DISTRIBUTION OF ENERGY AND MOBILITY OF ELECTRONS IN GASES AND SEMICONDUCTORS

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The form of the energy distribution of electrons in gases and semiconductors in a strong electric field is considered, taking simultaneous account of elastic collisions with atoms (or acoustic phonons) and electron-electron interactions. An approximate method for solving the kinetic equation is proposed, valid for an arbitrary ratio between the intensities of electron-atom (phonon) and electron-electron interactions. The distribution functions obtained are used to calculate the electron mobility. It is shown that in some cases the mobility may appreciably depend on the electron concentration.

THE question of the energy distribution of electrons in gases and semiconductors in a strong electric field was considered in[1,2] without taking into account electron-electron interactions. The interaction between electrons was treated approximately by A. V. Gurevich[3]. However, his method for solving the kinetic equation was suitable only for the case where, for electrons with energy $\epsilon \sim T$ (T is the electron temperature in units of energy), the electron-electron interaction predominates, and the energy distribution in this region is nearly Maxwellian.

In this paper we propose an approximate method for solving the kinetic equation, allowing for electron-electron interactions, free from the restriction mentioned above. The resulting distribution function is used to calculate the electron mobility.

1. THE ENERGY DISTRIBUTION

Suppose there is a constant homogeneous electric field E along the z axis. Representing the distribution function as usual in the form

$$f(v) = f_0(v) + \frac{v_z}{v} f_1(v)$$

and allowing for elastic scattering of electrons by atoms (acoustic phonons for semiconductors) and the electron-electron interaction, we have for $f_0(v)$ the well-known equation[3]:

$$\frac{1}{v^2}\frac{d}{dv}\left\{\frac{m}{M}\frac{v^4}{\lambda(v)}f_0 + \frac{e^2E^2}{3m^2}\lambda(v)v\frac{df_0}{dv}\right\} + \frac{1}{v^2}\frac{d}{dv}\left\{v^2\nu_{ee}(v)\left[A_1(f_0)\frac{df_0}{dv} + vA_2(f_0)f_0\right]\right\} = 0, \quad (1)$$

$$A_1(f_0) = \frac{4\pi}{3n}\left\{\int_0^v v_1^4 f_0(v_1)\,dv_1 + v^3\int_v^\infty v_1 f_0(v_1)\,dv_1\right\}, \quad (2)$$

$$A_2(f_0) = \frac{4\pi}{n}\int_0^v v_1^2 f_0(v_1)\,dv_1. \quad (3)$$

Here, $\lambda(v) = v/\nu(v)$ is the length of the electron free path; e and m are the electron charge and mass; M is the mass of an atom of gas in the case of a monoatomic gas, or $M = \Theta/v_S^2$ for a semiconductor (v_S is the speed of sound, Θ the lattice temperature); n is the electron concentration;

$$\nu_{ee}(v) = \frac{4\pi e^4 n}{m^2 v^3}\ln\frac{R_D T}{e^2}$$

is the frequency of electron-electron collision (R_D is the Debye radius). The function $f_0(v)$ is normalized by the equation

$$4\pi\int_0^\infty f_0(v)v^2\,dv = n. \quad (4)$$

The electron temperature T is determined from the mean energy by the relation

$$\langle {}^1/_2 mv^2\rangle = {}^3/_2 T. \quad (5)$$

It is easy to see that for $v \to \infty$

$$A_1(f_0) \to T/m, \quad A_2(f_0) \to 1. \quad (6)$$

In writing Eq. (1), we have neglected the thermal motion of the atoms (lattice), which is possible when $T \gg \Theta$. Equation (1) also ignores: a) loss of momentum in electron-electron processes; b) inelastic collisions in which electrons excite atoms, induce ionization etc. Process a) can be neglected if the condition

$$\nu_{ee}(T) \ll \nu(T). \quad (7)$$

is satisfied. In general, the distribution obtained neglecting the processes b) is valid only in the energy region $\epsilon < \epsilon_1$, and with the condition that

$$\varepsilon_1/T \gg 1. \quad (8)$$

Here ϵ_1 is the energy threshold for inelastic processes. A more exact criterion than Eq. (8) is given in the Appendix.

