THE GENERATION OF A LARGE-SCALE MAGNETIC FIELD BY A TURBULENT FLUID

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We obtain an equation for the magnetic field in two cases: 1) gyrotropic turbulence; 2) anisotropic turbulence. We use the method of selective summation of the perturbation theory series, we find an exact solution of these problems for a particular model of turbulence. The most important assumption which we use then is the neglect of the velocity correlation time. In the first case it follows from the equation obtained that gyrotropic turbulence can generate a large-scale magnetic field. We find the conditions for the generation of a field. Anisotropic turbulence does not generate a large scale field and, in contrast, leads to an anomalous anisotropic diffusion of the field. Isotropic turbulence also leads to an anomalous diffusion of the field.

RECENTLY many papers have appeared on the generation of regular magnetic fields. One direction in which these papers have moved is the problem of the generation of magnetic fields when turbulent motions are present. In connection with the work by Steenback et al.[1-3] interest was drawn to gyrotropic turbulence where the density of the probability distribution of the velocities is not invariant under reflections. Since in these papers it was assumed that the pulsation of the magnetic field $\mathbf{h} \ll \mathbf{H}_0$, where $\mathbf{H}_0$ is the initial field (a quadratic effect is calculated), the solution is essentially applicable to motions with a small magnetic Reynolds number R_m. This was stipulated in[3] (assumption 1). In the above mentioned papers it was shown that gyrotropic turbulence can act as a generator of magnetic fields with a scale which is significantly larger than the scale of the pulsations.

On the other hand, there is a class of motions for which the velocity probability distribution is anisotropic. Turbulent convection is an example of this. Tverskoĭ[4] has shown that a separate convective cell (toroidal vortex) can generate a large scale field. What will be the effect of a whole statistical ensemble of cells?

We have been able to obtain in the present paper for a particular model of turbulence an exact solution for the magnetic field when there are large pulsations for gyrotropic and anisotropic turbulence, by using the method of selective summation of the perturbation theory series.

GYROTROPIC TURBULENCE

1. Statement of the problem. We shall use the approximation of magnetic hydrodynamics, i.e., we shall start from the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}[\mathbf{vB}] + \nu_m \Delta \mathbf{B}. \qquad (1)^*$$

Here $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ the velocity, and ν_m the magnetic viscosity. Let the conducting fluid occupy some bounded volume. The magnetic field must vanish at infinity. This means that $\mathbf{B}$ is not maintained by any external sources. At the fluid-vacuum boundary the usual boundary conditions are satisfied which are obtained from the main Maxwell equations. The quantities $\mathbf{B}$ and $\mathbf{v}$ inside the volume of the fluid can be expanded in spatial Fourier components:

$$\mathbf{B}(\mathbf{r}, t) = \sum_{\mathbf{k}} \exp(i\mathbf{kr}) \mathbf{B}(\mathbf{k}, t), \qquad (2)$$

$$\mathbf{v}(\mathbf{r}, t) = \sum_{\mathbf{k}} \exp(i\mathbf{kr}) \mathbf{u}(\mathbf{k}, t).$$

In the Fourier-representation Eq. (1) has the following form

$$\frac{\partial \mathbf{B}(\mathbf{k}, t)}{\partial t} + \nu_m k^2 \mathbf{B}(\mathbf{k}, t) = i \sum_{\mathbf{k}'} [\mathbf{k}[\mathbf{u}(\mathbf{k}-\mathbf{k}')\mathbf{B}(\mathbf{k}')]]. \qquad (3)$$

In problems of this kind one assumes that the velocity field is given. We need the spectral tensor of the velocity field, T_{jl}. For gyrotropic and homogeneous turbulence the form of T_{jl} is known (see[5]): if $\mathbf{k} \neq \mathbf{k}'$, then

$$T_{jl} = \langle u_j(\mathbf{k}, t) u_l^*(\mathbf{k}', t') \rangle = 0, \qquad (4)$$

and if $\mathbf{k} = \mathbf{k}'$, then

$$T_{jl} = \langle u_j(\mathbf{k}, t) u_l^*(\mathbf{k}', t') \rangle = A(k, |t-t'|)[\delta_{jl} - k_j k_l / k^2] + iA_1(k, |t-t'|)\varepsilon_{jlf} k_f. \qquad (4')$$

