case, thus proving the correctness of the choice of the H-function as the zeroth approximation. The accuracy of formulas (26) and (27) can be estimated, just as in the case of photon scattering, by a comparison with the second approximation for $I(n_1, 2L)$ and $I_0(n_1, 2L)$, which can be readily obtained by substituting (26) and (27) in (7). As such an estimate, we can calculate $\rho_{\alpha\beta}^{(2)}(n_1, 2L)$ for $x_1 = 0$:

$$\rho_{\alpha\beta}^{(2)}(x_1 = 0, 2L) = \frac{1}{\sigma_0} \int_{\Omega+} d\mathbf{n} \langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}) \rho_{\gamma\nu}^{(1)}(\mathbf{n}, 2L) t_{\nu\beta}^{+}(\mathbf{n}_1\mathbf{n}) \rangle. \tag{28}$$

An estimate by means of (28) for the case of neutron scattering by oxygen shows that the accuracy of the method is not worse than 7% for the polarization and 8% for the angular distribution.

In concluding this section, let us discuss the applicability of the polarization-calculation method described above to those cases when the scattering cross section is strongly elongated forward (Coulomb scattering of particles, scattering of light by large dust particles). In this case the condition $2\tau_0 \gg 1$ is necessary but not sufficient for complete isotropization of the scattered radiation. A characteristic quantity describing the isotropization of the scattered radiation is the so-called transport mean free path of the particle in the medium[7,16]

$$\Lambda_T = \Lambda \left[\int d\mathbf{n} (1 - \cos\vartheta) \sigma(\vartheta) \sigma_0^{-1} \right]^{-1},$$

where Λ is the mean free path of the particle (or photon) in the medium, $\sigma(\vartheta)$ is the differential cross section for scattering through a given angle. If the layer thickness 2L is much larger than the transport mean free path, then the motion of the particles in the medium has a diffuse character, and the angular distribution of the particles is well described by formula (8). The polarization of the particles in this case turns out to be small, and is proportional to the ratio $(\Lambda/\Lambda_T) \ll 1$. The reason for the smallness of the polarization in this case is the effect of multiplicity of the scattering, the result of which is that practically isotropic radiation is incident on the layer, of thickness Λ, responsible for the polarization. The main contribution to the radiation scattered through a given angle is made by particles that are deflected by small angles. The number of such particles is proportional to Λ_T. Their polarization is practically zero. The number of particles scattered through a large angle and having an appreciable polarization is small, $\sim\Lambda$. Therefore the polarization is very small in this case, $\sim\Lambda/\Lambda_T$. In ordinary experiments with single scattering, the beam has a strictly fixed direction, and the greater part of the particles (or quanta), which is scattered at small angles and has no polarization, makes no contribution to the large-angle scattering intensity. And although the number of particles scattered through a large angle is small, their polarization can reach an appreciable value $\sim\sin\vartheta_1$. If the thickness of the layer satisfies the condition $\Lambda \ll 2L \ll \Lambda_T$, then the angular distribution of the scattered particles has a maximum in the region of small angles.[7,16] In this region of angles, the particle polarization is small. The electron polarization in the case of small-angle Coulomb scattering was calculated, for example, in [17].

5. CONCLUSION

One of the most important problems of scattering theory is the determination of the properties of a scattering medium from the intensity and polarization of the scattered radiation. In particular, an important problem is the determination of the invariants $a^{(0)}$, $a^{(s)}$, and $a^{(a)}$ of the scattering tensor of the medium. As is well known, in laboratory experiments with single scattering of natural light, the angular dependence and the polarization of the scattered radiation depend only on one parameter b_1, which is a linear combination of the invariants of the scattering tensor (Eq. (16)). In astrophysics we frequently deal with an optically thick scattering medium. In this case, as we have already seen, the angular dependence is practically independent of the pattern of scattering from an individual force center (Eq. (8)). Therefore the only source of information on the parameter b_1 of the scattering tensor is measurement of the degree of linear polarization of the scattered radiation, which does not depend on b_1. A second relation for the determination of the invariants is the connection between the probability of the true absorption p and the parameters b_1 and b_2, namely $1 - p = 8\pi(b_1 + 3b_2)/3$. A third relation can be obtained by measuring the circular polarization of the scattered light, under the condition that the degree of circular polarization of the radiation source is determined with the aid of an independent experiment. If we confine ourselves to the measurement of linear polarization, we cannot determine uniquely all three invariants of the scattering tensor, but nevertheless we can decrease the arbitrariness of the choice. Thus, we can obtain information concerning the presence of any particular type of scattering in the medium (scalar scattering: $b_1 = 3/8\pi$, $b_2 = 0$; symmetrical: $b_1 = 3/80\pi$, $b_2 = 3b_1$; antisymmetrical: $b_1 = -b_2 = 3/16\pi$).

As to neutrons, we have already shown that an optically thick target can serve as a source of neutrons with a high degree of polarization. Unlike the case of single scattering, we gain also in the total intensity of the neutrons, although we lose in the degree of polarization of the beam.

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