THE DYNAMICS OF BOLTZMANN SYSTEMS

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In the first part of this paper the spectrum of excitations of Boltzmann systems is studied. The momentum relaxation time and the diffusion trajectory are calculated in the case of self-diffusion of hard-sphere particles. A possibility of nonmonotonic relaxation is found for soft interactions. In the case of charged particles the influence of an electric field on the spectrum is considered. In the second part a macroscopic theory of quasicollective processes is developed on the basis of the use of nonequilibrium kinetic coefficients. A hydrodynamics of fast processes is constructed. Electric conductivity and relaxation for large perturbations are considered.

1. INTRODUCTION

THE study of the dynamics of linear Boltzmann systems in the present work follows two directions. In the first part, which is a continuation of[1,2], there is a further discussion of the spectrum of the Boltzmann equation. In particular, for charged particles the effect of a field is taken into account. Here also we examine the nonspectral singularities of the resolvent and the corresponding quasicollective excitations. We further touch on such questions as the exponential damping of a neutron pulse in a small assembly and the possibility of a nonmonotonic relaxation and of "runaway" in systems with soft interactions.

In the second part nonequilibrium kinetic coefficients (NKC) are defined and calculated for the description of processes whose space-time scales are comparable with the characteristic motions of the particles. On the basis of the NKC a hydrodynamics of fast processes (HFP) is constructed.

Expressions for NKC were proposed earlier[3,4] in the framework of the method of correlation functions. A different approach was developed by Zwanzig.[5] In the present paper the NKC are defined by means of a generalization of the thermodynamics of irreversible processes, in which a connection between flux and force is fixed by means of an operator.1) For the translationally invariant systems considered here this is of the form of a convolution.2) The kernels, and the corresponding NKC, are found from the solution of problems with special initial (boundary) conditions. Recent progress in linear Boltzmann kinetics enables us to give a detailed analysis of NKC of this kind.

2. SPECTRUM AND DYNAMICS

The dynamics of the systems considered is described by the equation

$$\partial\varphi / \partial t = S\varphi, \tag{1}$$

where φ is the deviation of the distribution function

$$f = f^{(0)}[1 + \varphi(\mathbf{v}, \mathbf{r}, t)], \qquad f^{(0)} = n_0(\pi v_T^2)^{-3/2} e^{-v^2/v_T^2} \equiv n_0 f_0. \tag{2}$$

The evolution operator S is the sum of a flow term, which takes account of nonuniformity and external forces, and the collision integral. The latter can be represented in the form[7]

$$J\varphi = -\nu(v)\varphi + \int K(\mathbf{v}, \mathbf{v}_1)\varphi(\mathbf{v}_1) f_{01}\, d\mathbf{v}_1. \tag{3}$$

In the case of an impurity relaxing in an equilibrium medium the kernel

$$K(\mathbf{v}, \mathbf{v}_1) = f_0^{-1}(v)\, w(\mathbf{v}_1 \to \mathbf{v}),$$

of the equation [where $w(\mathbf{v}_1 \to \mathbf{v})$ is the probability of the transition $\mathbf{v}_1 \to \mathbf{v}$] is symmetric owing to detailed balancing.

When there are no spatial gradients and external forces the dynamics of the system is determined by the spectrum of the collision operator. The general pattern of the spectrum is known.[1,8] It is clear from this pattern that for interactions with nonvanishing smallest value ν_* of the collision frequency only the first few eigenvalues are important for $t > 1/\nu_*$. Because the collision operator is isotropic, it is convenient to study it by going over to a spherical-harmonic representation. We then get the equations3)

$$\lambda_{rl}\varphi_{rl} = -\nu(v)\varphi_{rl} + \int_0^\infty K_l(v, v_1)\varphi_{rl}(v_1) f_{01}\, dv_1, \tag{4}$$

where

$$K_l(v, v_1) = 2\pi v v_1 \int K(v, v_1, \mu) P_l(\mu)\, d\mu, \qquad \mu = \frac{\mathbf{v}\mathbf{v}_1}{v v_1}. \tag{5}$$

The basic theory in the case of a collision frequency independent of v (Maxwell model) has been studied in detail. We shall examine below the case of a constant cross section (s-scattering). We have here $\sigma(g, \vartheta) = \sigma_0/4$, $\nu_0 = 2n_0\sigma_0 v_T(m/M\pi)$, where M is the mass of the particles of the medium. The series of eigenvalues λ_{r0} has been calculated by Takahasi[9] in treating the thermalization of neutrons. For a proton moderator $(m/M = 1)$ the smallest relaxation fre-

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1) This approach is akin to the phenomenology of the theory of relaxing media.

2) A formal theory of slow processes with such a connection is contained in [6].

