SOVIET PHYSICS JETP VOLUME 31, NUMBER 2 AUGUST, 1970

CALCULATION OF THE POLARIZATION CONTRIBUTION TO THE ENERGY OF INTERACTION OF A CHARGE WITH THE SURFACE OF A METAL

A. V. SIDYAKIN

Submitted March 11, 1969; resubmitted July 1, 1969

Zh. Eksp. Teor. Fiz. 58, 573–581 (February, 1970)

The polarization part of the energy of interaction between a fixed point charge and a semi-bounded metal is calculated in the approximation of high density as a function of the distance to its surface. At the metal's surface this quantity amounts to approximately 2/3 of the value inside the metal, and as the charge recedes from the metal it asymptotically goes over into the "potential of the image forces." The polarization contribution to the electron's energy agrees with the calculated result if a "Coulomb hole" gives the major contribution to the electron's mass operator. The results of the calculations are applied to a calculation of the field emission current from the metal. It is shown that the effect of broadening the barrier in comparison with that used for the derivation of the Fowler-Nordheim formula is to a considerable extent cancelled by the effect of penetration of the field into the metal; taking both of these effects into consideration does not lead to significant corrections to the Fowler-Nordheim formula.

1. In electrostatics the polarization part of the energy of interaction between a charge and a semi-bounded metal is determined by the well-known expression

$$U(z) = -e^2/4z \tag{1}$$

(e denotes the charge, and z is the distance between the charge and the metal's surface). Expression (1) becomes invalid as z → 0 and must be changed in this region.

A quantum mechanical calculation of the polarization contribution was made by Bardeen.[1,2] In[1] it was shown that the law (1) is valid asymptotically at large distances, but it was not established, starting at what distances does it begin to be violated. The correction to (1) due to the change of the kinetic energy of the metal's electrons was calculated in[2], and the distances were found at which this correction becomes comparable with (1); there it is noted that taking account of the fact that the electrons induced in the metal are not located exactly on its surface, as is assumed in electrostatics, plays a more important role. The computation itself, however, was not carried out in[2]. A different kind of correction to (1) was calculated in later articles,[3] but the whole behavior of the polarization contribution in the most interesting region z → 0 remained uninvestigated in the articles enumerated above.

In the present article the energy of the interaction of the charge with the metal is calculated in the approximation of high density, in which the polarization contribution is equal to half the product of the charge e times the potential created by the charges induced in the metal at the point where the charge e is located. When the charge is located outside the metal the same answer is obtained according to the method of images; the difference consists in the distribution of the induced charges in the metal.

In electrostatics it is assumed that the charges are situated exactly on the surface of the metal, and this finally leads to nonapplicability of the classical formula (1) near the metal. From the equation for the effective potential characteristic of the high-density approximation, it follows that when the charge is located near the metal the induced charges are distributed in a surface region of dimensions on the order of the Debye screening radius in the metal. Taking the latter fact into consideration leads to a revision of formula (1) at small distances from the metal.

2. Let us calculate the energy of interaction between the charge and the metal. Let E_0 be the ground state energy of the metal, and let E be the ground state energy of the system consisting of the metal and of a charge at rest inside or outside of it. The interaction energy of the charge with the metal is equal to $E - E_0$. In order to calculate this quantity it is convenient to use the well-known quantum mechanical formula giving the change in the ground state energy of a system when an additional interaction appears in it:

$$E - E_0 = \int_0^1 \frac{d\lambda}{\lambda} \langle \psi_\lambda | \lambda V | \psi_\lambda \rangle, \tag{2}$$

where ψ_λ denotes the exact wave function of the system when the interaction λV is switched on. In the case under consideration V is the Coulomb interaction of a point charge e with the ions and electrons of the metal, and one can represent expression (2) in the form

$$E - E_0 = \int_0^1 \frac{d\lambda}{\lambda} \int d\mathbf{r}' \frac{\lambda e}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}', \mathbf{r}, \lambda),$$

where $\rho(\mathbf{r}', \mathbf{r}, \lambda)$ denotes the density of ions and electrons at the point $\mathbf{r}'$ of the metal in the case when a charge λe is located at the point $\mathbf{r}$. Representing ρ in the form of a sum $\rho_0 + \delta\rho$, where ρ_0 is the unperturbed density (for $\lambda = 0$), let us write

