THEORY OF THE ELECTRICAL CONDUCTIVITY OF A WEAKLY IONIZED IMPERFECT PLASMA

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The electrical conductivity coefficient of a weakly ionized plasma, σ, is expanded in powers of the concentration of the neutral component N: $\sigma = \sigma_0 (1 + c_1 N + c_2 N^2 + c_3 N^2 \ln N)$, where σ_0 is the electrical conductivity in the Lorentz approximation. The corrections to σ_0 take account of the nonideality effects.

IN a weakly ionized perfect plasma the electrical conductivity is determined by the solution of the kinetic equation with account of the scattering of the electrons on the neutrals in the binary approximation. When the density is increased, the binary approximation becomes insufficient. When the wavelength of the electron is comparable with the average distance between the scatterers, the electron moving in an external field interacts simultaneously with many particles.

Kohn and Luttinger[1] have solved the problem of the electrical conductivity of a metal caused by the scattering on impurities. They obtained corrections to the collision integral due to non-binary scatterings. The subsequent papers used the Green's function method. In the general theory of slightly non-equilibrium processes the electrical conductivity is given in terms of the two-particle Green's function according to the Kubo formula.[2,3] Along this line, Abrikosov and Gor'kov[4] and Edwards[5] have calculated the residual resistance of a metal, and Konstantinov and Perel'[3] and Perel' and Éliashberg[6] have determined the electrical conductivity of a fully ionized plasma.

We shall consider a weakly ionized plasma under conditions where the wavelength of the electron λ and the scattering amplitude $\sqrt{q}$ are comparable with $N^{-1/3}$, where N is the concentration of the neutral particles. The problem consists in the description of transfer in the medium of these particles. The concentration of the free electrons n will be assumed known, since the problem of the number of carriers and their mobility can be solved separately if the effects of nonideality are not too large.

Assuming that $\lambda q N < 1$, we obtain an expression for the electrical conductivity in the form of an expansion in powers of the density. The first term of the series corresponds to the known formula for a Lorentz gas. We obtain the explicit form of the corrections to this formula. The limits of applicability of the formula are discussed.

BASIC EXPRESSIONS

The general expression for the conductivity in a spatially homogeneous electric field has the form[3]

$$\sigma_{\mu\nu}(\omega) = \left(\frac{e\hbar}{m}\right)^2 \int \frac{p_\mu p_\nu'}{i\omega} [K^R_{pp'}(\omega) - K^R_{pp'}(0)] \frac{d^3p d^3p'}{(2\pi\hbar)^6}, \qquad (1)$$

where ω is the frequency of the field, and $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of the electrons; $K^R_{pp'}$ is the retarded two-particle Green's function,

$$K^R_{pp'}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} K^R_{pp'}(t)\, dt,$$

$$K^R_{pp'}(t) = \frac{i}{\hbar}\theta(t)\langle [a_{\mathbf{p}'}^+ a_{\mathbf{p}'}(t),\ a_{\mathbf{p}}^+ a_{\mathbf{p}}(0)]_- \rangle,$$

$a_{\mathbf{p}}^+$ and $a_{\mathbf{p}}$ are creation and annihilation operators, and $\langle \ldots \rangle$ denotes the statistical average over an equilibrium state of the system.

For the calculation of $K^R_{pp'}(\omega)$ we use the temperature dependent graph technique.[7] $K^R_{pp'}(\omega)$ is the analytic continuation of the function $K_{pp'}(\omega_n)$ from the discrete points on the imaginary axis $i\omega_n = (2n+1)\pi i/\beta$, $n > 0$, to the real axis of ω. We have

$$K_{pp'}(\omega_n) = \frac{1}{2}\int_{\beta}^{-\beta} \exp(i\omega_n \tau) K_{pp'}(\tau)\, d\tau,$$

$$K_{pp'}(\tau) = \langle T\{a_{\mathbf{p}'}^+ a_{\mathbf{p}'}(\tau) a_{\mathbf{p}}^+ a_{\mathbf{p}}(0)\}\rangle,$$

where $\beta = 1/T$ is the inverse temperature.