3) Hereafter in this section, while keeping the same notation, we shall use dimensionless quantities: time in units $1/\nu_0$, velocity in units $v_t = (2k_B T/m)^{1/2}$, and frequency in units ν_0, with $\nu_0 = \nu(0)$.

quency is $\lambda_{10} = 0.82$. A simple gas relaxes somewhat more slowly; for it, according to[10], we have $\lambda_{20} = 0.67$. The study of anisotropic perturbations and the determination of the kinetic coefficients requires the use of further eigenvalues $\lambda_{r1}(1 \neq 0)$. These latter are always located not far from the continuum,4) and to calculate them it is expedient to use a modified moment method (the ν representation of the collision integral).[1,2,11] On this basis, for impurity particles mechanically similar to the main gas, the value $\lambda_{01} = 0.85$ was found in the two-term approximation. Accordingly, the relaxation times of an initial flux and of a temperature perturbation are practically the same (for $m/M = 1$).5)

Let us return to the case of interactions with frequency tending to zero for $v \to \infty$ $[\nu(v) \sim v^{-\gamma}]$. It was found earlier[2] that in this case for $t \to \infty$ the evolution follows the law $\exp(-\delta t^{2/2+\gamma})$. We shall consider the damping of an initial perturbation for moderately large values of t.

To determine the course of the relaxation, we study the singularities of the resolvent of the collision operator for complex values of p. Because the general case is complicated, we make the calculations for a model. We assume that the collision frequency has the form $\nu(v) = (1 + av)^{-1}$, where a is a small number. We shall assume that in the ν representation of the collision term[2] the matrix elements K_{mn} and the functions $\psi_m(v)$ can be represented by power series in a. It is obvious that for $a = 0$ we arrive at a known case, since $K_{mn}^{(0)} = (1 + \lambda_m)\delta_{mn}$, $\psi_m^{(0)} = \chi_m(v)$ (the quantities on the right correspond to the Maxwellian model). Let us consider the dispersion equation[2]

$$D(p) = \left\| \delta_{lm} - \sum_n K_{mn} T_{ln}(p) \right\| = 0. \tag{6}$$

As is well known,[2] the dispersion function has a cut $(-\nu^*, 0)$, $\nu^* = \max \nu(v)$, on the real axis. Its limiting values above (below) the cut are

$$D(\lambda \pm i0) = \tilde{D}(\lambda) \pm i\Gamma(\lambda), \tag{7}$$

$$\Gamma(\lambda) = \pi \nu^2 v^2 \left| \frac{dv}{d\nu} \right| \sum_{lmn} \tilde{D}^{ml}(\lambda) K_{ln} \psi_m(v) \psi_n(v) f_0 \big|_{v=v_\lambda}, \tag{8}$$

$$\psi_m(v)\psi_n(v) = \int \psi_m(\mathbf{v})\psi_n(\mathbf{v}) \sin\vartheta \, d\vartheta \, d\varphi;$$

the tilde indicates the principal value of the quantity; $v_\lambda = -(\lambda + 1)/a\lambda$ for the case in question. It is clear that near the eigenvalue closest to zero the Maxwellian model has a sharp resonance $\lambda_1(a)$, which is what determines the behavior for moderately large t. In the neighborhood of $\lambda_1(a)$ we have the approximate representation

$$D^\pm(\lambda) = \tilde{D}'(\lambda_1)(\lambda - \lambda_1) \pm i\Gamma(\lambda_1), \quad \lambda_1 = \lambda_1(a). \tag{9}$$

Using this, we get

$$\lambda_\pm = \lambda_1 \mp i\Gamma(\lambda_1) / \tilde{D}'(\lambda_1) \tag{10}$$

—the zeroes of the dispersion function continued analytically upward and downward, under the cut.6) Accordingly, for systems with a soft interaction, a nonmonotonic course of the relaxation is possible along with a damping of the form $e^{-\alpha t}$. It can be shown that the frequency increases and the decrement $-\lambda_1$ decreases with increasing a (softness). In the model in question the value of the frequency

$$\omega \approx \frac{1}{\sqrt{\pi}} \frac{(1 + \lambda_1)^2}{a^3 \lambda_1^4} \chi_1^2(v_{\lambda_1}) \exp\left[-\left(\frac{1 + \lambda_1}{a\lambda_1}\right)^2\right] \tag{11}$$

does not provide any possibility of observing oscillations. But the tendency which the model shows allows us to hope that in real cases there will be a more favorable ratio between the decrement and the frequency.

Let us consider the spectrum of a uniform charged impurity in the presence of an electric field. One is usually interested in the current mode; the influence of the field on the temperature relaxation time is also interesting. In the case of a field directed along the z axis the evolution operator has the form (in the natural representation)

$$Sf = a \frac{\partial f}{\partial v_z} - \nu f + \int w(\mathbf{v}_1 \to \mathbf{v}) f(\mathbf{v}_1) \, d\mathbf{v}_1, \tag{12}$$

where $a = eE/mv_T\nu_0$ is the ratio of the energy acquired by a particle in a mean free path to the thermal energy.