We introduce the dimensionless variable $u = mv^2/2T$ and a dimensionless distribution function $\tilde{f}_0 = n^{-1}(2T/m)^{3/2}f_0$ and rewrite Eqs. (2)–(6) as

$$\frac{d}{du}\left\{[A_1(\tilde{f}_0) + \gamma_1\tilde{\lambda}(u)u]\frac{d\tilde{f}_0}{du} + \left[A_2(\tilde{f}_0) + \gamma_2\frac{u^2}{\tilde{\lambda}(u)}\right]\tilde{f}_0\right\} = 0, \quad (9)$$

where

$$\tilde{\lambda}(u) = \frac{\lambda(u)}{\lambda(T)} \quad \gamma_2 = \frac{m}{M}\frac{\nu(T)}{\nu_{ee}(T)}, \quad \gamma_1 = \frac{M}{6m}\left(\frac{eE\lambda(T)}{T}\right)^2\gamma_2,$$

$$A_1(\tilde{f}_0) = \frac{4\pi}{3}\left\{\int_0^u u_1^{3/2}\tilde{f}_0(u_1)\,du_1 + u^{3/2}\int_u^\infty \tilde{f}_0(u_1)\,du_1\right\}, \quad (10)$$

$$A_2(\tilde{f}_0)=2\pi\int_0^u \sqrt{u_1}\tilde{f}_0(u_1)\,du_1, \tag{11}$$

$$2\pi\int_0^\infty \tilde{f}_0(u_1)\sqrt{u_1}\,du_1=1. \tag{12}$$

From (6), (10), and (11) we get

$$\begin{aligned} &A_1(\tilde{f}_0),\quad A_2(\tilde{f}_0)\to 1 \text{ as } u\to\infty,\\ &A_1(\tilde{f}_0),\quad A_2(\tilde{f}_0)\to u^{3/2} \text{ as } u\to 0. \end{aligned} \tag{13}$$

Since $\tilde{f}_0(u)$ decreases exponentially with increasing argument, in practice $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$ are close to unity starting at $u\approx 2.5-3$. This can be seen from Fig. 3, in which $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$ are depicted for different forms of $\tilde{f}_0(u)$.

Equation (9) can be formally solved, considering $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$ to be known functions. The formal solution is

$$\tilde{f}_0=C\exp\left[-\int_0^u \frac{A_2(\tilde{f}_0)+\gamma_2u^2/\tilde{\lambda}(u)}{A_1(\tilde{f}_0)+\gamma_1u\tilde{\lambda}(u)}\,du\right]. \tag{14}$$

From Eqs. (13) and (14), the form of the distribution function for large and small[1)] values of the argument u follows immediately:

$$\tilde{f}_0(u)=B\exp\left[-\int_{u_0}^u \frac{1+\gamma_2u^2/\tilde{\lambda}(u)}{1+\gamma_1u\tilde{\lambda}(u)}\,du\right],\quad \text{for } u\gg 1,$$

where C and B are constants and $u_0\gg 1$.

The nature of the solution of Eq. (9) depends on the form of the function $\lambda(\epsilon)$. We shall restrict our analysis to a dependence of the type $\lambda(\epsilon)=\lambda_0\epsilon^n$, $n\le 1/2$. Then $\tilde{f}_0(u)$ decreases exponentially at large energies.

In the case $n=1/2$, the exact solution of Eq. (9) is

$$\tilde{f}_0(u)=\pi^{-3/2}e^{-u},\quad T=\frac{M}{3}\left(\frac{eE}{m\nu}\right)^2,$$

where the collision frequency $\nu=v/\lambda$ is independent of the velocity. For different energy dependences of the free path, one cannot find an exact solution, and it is necessary to use approximate methods.

For γ_1 and $\gamma_2\ll 1$, when the electron-electron interaction predominates in the region $u\approx 1$, one may substitute in $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$, in the zeroth approximation, the Maxwellian function $\pi^{-3/2}e^{-u}$, and obtain an approximate solution of Eq. (9) in the form[3]

$$\tilde{f}_0(u)\approx C_0\exp\left[-\int_0^u \frac{A(\sqrt{u})+\gamma_2u^2/\tilde{\lambda}(u)}{A(\sqrt{u})+\gamma_1u\tilde{\lambda}(u)}\,du\right],$$

$$A(\sqrt{u})=\frac{2}{\sqrt{\pi}}\left(\int_0^{\sqrt{u}} e^{-t^2}dt-\sqrt{u}\,e^{-u}\right). \tag{15}$$

Equation (15) can be simplified, using the fact that for $\gamma_1,\gamma_2\ll 1$

$$\frac{A(\sqrt{u})+\gamma_2u^2/\tilde{\lambda}(u)}{A(\sqrt{u})+\gamma_1u\tilde{\lambda}(u)}\approx\frac{1+\gamma_2u^2/\tilde{\lambda}(u)}{1+\gamma_1u\tilde{\lambda}(u)}\equiv p_1(u)$$

for any value of u.