$$E - E_0 = e \int d\mathbf{r}' \frac{\rho_0(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + U(\mathbf{r}),$$

where the integral is the unperturbed electrostatic potential, which is not of interest here and will not be calculated, and the quantity

$$U(\mathbf{r}) = \int_0^1 d\lambda \int d\mathbf{r}' \frac{e}{|\mathbf{r} - \mathbf{r}'|} \delta\rho(\mathbf{r}', \mathbf{r}, \lambda) \tag{3}$$

represents the polarization contribution to the energy of interaction between the charge and the metal.

Formula (2) and its transform for the case of the interaction of a classical charge with any kind of quantum mechanical system are exact. In order to determine the function $\delta\rho$ we shall use the high-density approximation in which the classical charge e creates an effective potential $\gamma(\mathbf{r}_1, \mathbf{r}_2)$ which satisfies the equation[1)]

$$\gamma(\mathbf{r}_1, \mathbf{r}_2) = \frac{e}{|\mathbf{r}_1 - \mathbf{r}_2|} + \int d\mathbf{r}_3\, d\mathbf{r}_4 \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_3|} \Pi(\mathbf{r}_3, \mathbf{r}_4) \gamma(\mathbf{r}_4, \mathbf{r}_2), \qquad (4)$$

where $\Pi(\mathbf{r}_1, \mathbf{r}_2)$ is the polarization operator for an electron gas.[4] From the linear equation (4) it follows that the function $\delta\rho(\mathbf{r}', \mathbf{r}, \lambda)$ of interest to us has the form

$$\delta\rho(\mathbf{r}', \mathbf{r}, \lambda) = \lambda \int d\mathbf{r}_1 e^2 \Pi(\mathbf{r}', \mathbf{r}_1) \gamma(\mathbf{r}_1, \mathbf{r}).$$

Substituting the last relation into (3), integrating over λ, and taking (4) into account, we obtain the following result for the polarization contribution to the energy of a classical point charge:

$$U(\mathbf{r}) = \frac{e}{2} \left[\gamma(\mathbf{r}', \mathbf{r}) - \frac{e}{|\mathbf{r}' - \mathbf{r}|} \right]_{\mathbf{r}' = \mathbf{r}}. \qquad (5)$$

For an electron the quantity analogous to (5) is determined by the mass operator M which enters into the one-particle Schrödinger equation

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) \right] \psi + \int M(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') d\mathbf{r}' = E\psi, \qquad (6)$$

where $V(\mathbf{r})$ is the potential energy of an electron in the field of the ions and the self-consistent Hartree field of the remaining electrons. In connection with an expansion of the mass operator in a screened Coulomb potential, which was recently studied by Hedrin,[5] a "Coulomb hole" gives the major contribution to $M(\mathbf{r}, \mathbf{r}')$. The expansion has the form

$$M(\mathbf{r}, \mathbf{r}') = \frac{e}{2} \left[\gamma(\mathbf{r}, \mathbf{r}') - \frac{e}{|\mathbf{r} - \mathbf{r}'|} \right] \delta(\mathbf{r} - \mathbf{r}') + M', \qquad (7)$$

where the term M′ includes the effects of screened exchange. Their contribution to the electron energy was calculated in[5] and turned out to be small in comparison with the contribution from a "Coulomb hole." If the term M′ is neglected in comparison with the first term in (7), then from Eqs. (6) and (7) one can easily see that the polarization contribution to the electron energy agrees with the same quantity for a classical charge, determined by relation (5). In order to calculate expression (5) it is necessary to solve Eq. (4) for the effective potential created by a point charge located outside or inside the semi-bounded metal.