First let us study the continuous spectrum of the one-particle motions. According to known arguments (cf.[1,2]), the position of the continuum can be determined from the equation

$$pf = a \frac{\partial f}{\partial v_z} - \nu(v) f. \tag{13}$$

Its solution is

$$f(p, v_\perp, v_z) = \psi(p, v_\perp) \exp\left[\frac{1}{a}\left(v_z p + \int_0^{v_z} \nu(v_\perp, u) \, du\right)\right], \tag{14}$$

where $v_\perp$ means the components v_x and v_y. We find the region of the continuous spectrum from the condition that the solution of (14) be bounded with respect to v_z. We have

$$\operatorname{Re} p = -\frac{1}{v_z} \int_0^{v_z} \nu(v_\perp, u) \, du \tag{15}$$

(Im p is arbitrary). It is clear that in the case of a constant frequency the continuum is concentrated on the line Re p = −1: for soft interactions it fills the left half-plane. We note that the transition to the case a = 0 requires the use of the condition of stationary phase.

Let us consider the influence of the field on the relaxation times. We confine ourselves to weak fields ($a \sim 0$) and use perturbation theory. We represent the eigenfunctions of the collision operator for an arbitrary interaction in the form

$$\varphi_{rl} = \sum_s a_{rsl} \chi_{sl},$$

$$a_{rsl} = \int \chi_{sl}^* \varphi_{rl} f_0 \, d\mathbf{v}, \qquad a_{0s0} = \delta_{s0}, \tag{16}$$

where χ_{s1} are the normalized eigenfunctions for the

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4) Unlike the values λ_{r0}, which go to zero when the difference of the masses of the particles is large.

5) This agrees with the degeneracy $\lambda_{01} = \lambda_{10}$ found for Maxwellian particles (cf. [12], where the basic theory of many-component systems is given).

6) Similar zeroes, lying on the nonspectral sheet, are eigenvalues of the analytic continuation of the original operator.

Maxwellian model ($m = 0$). The coefficients a_{rsl} decrease very rapidly as we go away from the diagonal $r = s$. We have the relation

$$\frac{\partial}{\partial v_z}\chi_{rl} = N^{rl}_{0,-1}\chi_{r,l-1} - N^{rl}_{-1,1}\chi_{r-1,l+1}, \tag{17}$$

$$N^{rl}_{\Delta r,\Delta l} = (2l+1+\Delta l)\left[\frac{r+{}^1/_2(2l+1)(1+\Delta r)}{(2l+1)(2(l+\Delta l)+1)}\right]^{1/2}. \tag{18}$$

Using (16) and (17), we get for the matrix element of the perturbation

$$U^{r'l'}_{rl} = \int \varphi^*_{r'l'}\frac{\partial}{\partial v_z}(f_0\varphi_{rl})\,d\mathbf{v} \tag{19}$$

the expression

$$U^{r'l'}_{rl} = \sum_s a^*_{r's'l'}[a_{r,s-1,l}N^{sl'}_{-1,1}\delta^l_{l'+1} - a_{rsl}N^{sl'}_{0,-1}\delta^l_{l'-1}]. \tag{20}$$

The adjoint matrix is obtained by the interchange $r, l \rightleftarrows r', l'$. When we use the expressions that have been given, the formulas of perturbation theory show that: a) the zeroth eigenvalue is not perturbed; b) there is no perturbation of odd order in a; c) for the Maxwellian model, $a_{rsl} = \delta_{rs}$, the corrections are zero (the field does not affect the relaxation time).7) Since the quantities $U^{r'l'}_{rl}U^{rl}_{r'l'}$ are not of definite sign, it is hard to draw any general conclusion about the direction of the shift of the low eigenvalues (owing to the field). In the case of carriers of small mass it is very likely that the frequency λ_{10} is shifted to the left, and λ_{01} to the right.

From this analysis we can, in particular, draw the conclusion that an effect such as the "running away" of electrons8) is of a quasicollective character (unlike the usual "collective" instabilities). Furthermore the current cannot increase with time more rapidly than ct.

Let us go on to the case of spatially nonuniform systems with no field. The general pattern of the spectrum of the equation

$$p\varphi = -i\mathbf{k}\mathbf{v}\varphi + J\varphi \equiv S(\mathbf{k})\varphi(p,\mathbf{k}) \tag{21}$$

has been investigated earlier.[1,2]9) The intrinsic theory of the adjoint $S^+(\mathbf{k})$ of the evolution operator was also developed in those papers. Owing to the relations

$$S^+(\mathbf{k}) = S^*(\mathbf{k}) = S(-\mathbf{k}) \tag{22}$$

its eigenfunctions $\varphi^+_p(\mathbf{k})$ are given by

$$\varphi_p{}^+(\mathbf{k}) = \varphi_{p^*}{}^*(\mathbf{k}). \tag{23}$$

The functions $\varphi^+_p(\mathbf{k})$ are orthogonal to the eigenfunctions of the original operator, i.e.,

$$\int \varphi_p{}^{+*}\varphi_{p'}f_0\,d\mathbf{v} \equiv \langle\varphi_p{}^+, \varphi_{p'}\rangle = \delta_{pp'} \tag{24}$$