Here ϵ_{jlf} is the third rank tensor which is antisymmetric in all indices, $\epsilon_{123} = 1$; δ_{jl} is the Kronecker symbol. Moreover, we split off the large scale (slowly varying) component of $\mathbf{B}$: $\mathbf{B} = \mathbf{H} + \mathbf{h}$, $\langle \mathbf{B} \rangle = \mathbf{H}$. The average is taken here over a time interval which is large compared to the time of the pulsations but small compared to the time over which the large scale components change. Thus, $\langle \mathbf{u}(\mathbf{k}, t) \rangle = 0$.

In the following we shall use the perturbation theory series:

$$\mathbf{B}(\mathbf{k}, t) = \sum_{n=0}^{\infty} \mathbf{B}^{(n)}, \qquad \mathbf{B}^{(0)} = \mathbf{B}(\mathbf{k}, 0)\exp(-k^2\nu_m t), \qquad (5)$$

$$\mathbf{B}^{(n+1)} = i \int_0^t dt_1 \exp[k^2 \nu_m (t_1 - t)] \sum_{\mathbf{k}'} [\mathbf{k}[\mathbf{u}(\mathbf{k}-\mathbf{k}')\mathbf{B}^{(n)}(\mathbf{k}')]].$$

We shall consider the following model of turbulence: 1) $B^2/8\pi \ll \rho v^2/2$ which is usually assumed in problems of the generation of a field, i.e., one can assume that the velocity field is stationary; 2) the velocity probability distribution is Gaussian; 3) we assume that

$$A(k, |t-t'|) = u(k)\delta(t-t'), \quad A_1(k, |t-t'|) = u_1(k)\delta(t-t').$$

*$[\mathbf{vB}] \equiv \mathbf{v} \times \mathbf{B}$.

2. Derivation of the equation for the magnetic field H. We shall assume that the initial field is large scale i.e. k_L, the center of gravity of the product $B_i(\mathbf{k}, 0)B_i^*(\mathbf{k}, 0)$, occurs for small $k(k_L \ll k_1$, k_1 is the center gravity of $u(k)$ and $u_1(k))$. We shall be interested in the large scale component of the field:

$$\mathbf{H} = \sum_{n=0}^{\infty} \langle \mathbf{B}^{(n)} \rangle = \sum_{n=0}^{\infty} \mathbf{H}^{(n)}.$$

Using the second assumption we get from (5)

$$\langle \mathbf{B}^{(2n+1)} \rangle = \mathbf{H}^{(2n+1)} = 0,$$

in the even terms the average of the products of velocities will be split up into a sum of all possible pair combinations but by virtue of the third assumption there remains only one combination. This makes it possible to establish a recurrence formula for the averaged quantities:

$$\mathbf{H}^{(2n+2)} = -v_0k^2 \int_0^t \exp(\nu_m k^2(t_1 - t))\mathbf{H}^{(2n)}\,dt_1 + i\alpha \int_0^t \exp(\nu_m k^2(t_1 - t))[\mathbf{k}\mathbf{H}^{(2n)}]\,dt_1, \tag{6}$$

where

$$v_0 = {}^1\!/_3 \sum_k u(k), \quad \alpha = {}^1\!/_3 \sum_k k^2 u_1(k).$$

We must now express $\langle \mathbf{B}^{(2n+2)} \rangle$ in terms of $\mathbf{B}^{(0)}$ and it turns out that $\mathbf{H}^{(2n+2)}$ and hence $\mathbf{H}$ can be written in the form of a sum:

$$\mathbf{H} = A\mathbf{H}^{(0)} + Ci[\mathbf{k}\mathbf{H}^{(0)}],$$

where A and C depend on v_0, α, k, and t. We first evaluate A, the sum of the series. It is convenient to separate off partial sums of the form:

$$S_0 = \sum_n S_0^{(n)}, \quad S_2 = (\alpha^2 k^2)^2 \sum_n S_2^{(n)}, \ldots,$$

$$S_{2p} = (\alpha^2 k^2)^{2p} \sum_n S_{2p}^{(n)}.$$

One shows easily that the general term of the series S_{2p} has the form

$$(\alpha^2 k^2)^{2p} C_n^{2p} (-v_0 k^2)^{n-2p} \frac{t^n}{n!}$$

(we put $C_n^0 = 1$). The series S_{2p} converge for all k and t and we can easily find its sum:

$$S_{2p} = (\alpha^2 k^2)^{2p} t^{2p} (2p!)^{-1} \exp[-v_0 k^2 t].$$

We can now sum the partial sums:

$$A = \sum_p S_{2p} = \operatorname{ch}(k\alpha t)\exp(-v_0 k^2 t).$$

We can in the same way evaluate C and as a result we get

$$\mathbf{H} = \exp[-(v_0 + \nu_m)k^2 t]\{\mathbf{B}(\mathbf{k}, 0)\operatorname{ch}(k\alpha t) + i[\mathbf{k}\mathbf{B}(\mathbf{k}, 0)]k^{-1}\operatorname{sh}(k\alpha t)\}. \tag{7}$$

We see that in the given problem the series remaining when we perform a selective summation converges.

Differentiating (7) with respect to t we get

$$\frac{\partial \mathbf{H}}{\partial t} + (v_0 + \nu_m)k^2\mathbf{H} = \alpha i[\mathbf{k}\mathbf{H}] \tag{8}$$

or, in r-space

$$\frac{\partial \mathbf{H}}{\partial t} = \alpha \operatorname{rot}\mathbf{H} + (v_0 + \nu_m)\Delta\mathbf{H}. \tag{9}$$

By directly substituting (7) into (8) one can verify that (7) is a solution of Eq. (8). It follows from (7) that the magnetic field increases for small k: $k < k_m$ where

$$k_m = \alpha / (v_0 + \nu_m). \tag{10}$$

When $k \geq k_m$, the field decreases. This statement is true if $k_m \ll k_1$; in the opposite case $(k_m \approx k_1$, $k_m \gg k_1)$ the generation occurs in the entire large scale. We note that there is no stationary solution: the field is either generated or dissipated. There is no generation in the case when $k_m < 2\pi/L$, where L is the size of the body.

In the strictly uniform case of an unbounded turbulent medium we must also obtain an increase in the field. Then

$$T_{jl} \sim \delta(\mathbf{k} - \mathbf{k}'),$$

$$\langle B_j(\mathbf{k}, t)B_l^*(\mathbf{k}', t)\rangle \sim \delta(\mathbf{k} - \mathbf{k}'), \quad \langle \mathbf{B} \rangle = 0;$$

all summation signs in (2), (3), and (5) must be replaced by integrals over the whole of k-space. Moreover, we multiply the series (5) by $\mathbf{B}^*(\mathbf{k}', t)$, average, and integrate over $\mathbf{k}'$.

Let the initial field satisfy the isotropy condition:

$$\langle B_i(\mathbf{k}, 0)B_j^*(\mathbf{k}', 0)\rangle = B(k, 0)\delta(\mathbf{k} - \mathbf{k}')(\delta_{ij} - k_i k_j / k^2). \tag{11}$$

We shall assume that $B(\mathbf{k}, 0)$ is statistically independent of the velocity field and is large scale

$$\left(\int_0^\infty B(k,0)k\,dk \Big/ \int_0^\infty B(k,0)\,dk \ll \int_0^\infty u(k)k\,dk \Big/ \int_0^\infty u(k)\,dk\right).$$