Thus, for $\gamma_1,\gamma_2\ll 1$, a solution of Eq. (9) which is approximately valid for all values of u is given by

$$\tilde{f}_0^{(1)}=C_1\exp\left[-\int_0^u \frac{1+\gamma_2u^2/\tilde{\lambda}(u)}{1+\gamma_1u\tilde{\lambda}(u)}\,du\right]. \tag{16}$$

For arbitrary values of the parameters γ_1 and γ_2, we solve Eq. (9) by the general method of Galerkin[5]. It is natural to choose the function (16) as a trial function, since this function has many advantages. In fact, the function (16) is close to the exact solution of Eq. (9) for $\gamma_1,\gamma_2\ll 1$, and turns into the exact solution for $\gamma_1,\gamma_2\gg 1$ (Druyvstein distribution or its analog). For $\gamma_1,\gamma_2\approx 1$, the function (16) behaves like the exact solution for $u\ll 1$ and for $u>u_0\approx 3$. Substituting the trial function in the left hand side of Eq. (9), multiplying by udu and integrating from 0 to ∞, we obtain an equation determining the parameter T. Physically, this is the equation of energy balance, and it can be put in the form

$$T=\left(\sqrt{\frac{M}{6m}}eE\lambda_0\right)^{1/(1-n)}\left(\frac{\Psi(\tilde{f}_0)}{\Phi(\tilde{f}_0)}\right)^{1/2(1-n)}, \tag{17}$$

where

$$\Psi(\tilde{f}_0)=-\frac{1}{C}\int_0^\infty u\tilde{\lambda}(u)\frac{d\tilde{f}_0}{du}\,du, \tag{18}$$

$$\Phi(\tilde{f}_0)=\frac{1}{C}\int_0^\infty \frac{u^2}{\tilde{\lambda}(u)}\tilde{f}_0du. \tag{19}$$

For n = 0, Eq. (17) is satisfied in case a) with a Maxwellian distribution and in case b) with a Druyvstein distribution with $T=0.29\sqrt{M/m}\,eE\lambda_0$. For n = −1, in case a) $T=0.41(M/m)^{1/4}\sqrt{eE\lambda_0}$, and in case b) $T=0.37(M/m)^{1/4}\sqrt{eE\lambda_0}$.

Thus, the parameter T depends weakly on the form of the distribution, and so in the first approximation, we may take for any γ_1 and γ_2

$$n=0:\ T=T_0=0.29\sqrt{M/m}\,eE\lambda_0,\quad \gamma_1=2\gamma_2=2\gamma, \tag{20}$$

$$n=-1:\ T=T_0=0.39\,(M/m)^{1/4}\sqrt{eE\lambda_0},\quad \gamma_1=7.4\gamma_2=7.4\gamma,$$

$$\gamma=\frac{m}{M}\frac{\nu(T_0)}{\nu_{ee}(T_0)}. \tag{21}$$

The normalization constant C_1 for $\tilde{f}_0^{(1)}$ depends on the single parameter γ. The dependence of $C_1(\gamma)$ is shown in Fig. 1.

We may judge the accuracy of the approximate solution (16) by calculating the next approximation from the formula (14). Such calculations were carried out for the cases n = 0, γ = 1 and n = −1, γ = 0.1. The

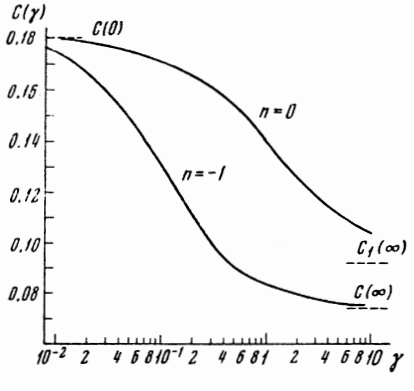

FIG. 1. Graph of the dependence of the normalization constant $C_1(\gamma)$.

[1)] For scattering by acoustic phonons, λ = const. [4]. In the case of low energy scattering by atoms, the fundamental role is played by s-wave scattering, and $1/\lambda\sim\epsilon^{-1}\sin^2\delta_0$. Thus, the integrand in Eq. (14) is always finite.

normalized functions in the first and second approximation are compared in Fig. 2. From this comparison it can be seen that the second approximation differs from the first by not more than 15% in the region of slow electrons. The approximation can be substantially improved if, in choosing the trial function, allowance is made of the difference from unity of the coefficients $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$ for $u \approx 1$. Figure 3 shows the coefficients $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$ calculated with the function (16) for the case $n = -1$ and three values of the parameter γ. From the drawing one sees that $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$ depend comparatively weakly on the form of the distribution.