(the normalization factor, which is not positive, has been taken equal to unity). When we use (24) the spectral resolution of the evolution operator can be written in the form

$$S(\mathbf{k})\cdot = \sum_n^{N(k)}\langle\varphi_n{}^+, \cdot\rangle\, p_n\varphi_n + \int\lambda\,\langle\varphi_\lambda{}^+, \cdot\rangle\,\varphi_\lambda\,d\lambda \tag{25}$$

[the integration is taken over the continuum: $\lambda + i\mathbf{k}\cdot\mathbf{v} + \nu(v) = 0$]. As is well known,[1] for sufficiently large k there is no discrete spectrum [i.e., $N(k) = 0$].

We consider the spectrum branch $p = p_1(k)$ with the smallest damping, which describes the diffusion of a small impurity in the equilibrium medium. We take the case $\sigma = \text{const}$, $m/M = 1$. To calculate $p_1(k)$ we use a three-term representation of the operator K. The corresponding dispersion equation[2] gives for small k

$$p_1(k) = -dk^2 + \ldots, \quad d = 0.542. \tag{26}$$

The value of the self-diffusion coefficient d corresponds to the third approximation of the Chapman-Enskog theory.[15] Numerical analysis leads to the respective values $-p_1 = 0.5, 0.8, 1.2$ for $k = 1.0, 1.32, 1.73$. For $k = k_{pr}$ ($=1.53$) the trajectory reaches the continuum, and for larger k the original equation has no solutions. The last point was obtained from an equation continued by continuity. The roots of this disappear in turn if k exceeds a value k^*_{pr} ($k^*_{pr} \approx 2$, $p_{pr} \approx 1.4$). The observed quiasicollective excitation of a type of resonance indicates that there is a possibility of describing the quasiexponential damping of a neutron pulse in small ($k > k_{pr}$) assemblies.[16] (Here k^*_{pr} will determine the smallest assembly in which it can be observed.)

In conclusion we shall look at the stationary problem [$p = i\omega$, $k = k(\omega)$]. In this case the characteristic singularities of the resolvent, associated with excitations of the collective type, can be found by using the method of "analytic propagation" of the results of the intrinsic theory, $p = p(k)$. According to this the resonance at $\omega = 0$, indicated in[2], is determined by the formula $\lambda_1 = -\kappa\bar{v}_z = \text{Im } k$, where $\bar{v}_z$ is some average value of the velocity. Taking $\bar{v}_z$ equal to $2^{-1/2}$, we get an approximate expression for the diffusion length, $\kappa \approx 2^{1/2}|\lambda_1|$. The more complicated case of propagation is studied by an analogous procedure of inverting the dependences $p = p(k)$ (known from the intrinsic theory) for $p = i\omega$.

3. THE MACROSCOPIC DESCRIPTION OF FAST PROCESSES

Using the results of[1,2] and the preceding section, we shall discuss the macroscopic approach to the study of fast processes in Boltzmann systems. It is based on the use of nonequilibrium transport coefficients (NKC), and is a generalization of the well known Chapman-Enskog theory[15] and of the relaxation theory, which describes slow processes, to the case of large space-time gradients.

In the problems with which we shall deal it is convenient to use a Fourier-Laplace representation. We define the nonequilibrium coefficients by starting from the relations

$$\pi_n(p,\mathbf{k}) = -\sum_m K_{nm}(p,\mathbf{k})\,s_m(p,\mathbf{k}), \tag{27}$$

where π_n is a flux, s_m is a thermodynamic force, and

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7)From this it follows that in the case of a smoothly varying collision frequency the perturbation will have an additional small factor [$\sim\nu'(v_T)$]

8)A stationary analysis of this is given by Levinson. [13]

9)The isotropic approximation was studied by Corngold. [14] We note that it is insufficient for the description of transport processes.

K_{nm} is the corresponding NKC. We obtain the expressions for the fluxes and the forces by means of solutions of problems with special initial (boundary) conditions.