Let us find a solution under the assumption that the polarization operator Π may be represented in the form

$$\Pi(\mathbf{r}_1, \mathbf{r}_2) = \begin{cases} \Pi_0(\rho, z_1 - z_2) + \Pi_0(\rho, z_1 + z_2), & z_1, z_2 < 0 \\ 0, & z_1 \text{ or } z_2 > 0 \end{cases}, \qquad (8)$$

where Π_0 is the polarization operator for a homogeneous electron gas, $\rho = \rho_1 - \rho_2$ is the difference between the projections of the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ on the plane z = 0 (the metal occupies the half-space z < 0). It will be shown below that the representation (8) for the polarization operator of an electron gas for a semi-bounded metal leads to the same result as the assumption about specular scattering of electrons at the boundary of the metal, which is used, for example, for a macroscopic description of the problem of the penetration of an electric field into a metal.[6] Relation (8) enables us to express the solution of Eq. (4) in terms of the characteristics of a homogeneous electron gas (in terms of the Fourier component $\Pi_0(\mathbf{k})$ of the polarization operator $\Pi_0(\mathbf{r} - \mathbf{r}')$ for a homogeneous electron gas or finally in terms of its dielectric constant $\epsilon(\mathbf{k})$ as a function of the wave vector $\mathbf{k}$).

Operating on both sides of Eq. (4) with the Laplacian operator with regard to the variable $\mathbf{r}_1$ and changing to the Fourier components $\gamma(\boldsymbol{\kappa}, z_1, z_2)$, $\Pi_0(\boldsymbol{\kappa}, z_1, z_2)$ with respect to the variable ρ, we obtain

$$\frac{d^2\gamma(\varkappa, z_1, z_2)}{dz_1^2} + 4\pi e^2 \theta(-z_1) \int_{-\infty}^{0} dz_3 [\Pi_0(\varkappa, z_1 - z_3) + \Pi_0(\varkappa, z_1 + z_3)] \gamma(\varkappa, z_3, z_2) - \varkappa^2 \gamma(\varkappa, z_1, z_2) = -4\pi e \delta(z_1 - z_2), \qquad (9)$$

where

$$\theta(z) = \begin{cases} 1, & z > 0 \\ 0, & z < 0 \end{cases}.$$

In the case when the charge is located outside the metal ($z_2 > 0$), we seek the solution of Eq. (9) in the form

$$\gamma(\varkappa, z_1, z_2) = (4\pi e / 2\varkappa) e^{-\varkappa |z_1 - z_2|} + f_1(\varkappa, z_2) e^{-\varkappa z_1}, \quad z_1 > 0; \qquad (10)$$
$$\gamma(\varkappa, z_1, z_2) = f_2(\varkappa, z_2) \int \frac{dk_z}{2\pi} \frac{4\pi e}{k^2 \varepsilon(k)} e^{ik_z z_1}, \quad z_1 < 0.$$

Here $\epsilon(\mathbf{k})$ is the dielectric constant of a homogeneous electron gas, which is connected to the Fourier component $\Pi_0(\mathbf{k})$ by the relation

$$\varepsilon(k) = 1 - (4\pi e^2 / k^2) \Pi_0(k) \quad (k^2 = k_z^2 + \varkappa^2),$$

and the functions f_1 and f_2 are determined from the conditions for continuity of γ and $d\gamma/dz_1$ at $z_1 = 0$.

The final answer has the form:

$$\gamma(\mathbf{r}_1, \mathbf{r}_2) = \frac{e}{|\mathbf{r}_1 - \mathbf{r}_2|} + \int \frac{d\boldsymbol{\varkappa}}{(2\pi)^2} \exp\{i\boldsymbol{\varkappa}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) - \varkappa(z_1 + z_2)\} \cdot \frac{4\pi e}{2\varkappa} \frac{\varkappa\varphi(\varkappa, 0) - 1}{\varkappa\varphi(\varkappa, 0) + 1}, \quad z_1 > 0, \qquad (11a)$$

$$\gamma(\mathbf{r}_1, \mathbf{r}_2) = \int \frac{d\boldsymbol{\varkappa}}{(2\pi)^2} \exp\{i\boldsymbol{\varkappa}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) - \varkappa z_2\} \frac{4\pi e \varphi(\varkappa, z_1)}{\varkappa\varphi(\varkappa, 0) + 1}, \quad z_1 < 0, \qquad (11b)$$

where

$$\varphi(\varkappa, z_1) = \frac{1}{\pi} \int dk_z \frac{e^{ik_z z_1}}{k^2 \varepsilon(k)}. \qquad (12)$$

In order to obtain expressions (11) the fact that

$$\left. \frac{d\varphi(\varkappa, z)}{dz} \right|_{z \to -0} = 1.$$

was taken into account. One can obtain the last condition by integrating the differential equation which the function φ obeys near z = 0.