For the description of the electron-atom interaction we introduce the Fourier component of scattering potential between the electron and the atom $V_{\mathbf{q}}$. Since the energy transfer between the electrons and the atoms can be neglected, the problem reduces to the study of the motion of the electrons in a field of fixed scatterers. The Hamiltonian has the form

$$H = \sum_{\mathbf{p}} \varepsilon_{\mathbf{p}} a_{\mathbf{p}}^+ a_{\mathbf{p}} + \sum_{\mathbf{p},\mathbf{q}} U_{\mathbf{q}} a_{\mathbf{p}+\mathbf{q}}^+ a_{\mathbf{p}},$$

where $\epsilon_{\mathbf{p}} = p^2/2m$, and $U_{\mathbf{q}}$ is the Fourier component of the scattering field, which depends on the coordinates of the scattering centers $\mathbf{r}_i$:

$$U_{\mathbf{q}} = \sum_{i=1}^{N} e^{-i\mathbf{q}\mathbf{r}_i} V_{\mathbf{q}}.$$

For not too large N one can assume that the atoms are distributed randomly, and average the Green's function over the $\mathbf{r}_i$.[5] The problem becomes spatially homogeneous. For the averaged one-particle Green's function $G_{\mathbf{p}}(\omega_n)$ we have the equation

$$[(G_p^{(0)})^{-1} - \Sigma_p] G_p = 1, \qquad (2)$$

$G_{\mathbf{p}}^{(0)}(\omega_n) = (i\omega_n - \epsilon_{\mathbf{p}} + \mu)^{-1}$. The self-energy part $\Sigma_{\mathbf{p}}$ is given by a sum of graphs, some of which are shown in Fig. 1. The solid line corresponds to $G^{(0)}(\omega_n)$, the dotted lines to $N|V_{\mathbf{q}}|^2$. By $|V_{\mathbf{q}}|^2$ we understand here (after

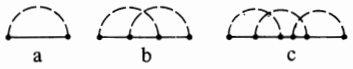

FIG. 1

summation over a definite subset of graphs[4,5]) the square of the exact amplitude for scattering on a single center. Depending on the number of vertices in the graph, we have for Σ

$$\Sigma = \Sigma_2 + \Sigma_4 + \dots \tag{3}$$

The first term in (3) corresponds to the approximation of binary scattering.

The integral over the two-particle Green's function in (1) can be written in the form

$$\int \frac{d^3p'}{(2\pi\hbar)^3} \mathbf{p}\mathbf{p}' K_{p'p}(\omega_n) = p^2 T \sum_k G_p(\omega_k) G_p(\omega_k + \omega_n)[1 + \Lambda_p(\omega_k, \omega_n)], \tag{4}$$

$$\mathbf{p}\Lambda_p(\omega_k, \omega_n) = \int \frac{d^3p_1}{(2\pi\hbar)^3} D_{p_1-p}\, \mathbf{p}_1 G_{p_1}(\omega_k) G_{p_1}(\omega_k + \omega_n)[1 + \Lambda_{p_1}(\omega_k, \omega_n)]. \tag{5}$$

It is clear that $\Lambda_p(\omega_k, \omega_n)$ depends weakly on p, so that[4]

$$\Lambda_p(\omega_k, \omega_n) \cong I_p(\omega_k, \omega_n)[1 - I_p(\omega_k, \omega_n)]^{-1},$$
$$I_p(\omega_k, \omega_n) = \int \frac{d^3p_1}{(2\pi\hbar)^3} D_{p_1-p} \frac{\mathbf{p}_1\mathbf{p}}{p^2} G_{p_1}(\omega_k) G_{p_1}(\omega_k + \omega_n). \tag{6}$$

The four-pole function D is given by a sum of graphs,

$$D = D_2 + D_4 + \dots \tag{7}$$

some of which are shown in Fig. 2.

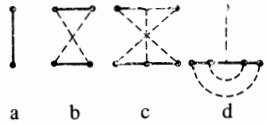

FIG. 2

The calculation of the electrical conductivity reduces to the calculation of the contribution of the graphs of the type shown in Figs. 1 and 2, their summation over ω_k, and analytic continuation.

