RELATIVISTIC EIKONAL APPROXIMATION

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An approximation analogous to the eikonal method in potential theory is constructed for the amplitude for the scattering of two relativistic particles. Applications of this approximation to the scattering of particles at high energies are discussed.

IN order to describe the scattering of high-energy particles at small angles in potential theory the eikonal approximation is usually used. In this approximation the amplitude for the scattering of a particle with mass m by a potential V(x) is given by an integral over the impact parameter:[1]

$$f(\mathbf{p},\mathbf{q})=\frac{ip}{2\pi}\int d^2x_\perp\, e^{-i\mathbf{q}\mathbf{x}_\perp}\left[1-\exp\left(-\frac{im}{p}\int_{-\infty}^{\infty}dx_3V(\mathbf{x})\right)\right],\qquad(1)$$

where p denotes the initial momentum of the particle and q denotes the momentum transfer. The basic object of the present article is a derivation of an analogous expression for the scattering of relativistic particles, when it is necessary to take the effect of retardation into account. Possible applications of the obtained formula to the scattering of particles at high energies are also discussed.

First of all we note that the eikonal approximation may be obtained if, in the perturbation theory expansion of the amplitude of potential scattering, the terms which are quadratic in the momentum transfer are neglected in each of the energy denominators (propagators). Such neglect is permissible for scattering at small angles (see in[2] for a similar treatment of the eikonal approximation for the wave field in optics). Let us consider therefore the analogous approximation for the amplitude of the scattering of two relativistic particles. For brevity of writing, we shall regard both the scattered particles as well as the quanta which they exchange during the scattering process as scalar particles. Then the modifications (extremely important) which must be made for fields possessing a more complicated spin structure will be indicated.

We shall start from the two-particle Green's function, taking the set of diagrams shown in the figure into account. Here the solid lines denote the propagators of the particles being scattered, and the dashed lines denote the propagation function D whose specific form, just like the form of the potential V(x) in Eq. (1), may be arbitrary. A Green's function in which the terms quadratic in the momentum transfer are neglected was constructed in[3]. In the case under consideration it has the form

$$G(p_1,p_2,q_1,q_2)=\int d^4x_1\int d^4x_2\exp\{-iq_1x_1-iq_2x_2\}$$
$$\times\int_0^\infty d\nu_1\int_0^\infty d\nu_2\exp\{-i\nu_1(p_1^2+m_1^2)-i\nu_2(p_2^2+m_2^2)\}$$
$$\times\exp\left\{ig^2\int_0^{\nu_1}d\nu'\int_0^{\nu_2}d\nu''D(x_1-x_2-2p_1\nu'+2p_2\nu'')\right\}.\qquad(2)$$

In order to determine the elements S_{fi} of the S-matrix, it is necessary to set up the expression

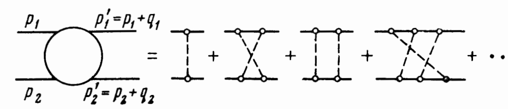
$$\tilde{S}_{fi}=\frac{1}{4\sqrt{p_{10}\,p_{20}\,p_{10}'\,p_{20}'}}(p_1^2+m_1^2)(p_2^2+m_2^2)(p_1^2+2p_1q_1$$
$$+m_1^2)(p_2^2+2p_2q_2+m_2^2)G(p_1,p_2,q_1,q_2)\qquad(3)$$

and then pass to the mass shell of the momenta of the external lines.

Introducing new variables

$$x_1'=x_1-2p_1\nu_1,\quad x_2'=x_2-2p_2\nu_2,$$

one can represent the product $(p_1^2+2p_1q_1+m_1^2)(p_2^2+2p_2q_2+m_2^2)$ as the result of differentiating the quantity $\exp[-i\nu_1(p_1^2+2p_1q_1+m_1^2)-i\nu_2(p_2^2+2p_2q_2+m_2^2)]$ with respect to the variables ν_1, ν_2 under the integral sign.



Then, having completed integration by parts in the integrals with respect to the variables ν_1 and ν_2, having returned to the previous variables x_1, x_2, and having carried out the limiting transition $(p_1^2+m_1^2)\to0$, $(p_2^2+m_2^2)\to0$, we obtain

$$S_{fi}=\delta_{fi}+\frac{(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')}{4\sqrt{p_{10}\,p_{20}\,p_{10}'\,p_{20}'}}\int d^4x\,e^{-iqx}\Big[ig^2D(x)$$
$$-g^2\int_0^\infty d\nu'D(x-2p_1\nu')g^2\int_0^\infty d\nu''D(x+2p_2\nu'')\Big]$$
$$\times\exp\left\{ig^2\int_0^\infty d\nu'\int_0^\infty d\nu''D(x-2p_1\nu'+2p_2\nu'')\right\},\qquad(4)$$

where $q\equiv q_1=-q_2$ is the momentum transfer and $x=x_1-x_2$. Now let us go to the center of momentum system, having chosen the x_3 axis along the direction of the momentum $\mathbf{p}\equiv\mathbf{p}_1=-\mathbf{p}_2$, and let us denote the energy of the particles by ϵ_1 and ϵ_2. It is not difficult to verify that in the argument of the exponential in (4) the variables x_0 and x_3 may be transferred to the limits of the integrals in the form of the combinations u′ and u″:

$$u'=(\varepsilon_2x_3+px_0)/2p(\varepsilon_1+\varepsilon_2),$$
$$u''=(\varepsilon_1x_3-px_0)/2p(\varepsilon_1+\varepsilon_2),\quad p=|\mathbf{p}|.\qquad(5)$$

At the same time the square bracket in (4) appears upon differentiation of the exponential with respect to the same variables u′ and u″. Therefore the scattering amplitude M_{fi}, defined by the relation