In the case when the charge is located inside the metal ($z_2 < 0$), in similar fashion we obtain

$$\gamma(\mathbf{r}_1, \mathbf{r}_2) = \int \frac{d\boldsymbol{\varkappa}}{(2\pi)^2} \exp\{i\boldsymbol{\varkappa}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) - \varkappa z_1\} \cdot \times \frac{4\pi e}{2} \frac{\varphi(\varkappa, z_2) + \varphi(\varkappa, 0) \varphi'(\varkappa, z_2)}{\varkappa\varphi(\varkappa, 0) + 1}, \quad z_1 > 0, \qquad (13a)$$

Here and below (Eq. (17)) the prime denotes the derivative of the function $\varphi(\kappa, z_1)$, defined by Eq. (12), with respect to the second argument.

[1)] In Eq. (4) the same symbol e is used for the classical charge and for the charge of an electron. This should not lead to confusion since the latter enters everywhere only in the combination $e^2\Pi$.

$$\gamma(\mathbf{r}_1,\mathbf{r}_2)=\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi e}{k^2\varepsilon(k)}\exp\{i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)\}-\int\frac{d\varkappa}{(2\pi)^2}\exp\{i\varkappa(\rho_1-\rho_2)\}$$
$$\times\frac{4\pi e}{2}\varphi(\varkappa,z_1)\frac{\varkappa\varphi(\varkappa,z_2)-\varphi'(\varkappa,z_2)}{\varkappa\varphi(\varkappa,0)+1},\quad z_1<0. \tag{13b}$$

Before going on to calculate the polarization contribution according to formula (5), let us clarify what kind of result relation (8) leads to in the problem of the penetration of a longitudinal electric field into a metal, which was solved by Shafranov[6] (also see[7]). One can obtain the solution of this problem from formula (11) if, instead of a point charge located at the point $\mathbf{r}_2$ one considers the charged plane $z = z_2$; for this purpose it is necessary to multiply expression (11) by the surface charge density σ and integrate with respect to the coordinate ρ_2. Denoting the average potential thus obtained by $V(z)$ and the quantity $4\pi\sigma$ by E, we obtain the following result outside the metal:

$$V(z)=\begin{cases}E(z+a), & 0<z<z_2\\ E(z_2+a), & z>z_2\end{cases}, \tag{14}$$

where the constant a, characterizing the effective depth of penetration of a homogeneous electric field into the metal, is determined by the relation

$$a=\frac{1}{\pi}\int dk_z\frac{1}{k_z^2\varepsilon(k_z)}.$$

In the long wavelength limit, the following expression is obtained for $\epsilon(k)$ (see, for example,[8])

$$\varepsilon(k)=1+k_0^2/k^2, \tag{15}$$

characterizing Deybe screening in a metal; the constant a is equal to the Debye screening radius $r_0 = k_0^{-1}$, which can be estimated by using the relationship $r_0 = (\pi a_0/4p_0)^{1/2}$ in which a_0 denotes the Bohr radius and p_0 denotes the limiting Fermi momentum of the electron gas of the metal (see, for example,[4]).

Inside the metal we obtain the following expression for $V(z)$:

$$V(z)=\frac{E}{\pi}\int_{-\infty}^{\infty}dk_z\frac{\exp\{ik_zz\}}{k_z^2\varepsilon(k_z)},\quad z<0. \tag{16}$$

Expression (16) follows from the results of article[6], which are obtained from macroscopic equations under the assumption of specular scattering of the electrons at the boundary of the metal.

3. With Eqs. (5), (11), and (13) taken into account the polarization contribution U(r) turns out to be given by

$$U(z)=\frac{1}{2}\int\frac{d\varkappa}{(2\pi)^2}e^{-2\varkappa z}\frac{4\pi e^2}{2\varkappa}\frac{\varkappa\varphi(\varkappa,0)-1}{\varkappa\varphi(\varkappa,0)+1},\quad z>0, \tag{17a}$$

$$U(z)=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi e^2}{k^2}\left(\frac{1}{\varepsilon(k)}-1\right)$$
$$-\frac{1}{2}\int\frac{d\varkappa}{(2\pi)^2}\frac{4\pi e^2}{2}\varphi(\varkappa,z)\frac{\varkappa\varphi(\varkappa,z)-\varphi'(\varkappa,z)}{\varkappa\varphi(\varkappa,0)+1},\quad z<0. \tag{17b}$$

Formula (17) determines U(z) in terms of the dielectric constant $\epsilon(k)$. From Eq. (17a) it is seen that for large values of z the function U(z) goes over into expression (1).