CALCULATION OF THE ELECTRICAL CONDUCTIVITY

In first approximation in the scattering terms (Σ_2, D_2) we have expressions for $G_p(\omega_k)$ and $\Lambda_p(\omega_k, \omega_n)$ of the type obtained earlier[4] in the calculation of the residual resistance of a metal:

$$G_p^{(2)}(\omega_k) = \left[i\left(\omega_k + \frac{\hbar\nu}{2}\operatorname{sign}\omega_k\right) + \mu - \varepsilon_p \right]^{-1}, \tag{8}$$

$$\Lambda_p(\omega_k, \omega_n) = \begin{cases} \hbar\nu_1 \left[\omega_n \operatorname{sign}\left(\omega_k + \frac{\omega_n}{2}\right) + \hbar\nu_t \right]^{-1}, & |\omega_k| < \frac{\omega_n}{2} \\ 0, & |\omega_k| > \frac{\omega_n}{2}. \end{cases} \tag{9}$$

Here ν and ν_t are collision frequencies: $\nu = Nqv$, $\nu_t = Nq_tv$, $\nu_1 = \nu - \nu_t$, $v = p/m$; q and q_t are the electron-atom scattering cross section and the transport cross section, respectively.

In this approximation, the electrical conductivity must be given by the known formula for a Lorentz gas. Let us find it. To this end, we first cast the integral over the function $K_{pp'}(\omega_n)$ in the following form, using (9):

$$\int \frac{d^3p'}{(2\pi\hbar)^3} \mathbf{p}\mathbf{p}' K_{pp'}(\omega_n) = \frac{\nu_t(\omega_n + \hbar\nu)}{\nu(\omega_n + \hbar\nu_t)} p^2 T \sum_k G_p(\omega_k) G_p(\omega_k + \omega_n). \tag{10}$$

The summation in (10) goes over all ω_k. The analytic continuation of the function

$$F_p(i\omega_n) = T\sum_k G_p(\omega_k) G_p(\omega_k + \omega_n)$$

from the points $i\omega_n$, $n > 0$ to the real axis of ω is carried out by the method developed in [6]. We obtain

$$F_p(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \Big[\operatorname{cth}\frac{\alpha}{2T} \cdot \operatorname{Im} G_p^R(\alpha) G_p^R(\alpha + \omega) + \operatorname{cth}\frac{\alpha + \omega}{2T} \cdot \operatorname{Im} G_p^R(\alpha + \omega) \cdot G_p^R(\alpha) \Big].$$

Using $\hbar\nu$, $\omega \ll T$ in our calculations, we find

$$\int \frac{d^3p'}{(2\pi\hbar)^3} \mathbf{p}\mathbf{p}' K_{pp'}(\omega) = \frac{i\hbar\nu_t}{\omega + i\hbar\nu_t} p^2 \frac{\partial n_p}{\partial \varepsilon_p}. \tag{11}$$

As a result we obtain the Lorentz formula for $\operatorname{Re}\sigma(\omega)$:[8]

$$\operatorname{Re}\sigma(\omega) = \left(\frac{e\hbar}{m}\right)^2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\partial n_p}{\partial \varepsilon_p} \frac{p^2}{3} \frac{\hbar\nu_t}{\omega^2 + (\hbar\nu_t)^2}. \tag{12}$$

The system is isotropic and in (12) $\sigma(\omega) = \sigma_{xx}(\omega)$; n_p is the momentum distribution function of the electrons.

Let us now turn to the calculation of the corrections to the Lorentz formula in the next approximations in the density N. The electrical conductivity depends on the momentum distribution of the electrons and on their mobility. We recall that these two quantities are determined by different processes. If the concentration of the electrons is not too small, n_p is established through electron-electron collisions and has Maxwellian form (the corresponding criteria are given below). At the same time the mobility is governed by collisions with the neutrals and must be determined. We shall therefore seek only Im Σ_p, which is responsible for the "damping."

The quantity Σ_4 consists of two graphs, and Σ_6 of ten. It turns out that some of the graphs of Σ_6 diverge. We avoid these divergences by summing over such subsets which lead to the replacement of the lines $G_p^{(0)}$ by $G_p^{(2)}$. An example is given in Fig. 3, where the heavy line corresponds to $G_p^{(2)}$. This procedure has an obvious physical meaning.