With this knowledge, we choose as the trial function

$$f_0^{(2)} = C_2 \exp\Big[-\int_0^u \frac{A_0(u)+\gamma_2 u^2/\bar{\lambda}(u)}{A_0(u)+\gamma_1 u\bar{\lambda}(u)}\,du\Big] \equiv C_2 \exp\Big[-\int_0^u p_2(u)\,du\Big],$$

$$\text{where } A_0(u) = \begin{cases} 0{,}385u & \text{for } u \leqslant 2{,}6, \\ 1 & \text{for } u \geqslant 2{,}6. \end{cases} \qquad (22)$$

As before, we shall suppose that the relation (20) is satisfied. Calculation of (22) for $n = -1$ and $\gamma = 0.1$ shows that this function differs in the second approximation (relative to $\tilde{f}_0^{(1)}$) by not more than 2.5% (Fig. 2). The normalization constant for the function (22) is

$$C_2 = \frac{1}{2\pi\varphi(\gamma)}, \quad \varphi(\gamma) = \int_0^\infty \sqrt{u}\exp\Big[-\int_0^u p_2(u)\,du\Big]du.$$

The $\varphi(\gamma)$ curves for $n = 0$ and $n = -1$ are shown in Figs. 4 and 5.

Substituting Eq. (22) in Eq. (17), we can compute the parameter T in the next approximation:

$$T = \Big(\sqrt{\frac{M}{6m}}\,eE\lambda_0\Big)^{1/(1-n)}\Big(\frac{\Psi(\gamma)}{\Phi(\gamma)}\Big)^{1/2(1-n)}, \qquad (23)$$

where $\Psi(\gamma) \equiv (\tilde{f}_0^{(2)})$, $\Phi(\gamma) = (\tilde{f}_0^{(2)})$.

The form of the functions $\Psi(\gamma)$ and $\Phi(\gamma)$ for the cases $n = 0$, $n = -1$ is shown in Figs. 4 and 5. For $n = -1$ we have $\Psi(\gamma) = 1$. The results of these calculations show that T differs little from T_0 for all γ.

2. MOBILITY

If the distribution is known as a function of energy $f_0(v)$, the mobility b_e can be found from the standard formula

$$b_e = -\frac{4\pi}{3}\frac{e}{mn}\int_0^\infty \lambda(v)v^2\frac{df_0}{dv}\,dv. \qquad (24)$$

Using Eq. (22), the expression (24) can be transformed to

$$b_e = \frac{2}{3}\frac{e}{m}\frac{\lambda(T_0)}{\sqrt{2T_0/m}}\frac{\Psi(\gamma)}{\varphi(\gamma)}, \qquad (25)$$

where for the cases $n = 0$ and $n = -1$ the functions $\Psi(\gamma)$ and $\varphi(\gamma)$ are given graphically (Figs. 4 and 5), and T_0 is determined as a function of E by Eq. (20).

From Eq. (25) it follows that the mobility depends not only on the applied field, but also on the electron concentration (through the parameter γ). For $n = 0$, this dependence turns out to be weak (the ratio $\Psi(\gamma)/\varphi(\gamma)$ changes by at most 15%), but for $n = -1$ it is already significant ($\Psi(\gamma)/\varphi(\gamma)$ changes by a factor 2.6 between $\gamma = \infty$).

The cases of energy dependence of the free path we have considered are approximately realized in a number of inert gases. Using the data[6] on the energy dependence of the transport cross-section, one finds that $n = 0$ is obtained approximately for neon ($\lambda = N_0/8N$), $n = -1$ for argon ($\lambda = N_0/5\epsilon N$), and $n = 1/2$ for helium ($\lambda = 3ON_0/\sqrt{\epsilon}N$, $\nu = 1.8\ 10^9 N/N_0$. Here ϵ is the energy of the electrons in electron volts, N the atomic concentration, and N_0 the atomic concentration at pressure 1 Torr and $\Theta = 273$ K. For scattering by acoustic phonons, $n = 0$.