Let us first consider the diffusion of a small impurity. In this case the force is the density gradient, and in accordance with (27) we have

$$\mathbf{j}(p, \mathbf{k}) = -d(p, \mathbf{k})\, i\mathbf{k}n(p, \mathbf{k}), \tag{28}$$

where

$$\mathbf{j}(p, \mathbf{k}) = n_0 \int \mathbf{v}\varphi(p, \mathbf{k}, \mathbf{v}) f_0\, d\mathbf{v} = n_0 \langle \mathbf{v}, \varphi \rangle. \tag{29}$$

$$n(p, \mathbf{k}) = n_0 \langle 1, \varphi \rangle. \tag{30}$$

The relation between the flux $\mathbf{j}$ and the gradient of the density perturbation determines the NKC for diffusion $d(p, \mathbf{k})$. We get the expression for it by using the solution of the problem with the initial condition $\varphi_{t=0} = \eta(\mathbf{k})g_1(\mathbf{v})$, $g_1(\mathbf{v}) = 1$. We write the solution by means of the resolvent

$$\varphi(p, \mathbf{k}) = R(p, \mathbf{k})\varphi_0(\mathbf{k}). \tag{31}$$

By (28) we will have

$$d(p, \mathbf{k}) = \frac{i\mathbf{k}\langle \mathbf{v}, R1 \rangle}{k^2 \langle 1, R1 \rangle}. \tag{32}$$

Accordingly, the NKC is essentially expressed as the ratio of the corresponding correlation functions.10) As is clear from (32), the NKC does not depend on the form of the initial perturbation in space. In the limit $p \to 0$, $\mathbf{k} \to 0$ the quantity $d(p, \mathbf{k})$ given by (32) agrees with the known (equilibrium) value.[15]

The law of conservation of density

$$pn(p, \mathbf{k}) + i\mathbf{k}\mathbf{j} = n_0(\mathbf{k}) \tag{33}$$

and the symmetry (for real molecular attributes)

$$\langle h, Rg \rangle = \langle g, Rh \rangle \equiv R_{g,h}(p, \mathbf{k}) \tag{34}$$

which follows from (22) lead to connections between the correlation functions. In particular, the relations

$$p[1 - p\langle 1, R1 \rangle] = ip\langle \mathbf{k}\mathbf{v}, R1 \rangle = \langle \mathbf{k}\mathbf{v}, R\mathbf{k}\mathbf{v} \rangle \tag{35}$$

give convenient expressions for the NKC of diffusion in various cases. Owing to the translational properties of the resolvent

$$R(p, -\mathbf{k}, -\mathbf{v}) = R^*(p^*, \mathbf{k}, -\mathbf{v}) = R(p, \mathbf{k}, \mathbf{v}) \tag{36}$$

in **k** and **v** the dependence of the quantities $R_{h,g}(p,\mathbf{k})$ on p and $\mathbf{k}$ can be fixed more precisely. According to (36) we thus have $R_{1,1}(p, \mathbf{k}) = R_{1,1}(p, -\mathbf{k})$. In virtue of (36) the NKC for diffusion is real for real p.

Accordingly, the equation proposed for the macroscopic description of fast diffusion is of the form

$$[p + k^2 d(p, k)]n(p, \mathbf{k}) = n_0(\mathbf{k}), \tag{37}$$

where $d(p, \mathbf{k})$ is calculated according to (32). The spectrum of the excitations is found from the equation

$$D_{\mathrm{HFD}}(p, k) \equiv p + k^2 d(p, k) = R^{-1}_{1,1}(p, k) = 0 \tag{38}$$

[the first of the equations (35) has been used]. Accordingly, the collective modes of the hydrodynamics of fast diffusion (HFD) are the poles of the density correlator. A knowledge of the analytic structure of the resolvent (see[1,2] and the preceding section) gives us an idea of the general pattern of the spectrum of Eq. (37). In particular to describe gradients larger than k_{pr} one can use an analytic continuation of the trajectories.

Besides the qualitative results, one can also get some quantitative comparisons with the results of kinetic theory. Let us compare the diffusion trajectories. Using the ν representation of the collision integral, with some simple algebra we get

$$D_{\mathrm{HFD}}(p, k) = \frac{D}{D^{11} - D + \Delta}, \tag{39}$$

where $D(p, k)$ ($= \| d_{im} \|$) is the dispersion function of kinetic theory, and $D^{11}(p, k)$ is the algebraic complement of the element d_{11}. The quantity $\Delta(p, k)$, for which an expression can easily be written, vanishes when the arguments go to zero.11) Since $D^{11}(0, 0) = 0$, the diffusion trajectories for small k agree to arbitrary order in k.

We shall give approximate expressions for the NKC of diffusion for small values of the arguments. Here there are two very simple cases $p \gg kv_T$ and $kv_T \gg p$. Let us consider the first of these cases. Starting from the second equation in (25) and calculating the correlators by using the spectral resolution of the resolvent that follows from (25) for $k \to 0$, we get

$$d(p, 0) = \sum_n \frac{\langle \varphi_n, v_z \rangle \langle v_z, \varphi_n \rangle}{p - \lambda_n} + \int \frac{\langle \varphi_\lambda, v_z \rangle \langle v_z, \varphi_\lambda \rangle}{p - \lambda}\, d\lambda \tag{40}$$

(the z axis of the **v** space is directed along **k**). Here φ_n are the eigenfunctions of the collision integral. An idea of the number of terms that must be used is given by the case ν = const. For it we have

$$d(p, 0) = \frac{v_T^2}{2} \frac{1}{p - \lambda_{01}}. \tag{41}$$

Setting $p = i\omega$, we arrive at a formula analogous to that used in the phenomenological theory of relaxation phenomena (now, of course, with a quite definite value of the "relaxation time," namely $1/\lambda_{01}$).