For a detailed investigation of the behavior of U(z) we shall use expression (15) for $\epsilon(k)$. In this approximation

$$\varphi(\varkappa,z)=\frac{e^{-\mu|z|}}{\mu},\quad \mu=\sqrt{k_0^2+\varkappa^2}$$

and

$$U(z)=\begin{cases}-\dfrac{e^2}{2}\displaystyle\int_0^\infty d\varkappa\, e^{-2\varkappa z}\frac{\mu-\varkappa}{\mu+\varkappa}, & z>0,\\[2ex] -\dfrac{k_0e^2}{2}+\dfrac{e^2}{2}\displaystyle\int_0^\infty d\varkappa\, e^{2\mu z}\frac{\mu-\varkappa}{\mu+\varkappa}\frac{\varkappa}{\mu}, & z<0.\end{cases} \tag{18}$$

The integrals in formulas (18) can be expressed in terms of special functions

$$U(z)=\begin{cases}-\dfrac{e^2}{4z}\left[1+\dfrac{4}{\xi^2}+\pi E_2(\xi)+\pi N_2(\xi)\right], & z>0,\\[2ex] \dfrac{k_0e^2}{2}\left[-1+\dfrac{(\xi-2)^2}{\xi^3}e^{\xi}+\dfrac{2}{\xi}K_2(-\xi)\right], & z<0.\end{cases}$$

where $\xi = 2k_0z$, and $E_2(\xi)$, $N_2(\xi)$, and $K_2(\xi)$ denote the Weber, Neumann, and Macdonald functions, respectively (see, for example,[9]). For $\xi \gg 1$, i.e., $z \gg r_0$ the following expansion is valid

$$U(z)=-\frac{e^2}{4z}\left[1-\frac{2}{\xi}+\frac{4}{\xi^2}-\frac{6}{\xi^3}+\ldots\right],$$

and as $z \to 0$

$$U(z)=-\frac{k_0e^2}{3}\left[1+\frac{3}{8}\xi\left(\ln\frac{C\xi}{2}-\frac{3}{4}\right)+\ldots\right],$$

where C is Euler's constant (C = 0.577 ...). Outside the metal U(z) varies from 0 as $z \to \infty$ to a value $(2/3)U_0$ on its surface, and inside the metal U(z) rapidly reaches the limiting value $U_0 = -k_0e^2/2$, which is the polarization contribution to the energy of a point charge in the depths of the metal using the approximation (15) for $\epsilon(k)$.

The values of the integrals (18) calculated on an electronic calculating machine are shown in Fig. 1, and the approximation

$$U(z)=\begin{cases}-\dfrac{e^2}{4z+{}^3/_2e^2/|U_0|}, & z>0\\[1ex] U_0+{}^1/_3|U_0|e^{2k_0z}, & z<0\end{cases}. \tag{19}$$

is shown. In the case of other approximations for $\epsilon(k)$ one can obtain the function U(z) as a result of numerical integration of expressions (17), and for approximations one can use expression (19) where as U_0 it is necessary to substitute the value of the polarization contribution in the interior of the metal for a specific $\epsilon(k)$:

$$U=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi e^2}{k^2}\left(\frac{1}{\varepsilon(k)}-1\right).$$

Values of U in the approximation of high density are given, for example, in[5].