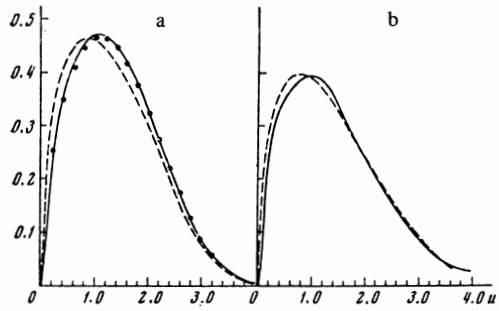

FIG. 2. Comparison of the various approximations for the electron distribution function: a) $n = -1$, $\gamma = 0.1$; dotted – the function $\sqrt{u}\,f_0^{(1)}$; solid line – solution of Eq. (9) in the next approximation; points – the function $\sqrt{u}\,f_0^{(2)}$; b) $n = 0$, $\gamma = 1$; dotted – $\sqrt{u}\,f_0^{(1)}$; solid line – solution of Eq. (9) in the next approximation.

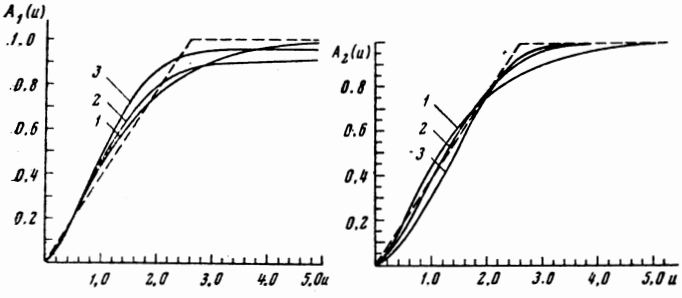

FIG. 3. Form of the coefficients $A_1(\tilde{f}_0)$ and $A_2(\tilde{f}_0)$ for various distribution functions. Curve 1, $\gamma = 0$; 2, $\gamma = 0.1$; 3, $\gamma = \infty$. Dotted line, the chosen approximation, $A_0(u)$.

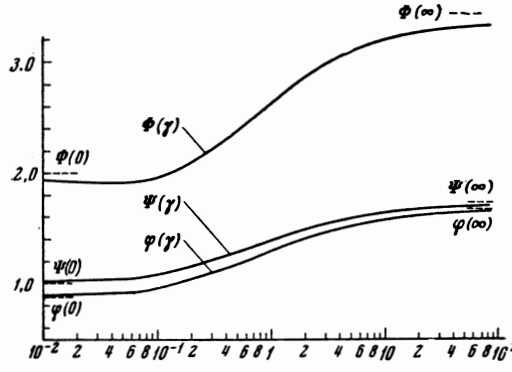

FIG. 4. Graphs of the functions $\varphi(\gamma)$, $\Psi(\gamma)$, and $\Phi(\gamma)$ for $n = 0$.

APPENDIX

The contribution of inelastic processes to the energy balance can be neglected if the condition

$$\Delta = \frac{\varepsilon_1 \langle vq \rangle N}{2(m/M)\langle \nu(v)\varepsilon \rangle} \ll 1 \qquad (26)$$

is satisfied, where q is the cross-section for inelastic processes. We calculate the quantity $\langle vq \rangle$ using the

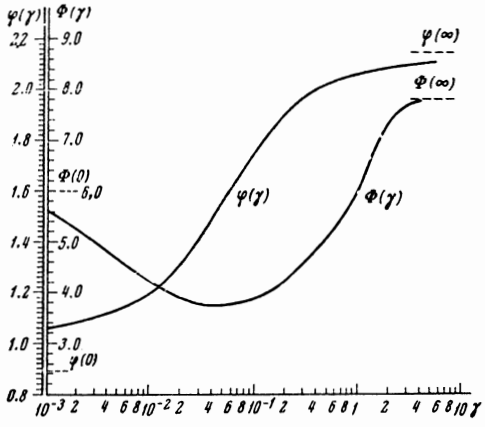

FIG. 5. Graphs of the functions $\varphi(\gamma)$ and $\Phi(\gamma)$ for n = −1.

function (16). In this way we over-estimate the role of the inelastic processes, since for $\epsilon > \epsilon_1$ the distribution should decrease faster than it does according to Eq. (16). Taking the estimate $q = (\epsilon - \epsilon_1)q_0$ and noting that $2(m/M)\langle \nu(v)\epsilon \rangle = eE^2 b_e$ disregarding inelastic processes, the condition (26) can be rewritten

$$\Delta = \frac{M}{2m\Phi(\gamma)p_1^2(u_1)} \exp\left[-\int_0^{u_1} p_1(u)\,du\right] \lambda(T_0)(q_0\varepsilon_1)N \ll 1, \qquad (27)$$

where

$$u_1 = \varepsilon_1 / T_0.$$

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