Because of Eq. (39) the second case leads to equations with spatial derivatives of increasing orders. For diffusion there will be fourth-order derivatives, so that (37) is reduced to the form

$$pn - n_0 = -k^2(d_0 - d_1 k^2 + \ldots)n. \tag{42}$$

Here d_0 is obviously the equilibrium diffusion coefficient; d_1 (>0) is the "cooling-down diffusion coefficient" generally used in the theory of neutron transport, and so on. The constants d_n effectively represent the NKC of diffusion only for sufficiently small k; as a rule the expansion in k is an asymptotic one.12) It must be noted that the expansion (42) corresponds to the linear part of the old approximations of the Chapman-Enskog theory for the case in question.

In the case of intermediate values of p and k it is

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10) The quantities $\langle h, Rg \rangle$ owe their name to the Onsager hypothesis.

11) In the case of a constant collision frequency $\Delta(p, k) = 0$.

12) The series in k will converge, as can be shown from the analytic properties of the resolvent near zero, in the increase of the frequency $\nu(v)$ for $v \gg v_T$ is not slower than bv. For the case σ = const the series converges for $k > b$ (cf. [1]).

natural to use for the calculation of the NKC a finite number of terms of the ν representation of the collision operator. For certain conditions [when values $p \approx p_1(k)$ are important] it is convenient to use the "one-mode" approximation, which is obtained if we keep the main terms in the spectral expansion of the correlation functions. Then for $p \approx p_1(k)$, $d(p, k)$ is given by

$$d(p,k) \approx \frac{i\langle v_z, \varphi_1(k)\rangle}{k\langle 1, \varphi_1(k)\rangle} \qquad (k \equiv k_z). \tag{43}$$

Let us proceed to the construction of the hydrodynamics of fast processes (HFP) in a simple gas. First, we choose as the thermodynamic forces the quantities (in the p, k representation)

$$s_1 = ik\,n(p,\mathbf{k}), \quad s_{2ij} = ik_i u_j,$$
$$s_3 = i\mathbf{k}\frac{m}{3k_B}\int\left(v^2 - \frac{3}{2}v_T^2\right)\varphi f_0\,d\mathbf{v}, \tag{44}$$

The viscous-stress tensor π_{ij} and the heat flux $\mathbf{q}$, for which the basic quantities are

$$h_{2ij} = \rho_0\left(v_i v_j - \delta_{ij}\frac{v^2}{3}\right), \quad \mathbf{h}_3 = \frac{\rho_0}{2}\mathbf{v}\left(v^2 - \frac{5}{2}v_T^2\right), \quad \rho_0 = mn_0, \tag{45}$$

can be expressed in terms of the forces by means of the relations

$$\pi = -[ik\overset{(3)}{\mathrm{K}}_{21}\cdot s_1 + \overset{(4)}{\mathrm{K}}_{22} : s_2 + ik\overset{(3)}{\mathrm{K}}_{23}\cdot s_3], \tag{46}$$

$$\mathbf{q} = -[\overset{(2)}{\mathrm{K}}_{31}\cdot s_1 + ik\overset{(3)}{\mathrm{K}}_{32} : s_2 + \overset{(2)}{\mathrm{K}}_{33}\cdot s_3] \tag{47}$$

(the ranks of the tensors are shown in the upper parentheses; points mean contractions). Here for convenience some NKC are defined differently than in (27). Substituting the expressions (46) and (47) for the fluxes in the conservation equations and using the equation of state (for small deviations δ),

$$\delta p = p_0\left(\frac{n}{n_0} + \frac{\delta T}{T_0}\right), \quad p_0 = k_B n_0 T_0. \tag{48}$$

we arrive at a closed system of equations of HFP.

As before, we get the expressions for the NKC by solving a problem with initial conditions. These are chosen so that they correspond to a perturbation of one of the hydrodynamic parameters (n, δT, or u).[13)] The result is that according to (46) and (47) we will have the following expressions:

$$\mathrm{K}_{22zx} = \frac{\rho_0 i\langle v_z v_x, Rv_x\rangle}{k_z\langle v_x, Rv_x\rangle}, \tag{49}$$

for the NKC of "shearing" viscosity;

$$\mathrm{K}_{33zz} = \frac{3n_0k_B}{2}\frac{i\langle v_z(v^2 - {}^5/_2 v_T^2), R(v^2 - {}^3/_2 v_T^2)\rangle}{k_z\langle v^2 - {}^3/_2 v_T^2, R(v^2 - {}^3/_2 v_T^2)\rangle} \tag{50}$$

for the thermal conductivity coefficient;

$$\mathrm{K}_{32zx} = \frac{\rho_0}{2}\frac{\langle v_z(v^2 - {}^5/_2 v_T^2), Rv_z\rangle}{k_x^2\langle v_z, Rv_z\rangle} \tag{51}$$

for the cross NKC of "thermoviscosity," and so on.