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')M_{fi}/4\varepsilon_1\varepsilon_2,\qquad(6)$$

may be written in the form

$$M_{fi} = -i \int d^4x e^{-iqx} \frac{\partial^2}{\partial u' \partial u''} \exp\left\{ ig^2 \int_{-u'}^{\infty} dv' \int_{-u''}^{\infty} dv'' D(\mathbf{x}_\perp - 2p_1 v' + 2p_2 v'') \right\}. \tag{7}$$

Neglecting the longitudinal momentum transfer $q_3 \sim q^2/p$, one can perform the integrations with respect to the variables x_0 and x_3 (i.e., with respect to the variables u' and u''). As a result we obtain an expression for the amplitude in the form of an integral over the impact parameter:

$$M_{fi} = 4ip(\varepsilon_1 + \varepsilon_2) \int d^2x_\perp \, e^{-i\mathbf{q}\mathbf{x}_\perp} \{1 - e^{2i\chi(p,\,\mathbf{x}_\perp)}\}, \tag{8}$$

where

$$\chi = \frac{g^2}{2} \int_{-\infty}^{\infty} dv' \int_{-\infty}^{\infty} dv'' D(\mathbf{x}_\perp - 2p_1 v' + 2p_2 v'') = \frac{g^2}{8p(\varepsilon_1 + \varepsilon_2)} \int \frac{d^2k_\perp}{(2\pi)^2} D(\mathbf{k}_\perp) e^{i\mathbf{k}_\perp \mathbf{x}_\perp}. \tag{9}$$

Relations (8) and (9) replace formula (1) of potential scattering.

Recently the representation of the amplitude in the form (8) has been widely used for a phenomenological description of the interaction of high-energy hadrons. In this connection, either the Regge pole contribution[4] or the Fourier transform of the product of the form factors of the colliding particles,[5] or a combination of one and the other, is taken as the function χ. Within the framework of the approximation under consideration in its scalar version, the phase χ, and together with it the invariant cross section

$$\frac{d\sigma}{d|t|} = \frac{1}{64\pi p^2(\varepsilon_1 + \varepsilon_2)^2} |M_{fi}|^2, \quad |t| \approx q^2, \tag{10}$$

tend to zero according to a power law with an increase of the energy for fixed momentum transfer. Therefore scalar exchange is unsuitable as a model for the elastic scattering of hadrons at high energies.

In the case of vector exchange, we must make the following substitution in the preceding formulas:

$$D(k) \to 4p_{1\mu} D_{\mu\nu}(k) p_{2\nu} = 4p_{1\mu} p_{2\nu} (g_{\mu\nu} - k_\mu k_\nu / M^2) D_0(k). \tag{11}$$

Here it turns out to be immaterial whether the particles being scattered are scalar or spinor. The part of the propagation function which contains the product $k_\mu k_\nu$ and leads to divergences (and unrenormalizability) does not give a contribution in the approximation under consideration, the phase

$$\chi = -\frac{g^2}{2} \frac{p^2 + \varepsilon_1 \varepsilon_2}{p(\varepsilon_1 + \varepsilon_2)} \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp \mathbf{x}_\perp} D_0(k_\perp), \tag{12}$$

and the differential cross sections (10) for fixed q are asymptotically constant, i.e., they have essentially the maximum possible degree of growth. Thus, we arrive at the dominance of vector exchange in elastic scattering at high energies, which corresponds to the maximum possible value $l = 1$ for the position of the singularity of the amplitude in the angular momentum plane. We note that for pseudoscalar exchange (π mesons) the limit corresponding to the eikonal approximation is completely absent for the amplitude.

Within the framework of the adopted approximation, it is of interest to calculate the amplitude for the scattering of two particles in the case of the electromagnetic interaction. Considering the scattering at small angles, as above, we do not begin to carry out symmetrization. For the photon propagator we have

$$D_{\mu\nu}(k) = 4\pi g_{\mu\nu} / (k^2 - i\varepsilon) \tag{13}$$

(as before the part of $D_{\mu\nu}(k)$ containing $k_\mu k_\nu$ here does not give any contribution to the amplitude in accordance with gauge invariance). In general the integral (12) containing the propagation function (13) diverges at large distances. Therefore we introduce a photon mass λ, $k^2 \to k^2 + \lambda^2$, and we keep only the terms that do not decrease with λ. Then one can easily calculate the amplitude (8) and the scattering cross section (10). We have

$$M_{fi} = -\frac{16\pi e^2(p^2 + \varepsilon^2)}{q^2} \frac{\Gamma(1 + i\beta)}{\Gamma(1 - i\beta)} e^{-2i\beta \ln(q/\gamma\lambda)}, \tag{14}$$

where $\beta = e^2(p^2 + \epsilon^2)/2p\epsilon$, γ is Euler's constant, $\epsilon_1 = \epsilon_2 = \epsilon$, $q = |\mathbf{q}|$, and

$$\frac{d\sigma}{d|t|} = \frac{\pi e^4(p^2 + \varepsilon^2)^2}{p^2 \varepsilon^2 q^4}. \tag{15}$$

The entire dependence on the photon mass λ appeared in the phase of the amplitude. We see that here, just as in potential theory, the expression for the cross section agrees with the result of lowest order perturbation theory, and taking account of higher order corrections only leads to the appearance of a phase (which becomes infinite as $\lambda \to 0$) in the scattering amplitude.

It is curious that the poles of the amplitude (14) give the Coulomb energy levels $E_n = -me^4/4n^2$ in the lowest approximation (more precisely, here it is necessary to take the amplitude with the substitution $e^2 \to -e^2$ in order to pass to the case of charges of opposite sign). Apparently this is related to the fact that the considered approximation correctly describes states with large angular momenta, but the Coulomb energy levels have the same form for all momenta.

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