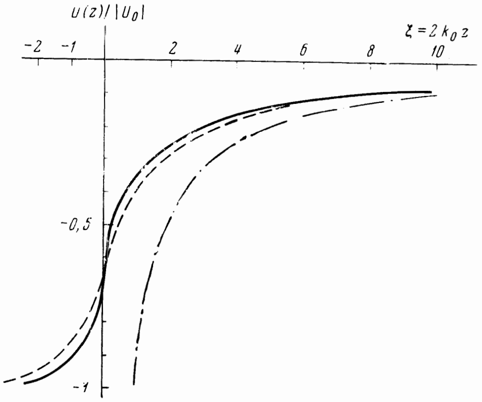

FIG. 1. The polarization contribution (18) (in the units $|U_0| = k_0e^2/2$) outside and inside of the metal. The dashed line indicates the approximation (19), and the dot-dash line is a graph of the function $-e^2/4z|U_0|$.

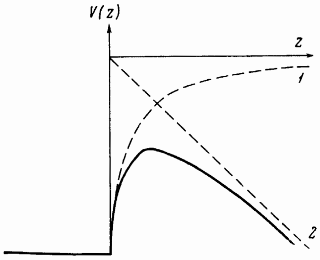

FIG. 2. The barrier used for the derivation of the Fowler-Nordheim formula. The dashed curves correspond to: 1 – the function $-e^2/4z$, 2 – the function $-eEz$.

A "jelly" model for the metal was considered above, in which the ionic charges are uniformly distributed over the volume of the metal. A more accurate account of the influence of the lattice would lead to periodicity of the function $U(\mathbf{r})$, for example, periodicity in a plane parallel to the boundary of the metal. One can qualitatively estimate the value of the Fourier components of the function $U(\mathbf{r})$ in the following way.

From Eq. (4) one can easily see that the "Coulomb hole" accompanying the charge e contains a charge $-e$ so that the quantity $U(\mathbf{r})$ is determined by the distribution of the charge $-e$ in the hole. In a homogeneous medium with a density of electrons n, the hole which accompanies the electron has an effective radius somewhat exceeding $r_s = (3/4\pi n)^{1/3}$, and the polarization contribution to the electron's energy is close to the value $-(3/4)e^2/r_s$. For a constant density of conduction electrons in a metal having a body-centered cubic lattice, $r_s \approx d/2$ where d denotes the lattice constant. Now let the density of conduction electrons change by a factor of two over the distance d; one can easily estimate that the relative deviations of $U(\mathbf{r})$ from its average value will be on the order of 0.1. Upon the emission of an electron from the metal, the Coulomb hole "spreads out" over the surface, and due to averaging the deviations in $U(\mathbf{r})$ from the values (17) become still smaller. Apparently averaging is the reason that, for example, numerous experimental results on field emission are explained very well by one-dimensional models of the potential barrier (see, for example,[10]).

4. Let us dwell briefly on one question[11] which arises in connection with the interpretation of the experimental data on field emission. The barrier shown in Fig. 2 is used for the derivation of the basic formula for the field emission current, the Fowler-Nordheim (F–N) formula. In connection with careful experimental verification[12] of the F–N formula it was observed that it describes the experimental results well for values of the field up to 5×10^7 V/cm, but for very high values of the field graphs of the logarithm of the current as a function of the reciprocal of the intensity deviate from the F–N straight lines. These deviations might be caused by nonfulfillment of the assumptions utilized in the derivation of the F–N formula, for example, a deviation of the dispersion law for the electron's energy from a quadratic law or a difference of the real barrier from the one shown in Fig. 2. Changes introduced into the F–N formula by the dispersion law were investigated by Itskovich.[13] It was shown by him that in certain cases it is necessary to add a correction to the work function in the F–N formula; however, the form of the formula remains the same as before, and deviations from it are not explained by a complicated dispersion law.

Two explanations of the cited deviations exist in the literature. The first[12] consists in the fact that because of the influence of the space charge, the field at the surface of the emitter is smaller than follows from the geometry of the device, and correspondingly the field emission current is smaller. According to this explanation the F–N formula, in which the intensity of the electric field E at the emitter enters, is valid.

The second explanation[11] is based on the fact that the real barrier must be broader than the one shown in Fig. 2 because the energy of the electron's interaction with the metal does not tend to infinity as in Eq. (1). This broadening of the barrier leads to a decrease of the field emission current in comparison with the values given by the F–N formula, which becomes noticeable in strong fields.