We break off the obtained series, restricting ourselves to terms of fourth order in the velocities. Moreover, we shall be interested in the large scale components of the field, i.e., $B(k, t)$ with $k \approx k_L$. The largest contribution to the sum comes then from terms for which $B(k, 0)$ can be taken out from under the integral sign:

$$\int \langle B_i^{(0)}(\mathbf{k})B_i^{*(0)}(\mathbf{k}')\rangle\,d\mathbf{k}' = 2B(k, 0)\exp(-2\nu_m k^2 t);$$

$$\int (\langle B_i^{(0)}(\mathbf{k})B_i^{*(2)}(\mathbf{k}') + B_i^{*(0)}(\mathbf{k})B_i^{(2)}(\mathbf{k}')\rangle)\,d\mathbf{k}' = -2B(k, 0)\exp(-2\nu_m k^2 t)2k^2 v_0 t;$$

$$\int \langle B_i^{(2)}(\mathbf{k})B_i^{*(2)}(\mathbf{k}')\rangle\,d\mathbf{k}' = \int (\langle B_i^{(0)}(\mathbf{k})B_i^{*(4)}(\mathbf{k}') + B_i^{*(0)}(\mathbf{k})B_i^{(4)}(\mathbf{k}')\rangle)\,d\mathbf{k}' = 2B(k, 0)\exp(-2\nu_m k^2 t)[{}^1\!/_4(2k^2 v_0 t)^2 + k^2 t^2 \alpha^2].$$

From these expressions it is clear that the contribution from $u_1(k)$ only appears in the fourth order. Thus

$$B(k, t) = B(k, 0)\exp(-2\nu_m k^2 t) \times [1 - 2k^2 v_0 t + \frac{1}{2!}(2k^2 v_0 t)^2 + 2k^2 \alpha^2 t^2]. \tag{12}$$

Equation (12) is, generally speaking, valid when $t \ll (2k^2 v_0)^{-1}$ so that for the operation of the dynamo it is necessary that the condition $k \ll \alpha/(v_0 + \nu_m)$ is satisfied. Indeed, then we have

$$2k^2\alpha^2 t^2 \gg \frac{1}{2!}(2k^2 v_0)^2$$

and amplification is obtained when $t \ll (2k^2 v_0)^{-1}$. This result agrees with (10) but in view of the fact that in the latter case the problem was solved approximately we do not obtain the threshold value k_m (i.e., the value k_m such that $B(k, t)$ increases when $k < k_m$ and decreases when $k > k_m$).

Turning to the exact solution (7) (for a body of size L) we note that (7) confirms in the general form the conclusions of Steenbeck. As to the third assumption, it will be justified if the characteristic time of variation of the field

$$t_0 = (\alpha k_L - (v_0 + \nu_m) k_L^2)^{-1} \gg t_1 = (k_1^2 v_0)^{-1},$$

t_1 is the correlation time. Of most interest is the case when $v_0 \gg \nu_m$ ($R_m \gg 1$). We introduce the quantity $f = \alpha / v_0 k_1$; it has the meaning of the ratio of the third term to the first two in (4) as far as order of magnitude is concerned, i.e., it characterizes "the role of the gyrotropy." If $f \ll 1$, the condition $t_0 \gg t_1$ is satisfied automatically, but if $f \gg 1$, the following condition must hold:

$$(\alpha k_L)^{-1} \gg (k_1^2 v_0)^{-1}.$$

ANISOTROPIC TURBULENCE

Anisotropic turbulence may serve as a model for turbulent convection (whether or not it consists of toroidal cells). In that case T_{jl} has the following form (see[5]):

$$T_{jl} = A_1 k_j k_l + A_2 \lambda_j \lambda_l + A_3 \delta_{jl} + A_4 \lambda_j k_l + A_5 \lambda_l k_j, \tag{13}$$

where $A_p = A_p(k, (\mathbf{k}\lambda), |t - t'|)$; $p = 1, \dots, 5$; λ is a unit vector parallel to the preferred direction (in the case of convection this is the vertical). The solenoidality condition leads to the following form of T_{jl} when $\mathbf{k} = \mathbf{k}'$:

$$T_{jl} = A_1 k_j k_l + k^2 \frac{k^2 A_1 + A_3}{(\lambda \mathbf{k})^2} \lambda_j \lambda_l + A_3 \delta_{jl} - \frac{k^2 A_1 + A_3}{(\lambda \mathbf{k})} (k_j \lambda_l + k_l \lambda_j). \tag{14}$$