Let us briefly discuss the properties of the NKC. First we note that the absence of an NKC of volume viscosity in (46) is justified analogously with the equilibrium case (see[17], p. 200). With obvious changes, everything said about the nonequilibrium diffusion coefficient holds for the diagonal NKC. In particular, let us consider the viscosity. Separating the longitudinal and transverse velocity components as in ordinary hydrodynamics, we get a diffusion equation for the transverse components. Working with this and considering for simplicity a flow in the x direction which is inhomogeneous in the z direction, we arrive at an expression like (39):

$$D_{\mathrm{HFP}}(p, k_z) = \frac{D}{D^{22} - D + \Delta}. \tag{52}$$

where $D^{22}(p, k_z)$ is the corresponding algebraic complement (we recall that $\psi_2 \sim v_x$). It is clear from (52) that the transverse branch of HFP is identical with the kinetic branch. We have in analogy with (41)

$$\mathrm{K}_{22}(p, 0) = \frac{\rho_0 v_T^2}{2}\frac{1}{p - \lambda_{02}}. \tag{53}$$

The cross NKC's: $\mathrm{K}_{21}^{(3)}$(viscodiffusion), $\mathrm{K}_{32}^{(3)}$, $\mathrm{K}_{23}^{(3)}$ are different from zero for $p = 0$, $k = 0$. But the corresponding terms in (46) and (47) are of order k^2, and are not taken into account in ordinary hydrodynamics. The equilibrium values of these coefficients ($p = 0$, $k = 0$) appear in the linear version of the theory of Barnett.[15] The coefficient $\mathrm{K}_{31}^{(2)}(p, k)$ (thermal diffusion), being of the order k^2, is important when gradients are large. Connections between the correlation coefficients, similar to (35), lead to various relations between the cross NKC. Here we note only the inequality $\mathrm{K}_{23}^{(3)} \neq \mathrm{K}_{32}^{(3)}$, which does not agree with the symmetry principle. Application of HFP to the problem of sound propagation leads, unlike ordinary hydrodynamics, to a finite speed of sound for $\omega \to \infty$. It must be pointed out that the expressions for the nonequilibrium coefficients obtained above by means of problems with initial conditions keep the same form for the stationary problem. The boundary conditions for the latter take the form of a point source of one of the hydrodynamic quantities, with a harmonic time dependence.

Continuing the study of NKC, we shall discuss the nonequilibrium conductivity in an inhomogeneous system. Let a field E along the z axis be applied to a system which is in equilibrium at the initial time. We shall calculate the current produced and use the relation

$$j(p) = \sigma(p)E(p), \quad E(p) = E/p, \tag{54}$$

to define the nonequilibrium conductivity $\sigma(p, E)$. The latter can also be used for the description of essentially nonstationary processes (for example, the L "runaway" of electrons), and for the calculation of stationary states in a strong field.

Accordingly, we require the solution of the equation

$$pf - f_0 = a\frac{\partial f}{\partial v_z} + Jf, \quad a = \frac{eE}{m}. \tag{55}$$

To obtain the solution, we use the ν representation of the collision term. We have

$$f(p, v_\perp, v_z) = \frac{1}{a}\int_{v_z}^{\infty} du \exp\left[-\frac{1}{a}\Lambda(p, v_\perp, u, v_z)\right] f_0(v_\perp, u)$$
$$\times\left[1 + \nu(v_\perp, u)\sum_{MN} c_M(p)K_{MN}\psi_N(v_\perp, u)\right],$$

[13)] The choice of "pure" initial conditions along with the possibility of superposition is based on arguments that the NKC are independent of the form of the perturbation in space.

$$\Lambda(p, v_\perp, u, v_z) = \int_{v_z}^{u} [p + \nu(v_\perp, u')]\, du'; \tag{56}$$

the $c_M(p)$ are determined from the system

$$\sum_M c_M \Big[\delta_{LM} - \sum_N K_{MN} Q_{LN}(p) \Big] = b_L(p), \tag{57}$$

where

$$b_L(p) = \frac{1}{a} \int d\mathbf{v}\, \nu(v) \psi_L^*(\mathbf{v}) \int_{v_z}^{\infty} du \exp\Big\{ -\frac{1}{a} \Lambda(p, v_\perp, u, v_z) \Big\} f_0(v_\perp, u), \tag{58}$$

$$Q_{LN}(p) = \frac{1}{a} \int d\mathbf{v}\, \nu(v) \psi_L^*(\mathbf{v}) \int_{v_z}^{\infty} du \exp\Big\{ -\frac{1}{a} \Lambda(p, v_\perp, u\ v_z) \Big\} \times f_0(v_\perp, u)\, \nu(v_\perp, u)\, \psi_N(v_\perp, u). \tag{59}$$

To calculate $\sigma(p, E)$ for a broad class of interactions it suffices to take a few terms, including two or three harmonics. We note that for carriers of small mass the use of two harmonics ($l = 0, 1$) allows us to take into account terms of order m/M.