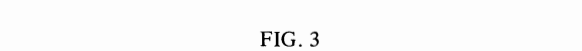

FIG. 3

In this way part of the graphs for Σ_6 reduce to Σ_4 with $G_p^{(0)}$ replaced by $G_p^{(2)}$. The remaining graphs, such as the one of Fig. 1c, are calculated independently. The values $\Sigma_p(\omega_k)$ are taken at the poles of the corresponding Green's functions. As a result, we obtain in third order in the collision frequency

$$\operatorname{Im}\Sigma_p = \operatorname{Im}\Sigma_{2p}\left\{1 + a_1\frac{\hbar\nu}{\varepsilon_p} - \left(a_2 + \frac{1}{128}\ln\frac{16\varepsilon_p}{\hbar\nu}\right)\left(\frac{\hbar\nu}{\varepsilon_p}\right)^2\right\}, \tag{13}$$

$$a_1 = \frac{1}{8\pi}\int_0^{\infty} \frac{a\,da}{1-a^2}[(|\sqrt{1-2a+a^2}|+2)\ln(|\sqrt{1-2a+a^2}|+2) - (|\sqrt{1-2a+a^2}|-2)\ln(|\sqrt{1-2a+a^2}|-2) - (3+a)\ln(3+a) - (a-1)\ln(a-1)] \cong \frac{6.3}{8\pi};$$

$$a_2 = \frac{1}{4}\int_0^{\sqrt{2}} \frac{dz}{z}\ln^2\frac{\sqrt{2}+z}{\sqrt{2}-z} \cong \frac{3.3}{4}.$$

Expression (13) has been obtained under the assumption of short-range electron-atom interactions. Therefore $|V_p|^2$ has been regarded as a function which depends weakly on p. The scattering was taken to be isotropic.

Let us calculate Re $I_p(\omega_k, \omega_n)$. We are not interested in the imaginary part of $I_p(\omega_k, \omega_n)$. The contribution of D_4 consists of three graphs. A contribution of second order in $\hbar\nu/\epsilon_p$ to Re $I_p(\omega_k, \omega_n)$ comes only from the graph of Fig. 1b. For $|\omega_k| > \omega_n/2$ it vanishes. For $|\omega_k| > \omega_n/2$ it amounts to

$$-\frac{\hbar\nu}{\omega_n+\hbar\nu}\left(\frac{\hbar\nu}{\varepsilon_p}\right)\frac{\pi}{12}.$$

The contribution of D_4 to $I_p(\omega_k, \omega_n)$ in the next order in $\hbar\nu/\epsilon_p$ is summed with the contribution from D_6.

The term D_6 consists of twenty graphs. Many of them do not give a contribution of the considered order in $\hbar\nu/\epsilon_p$. The terms of most importance are the ones of the type of Fig. 2, c and d, which are approximately given by products of the quantities D_4:

$$D_6(\varepsilon) \cong (N|V_p|^2)^{-1}[D_4(\varepsilon)]^2.$$

For the other contributions the region of integration over the angles is strongly limited, and the integrated quantities are small.

As a result we obtain in third order in $\hbar\nu$

$$\mathrm{Re}\, I_p(\omega_k, \omega_n) = -\frac{\hbar\nu}{\omega_n+\hbar\nu}\left(\frac{\hbar\nu}{\varepsilon_p}\right)\left[\frac{\pi}{12}-\left(\frac{\hbar\nu}{\varepsilon_p}\right)\left(a_4+a_3\ln\frac{16\varepsilon_p}{\hbar\nu}\right)\right]$$

$$|\omega_k| < \frac{\omega_n}{2},$$

$$a_3=\frac{\pi^2}{32}+\frac{1}{64},\quad a_4=\frac{\pi^2}{16}+\frac{1}{16}\int_0^{\sqrt{2}}\frac{1-z^2}{z}\ln^2\frac{\sqrt{2}+z}{\sqrt{2}-z}\,dz\cong\frac{\pi^2}{16}+\frac{0.74}{32}. \tag{14}$$