All three assumptions of the preceding problem remain in force and the third one takes the form

$$A_1 = u_1(k, (\mathbf{k}\lambda)) \delta(t - t'),$$
$$A_3 = u_3(k, (\mathbf{k}\lambda)) \delta(t - t').$$

Using these assumptions we obtain easily the recurrence relation

$$\mathbf{H}^{(2n+2)} = -\chi k^2 \int_0^t \mathbf{H}^{(2n)} \exp(t_1 - t)\, dt_1, \tag{15}$$

where

$$2k^2\chi = v_3(k^2 - 2(\mathbf{k}\lambda)^2) + (v_1' + v_3')(\mathbf{k}\lambda)^2 + v_1(\tfrac{1}{2}k^2 - \tfrac{5}{2}(\mathbf{k}\lambda)^2) + v_1''(\tfrac{3}{2}(\mathbf{k}\lambda)^2 - \tfrac{1}{2}k^2) + 2\nu_m k^2,$$

$$v_1 = \sum_{\mathbf{k}} k^2 u_1, \quad v_1' = \sum_{\mathbf{k}} \frac{k^4}{(\mathbf{k}\lambda)^2} u_1, \quad v_1'' = \sum_{\mathbf{k}} (\mathbf{k}\lambda)^2 u_1;$$

$$v_3 = \sum_{\mathbf{k}} u_3, \quad v_3' = \sum_{\mathbf{k}} \frac{k^2}{(\mathbf{k}\lambda)^2} u_3.$$

Equation (15) reminds us of (6) but is simpler in form. This series also converges and the sum is easily found:

$$\mathbf{H} = \mathbf{B}(\mathbf{k}, 0) \exp(-\chi k^2 t). \tag{16}$$

Hence we get an equation for $\mathbf{H}$:

$$\frac{\partial \mathbf{H}}{\partial t} + k^2 \chi \mathbf{H} = 0. \tag{17}$$

One sees easily that $\chi > 0$ and hence (17) describes an anomalous diffusion of the field with an anisotropic character. Anisotropic turbulence thus does not generate a large-scale field.

Let us consider limiting cases. If we can neglect the anisotropy, we have

$$A_3 + A_1 k^2 = 0$$

and

$$T_{lj} = u_3(k)(\delta_{lj} - k_l k_j / k^2) \delta(t - t'),$$
$$v_3' + v_1' = 0, \quad v_1'' = -\tfrac{1}{3} v_3, \quad v_1 + v_3 = 0,$$

whence

$$\chi = \tfrac{1}{3} v_3 + \nu_m.$$

It is clear that turbulent diffusion turns out to be ohmic, if $v_3 \gg \nu_m$ ($R_m \gg 1$). In the other limiting case of strong anisotropy, i.e., if

$$u_1(k, (\mathbf{k}\lambda)), \quad u_3(k, (\mathbf{k}\lambda))$$

are non-vanishing only when $(\mathbf{k}\lambda) = \pm k$, we shall have

$$v_1 = v_1' = v_1'', \quad v_3 = v_3', \quad \chi = \tfrac{1}{2}(1 - (\mathbf{k}\lambda)^2 / k^2) v_3 + \nu_m.$$

We note that the time over which the field energy is transferred to small scale motion is $t_0 = (k_L^2 \chi)^{-1}$, while the velocity correlation time is $t_1 \approx (k_1^2 \chi)^{-1}$. As $t_0 \gg t_1$, the neglect of the correlation time (third assumption) is fully justified.

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