Let us examine in more detail the case of constant frequency. Here the relations are simpler, and when we use (17), Eq. (57) takes the form

$$c_{rl} \Big[1 - \frac{\nu_0 K_{rl}^{rl}}{p + \nu_0} \Big] + \frac{a}{p + \nu_0} [N_{0,-1}^{rl} c_{r,l-1} - N_{-1,1}^{rl} c_{r-1\ l+1}] = \frac{\delta_{rl}^{00}}{p + \nu_0}. \tag{60}$$

The conductivity is given by

$$\sigma(p) = \sqrt{2} e^2 / m v_T^2 (p - \lambda_{01}). \tag{61}$$

Accordingly, for the Maxwellian interaction the NKC does not depend on the strength of the applied field.14)

In conclusion let us consider nonequilibrium coefficients of another kind, which generalize the relaxation times which are commonly used. NKC of relaxation are designed for the description of the approach of quantities to their equilibrium values, beginning with times $t \approx 1/\nu_0$, when the initial deviations are large. We shall discuss the equalization of an initial temperature perturbation of an impurity (thermalization), in the spatially uniform case. The NKC for thermalization $\tau(p)$ is defined by the relation

$$p\Delta T - \Delta_0 T = -\Delta T / \tau(p), \tag{62}$$

where $\Delta T(p) = T(p) - T_0$, T_0 being the temperature of the medium; $\Delta_0 T$ is the initial temperature difference. To get $\tau(p)$ we solve the equation

$$p\Delta f - \Delta_0 f = J_0 \Delta f, \qquad \Delta f = f(p, v) - f_0; \tag{63}$$

the index zero on the collision operator indicates the spherically symmetric part of it [see (5)]. Owing to conservation of density Δf is orthogonal to unity. As the initial condition we take the Maxwell distribution with the temperature $T_0 + \Delta_0 T$. According to (62) we get for the NKC of thermalization the expression

$$\tau(p) = \frac{(v^2, R(p) \Delta_0 f)}{(v^2, \Delta_0 f) - p(v^2, R(p) \Delta_0 f)}$$

$$(h, g) = \int h^* g\, d\mathbf{v}. \tag{64}$$

Obviously $\tau(p)$ depends on the initial deviation $\Delta_0 T/T_0$. Its magnitude determines the number of terms of the ν representation required for the calculation of $\tau(p)$. For $\Delta_0 T \to 0$, in the case of constant frequency we arrive at the formula $\tau(p) = -1/\lambda_{10}$, whose consequences are obvious. In concluding the discussion, we point out the inadequacy of the usual (equilibrium) relaxation approach in the case in which one-particle motions are important. For example, for a Lorentz plasma the usual theory using a slowing time τ_s gives a law of damping of an electron velocity perturbation of the form $\exp(-t/\tau_s)$. Use of a formula like (64) leads for $t \to \infty$ to a form $\exp(-\alpha t^{2/5})$, which agrees with the prescription of kinetic theory.[2]

4. CONCLUDING REMARKS

1. The analysis of the spectrum of Boltzmann systems made in[1,2] and the present paper has enabled us to show a correspondence between spectral singularities and observed excitations, has revealed possible excitations, (in particular, an oscillating relaxation mode in the case of a soft interaction), and has indicated some new aspects of the dynamics of such systems. The nonspectral poles of the resolvent have been brought in to describe quasicollective excitations. This is possible only when these poles are associated with sharp and very long-lived resonances. Resonances are of a more general character than nonspectral poles, for which analytic continuation is necessary. For example, there is a sharp resonance corresponding to the sound excitation for $k \gtrsim k_{pr}$ (in the case of a soft interaction for $k < k_{pr}$). On the other hand analytic continuation is possible only for the special model with constant collision frequency.

2. The complexity of the study of quasicollective processes15) brings about the resort to various simplifications. One of these is the spectral approach to the analysis of the dynamics. Another approach, developed in this paper, is the construction of a macroscopic theory of such processes on the basis of the use of nonequilibrium kinetic coefficients. Besides its computational aspect (for example, the application of hydrodynamics of fast processes to calculate the flow of rarefied gases), the study of NKC has a broader meaning. It provides analogies for an approach to many-particle systems of greater physical significance.

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14) Although this property of the model, along with the absence of thermal diffusion, seems "unphysical", still it gives a qualitatively correct description of the weak dependence of the conductivity of E in a wide range of conditions.

15) We remark that the treatment of such processes broadens the scheme of the kinetic analysis of nonequilibrium processes formulated by L. É. Gurevich. [17]

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237