Actually, if for example the approximations (19) (see Fig. 1) are used in order to estimate the influence of broadening the barrier, then one can easily show that a broadening of the barrier leads to an effective increase of the work function in the F–N formula by an amount

$$\delta\varphi_1 \approx eE\frac{3}{8}\frac{e^2}{|U_0|}$$

and leads to a corresponding decrease of the current. The quantity $\delta\varphi_1$ becomes appreciable in strong fields of the order of 5×10^7 to 10^8 V/cm. However, in such fields the effect of penetration of the field into the metal also becomes appreciable. From expressions (14) and (16) and the derivation of the F–N formula one can easily see that the corresponding decrease in the height of the barrier leads to an effective decrease of the work function in the F–N formula by an amount $\delta\varphi_2 \approx eEa$, where a denotes the penetration depth of the field into the metal, which is defined above. The quantities $\delta\varphi_1$ and $\delta\varphi_2$ almost completely cancel each other; using the approximation (15) for $\epsilon(k)$ the resulting change $\delta\varphi$ is given by

$$\delta\varphi \approx -{}^1\!/_4 eEr_0 \qquad (20)$$

and is small up to fields ~ 10^8 V/cm.

Here one should also estimate the effect which would be made on this derivation by taking account of the fact that the concentration of electrons inside the surface layer of the metal changes upon imposition of the electric field and, correspondingly, the interaction law $U(z)$ is somewhat changed. It is easy, however, to see that in the fields under consideration of 5×10^7 to 10^8 V/cm, the electron concentration is increased all together by 10 to 20%, and $U(z)$ by a few per cent. The corresponding narrowing of the barrier is small in comparison with that change being taken into consideration here; qualitatively this narrowing gives an effect of the same sign as the penetration of the field into the metal, i.e., it increases somewhat the numerical coefficient in formula (20).

Thus, one can draw the conclusion that together with taking account of the narrowing of the barrier, carried out by Lewis,[11] in field emission problems one should also take the effect of field penetration into the metal into consideration. On the basis of the estimates which have been made, one can anticipate that taking both of these effects into account only leads to a small change (to a decrease, as is clear from Eq. (20)) of the work function in the F—N formula, a change which may become appreciable only in very strong fields close to 10^8 V/cm. The Fowler-Nordheim formula is apparently valid up to these values of the field, notwithstanding the simplifying assumptions made in connection with its derivation.

---

[1] John Bardeen, Phys. Rev. **49**, 640 (1936).

[2] J. Bardeen, Phys. Rev. **58**, 727 (1940).

[3] R. G. Sachs and D. L. Dexter, J. Appl. Phys. **21**, 1304 (1950). A. Modinos, Brit. J. Appl. Phys. **18**, 531 (1967).

[4] D. A. Kirzhnits, Polevye metody teorii mnogikh chastits (Field Theoretical Methods in Many-body systems), Gosatomizdat, 1963 (English Transl., Pergamon Press, 1967).

[5] Lars Hedin, Phys. Rev. **139**, A796 (1965).

[6] V. D. Shafranov, Zh. Eksp. Teor. Fiz. **34**, 1475 (1958) [Sov. Phys.-JETP **7**, 1019 (1958)].

[7] Yu. I. Balkereĭ and V. B. Sandomirskiĭ, Zh. Eksp. Teor. Fiz. **54**, 808 (1968) [Sov. Phys.-JETP **27**, 434 (1968)].

[8] David Pines and Philippe Nozières, The Theory of Quantum Liquids, W. A. Benjamin, Inc., 1966 (Russ. Transl., Izd. Mir, 1967).

[9] I. S. Gradshteĭn and M. N. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1963 (English Transl., Academic Press, 1966).

[10] M. I. Elinson and G. F. Vasil'ev, Autoélektronnaya émissiya (Field Emission), Fizmatgiz, 1958.

[11] T. J. Lewis, Phys. Rev. **101**, 1694 (1956).

[12] J. P. Barbour, W. W. Dolan, J. K. Trolan, E. E. Martin, and W. P. Dyke, Phys. Rev. **92**, 45 (1953); W. P. Dyke and J. K. Trolan, Phys. Rev. **89**, 799 (1953).

[13] F. I. Itskovich, Zh. Eksp. Teor. Fiz. **50**, 1425 (1966) [Sov. Phys.-JETP **23**, 945 (1966)].

Translated by H. H. Nickle
69