The summation over ω_k and the analytic continuation are carried out in the same way as in the derivation of (12). The result for the static conductivity is conveniently written in terms of an "effective" collision frequency $\hbar\nu_{eff}$:

$$\mathrm{Re}\,\sigma(0)=\left(\frac{e\hbar}{m}\right)^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{\partial n_p}{\partial\varepsilon_p}\frac{p^2}{3}\frac{1}{\hbar\nu_{ef}},$$

$$\hbar\nu_{ef}=\hbar\nu\left[1+\left(\frac{\pi}{12}+a_1\right)\frac{\hbar\nu}{\varepsilon_p}-\left(a_2+a_4-\frac{\pi a_1}{12}\right)\left(\frac{\hbar\nu}{\varepsilon_p}\right)^2-\left(a_3+\frac{1}{128}\right)\left(\frac{\hbar\nu}{\varepsilon_p}\right)^2\ln\frac{16\varepsilon_p}{\hbar\nu}\right]. \tag{15}$$

The frequency $\hbar\nu$ needed in calculations using (15), can be obtained, for example, from experiments on the scattering of electrons on isolated atoms (molecules).

DISCUSSION

The expression obtained above for the electrical conductivity coefficient of a weakly ionized plasma has the form of an expansion of the virial type. It establishes the limits of applicability of the Lorentz approximation and allows one to calculate corrections to it. Being an expansion of the electrical conductivity in the parameter

$$\hbar\nu/\varepsilon_p \sim \lambda qN,$$

formula (15) takes account of the interference in the simultaneous scattering of an electron on several centers. We have considered binary, three-, and four-particle interactions. Clearly, we have taken even somewhat more into account in the summation over several subsets of graphs, since the expansion contains a term with $N^3 \ln N$.

Let us discuss the limits of applicability of our result.

The plasma must be slightly imperfect, so that one can speak of free electrons, the electrical conductivity being determined by their mobility. The requirement of weak non-ideality is reflected in the inequalities

$$e^2T^{-1}n^{1/3} \ll 1, \quad \lambda qN \ll 1, \quad RN^{1/3} \ll 1. \tag{16}$$

The first inequality takes account of the Coulomb interaction, the second of the electron-atom interaction, and the third, of the atom-atom interaction (R is the radius of the outer electron shell). When the conditions (16) are fulfilled, the concentration of the electrons can be determined with the help of perturbation theory, and the electrical conductivity can be sought in the form of an expansion in the density. In the opposite case, one must use a different approach in calculating σ. Thus, for example, one must consider the fole of the conduction mechanism which is caused by the "jumping" of an electron from an atom to an ion.[9]

The following restrictions are connected with the concrete method of including the interaction in a slightly imperfect plasma.

The neglect of the electron-ion scattering is justified if

$$qN \gg \frac{\pi e^4}{T^2}\Lambda n, \tag{17}$$

where Λ is the Coulomb logarithm. At the same time, if

$$\frac{m}{M}qN \ll \frac{\pi e^4}{T^2}\Lambda n, \tag{18}$$

(m/M is the ratio of the masses of the electron and the atom) the Coulomb collisions between the electrons determine the momentum distribution of the electrons n_p. If the first inequality (16) is satisfied, this distribution is Maxwellian. Owing to the short range of the interaction between the electrons and the neutrals, the electron-electron interaction has no effect on the mobility of the electrons in a weakly ionized plasma.

In Fig. 4 we show, in terms of the coordinates n and

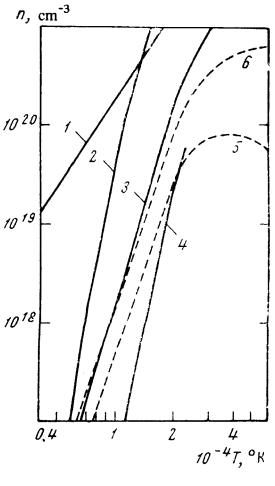

FIG. 4. Characteristics of a hydrogen plasma in equilibrium. Curve 1: the condition $e^2T^{-1}n^{1/3} = 1$, Curve 2: $RN^{1/3} = 1$, Curve 3: $qN = 1$, Curve 4: $qN = e^4T^{-2}\Lambda n$, Curve 5: the 1000 atm isobar, Curve 6: the 10 000 atm isobar.

T, the range of parameters of a hydrogen plasma in equilibrium over which expression (15) is applicable. For small T it comprises the conditions where the effects of non-ideality are important. For larger T and n the plasma becomes strongly ionized. In calculating its electrical conductivity one must delve deeper into the interaction than was done in [3]. Effects of the type considered above must be taken into account.

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119