PARAMETRIC EXCITATION OF LONGITUDINAL OSCILLATIONS IN A PLASMA BY A WEAK FREQUENCY ELECTRIC FIELD

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At external field microwave frequencies close to the electron Langmuir frequency, parametric instability of a plasma occurs at comparatively small values of the external field strength. Expressions are derived for the plasma oscillation increments and frequencies. Relations between the amplitudes, which determine the phase shift of the high frequency plasma waves relative to the phase of the external field, are also obtained. The threshold values of the microwave field strength are determined in a broad range of parameters. Values of the perturbation wave vectors characteristic of the beginning of instability are also determined.

It is known that in a plasma situated in a strong high-frequency (RF) field, under certain conditions, instability sets in with respect to growth of oscillations of the longitudinal electric field.[1-3] In this case, as shown by one of the authors[4,5] the threshold intensity of the external RF electric field, at which such an instability can arise, decreases when the frequency of the external field approaches the electron Langmuir frequency. Bearing in mind the anomalous character of the interaction between the external field and the plasma, connected with the development of instability and with the occurrence of the turbulent state of the plasma, it is of particular interest to consider the criteria for the development of instability in the case of very weak fields, when it is difficult to expect any anomalies whatever from the point of view of the usual ideas. In this connection, we shall study below the parametric properties of a plasma situated in a spatially-homogeneous electric field whose frequency is close to the electron Langmuir frequency. The amplitudes of the excited oscillations will be assumed to be small, and the wavelengths of such oscillations will be assumed not to exceed the wavelength of the external field and the characteristic dimensions of the plasma inhomogeneity, thus justifying the assumption that the external field and the plasma are homogeneous. At the same time, bearing in mind that the relatively long-wave oscillations grow when the external-field frequency approaches the electron Langmuir frequency, we shall assume the wavelength of this oscillation to be much higher than the electron Debye radius and the amplitude of the electron oscillations in the external RF field.

It turns out in this case that perturbations of zero frequency arise in the plasma, and also perturbations with the frequency ω_0 of the external field. For the minimum threshold value of the external field at which such oscillations become unstable we get the following relation, which depends on the frequency ν_{ei} of the electron-ion collisions:

$$\frac{E_0^2}{4\pi n_e T_e} = 4\frac{\nu_{ei}}{\omega_{Le}},$$

This condition is satisfied when $\omega_0 - \omega_p[1 + {}^3/_2 k^2 r_{De}^2] = \nu_{ei}/2$.

If the collision frequency is not too high, then the minimum threshold intensity of the external field for a strongly non-isothermal plasma is given by

$$\frac{E_0^2}{4\pi n_e T_e} = \sqrt{8\pi}\,\frac{\nu_{ei}\omega_{Li}}{\omega_{Le}^2}Z, \qquad Z = 1 + \frac{r_{De}^2 v_{Te}}{r_{Di}^2 v_{Ti}}\exp\left(-\frac{r_{De}^2}{2r_{Di}^2}\right)$$

at

$$\frac{\nu_{ei}}{\omega_{Le}} < \frac{5}{8}\sqrt{\frac{\pi}{2}}\left(\frac{m_e T_i}{m_i T_e}\right)^{1/2}\left(\ln\frac{\omega_{Le}}{\omega_{ei}}\right)^{-1/2}Z$$

and

$$\frac{E_0^2}{4\pi n_e T_e} = \frac{32}{5}\frac{r_{Di}^2\nu_{ei}\nu_{ii}}{r_{De}^2\omega_{Le}\omega_{Li}}\left(2\ln\frac{\omega_{Le}}{\nu_{ei}}\right)^{1/2}$$

at

$$\frac{\omega_{Li}}{\omega_{Le}}\left(2\ln\frac{\omega_{Le}}{\nu_{ei}}\right)^{-1/2} > \frac{\nu_{ei}}{\omega_{Le}} > \frac{5}{8}\sqrt{\frac{\pi}{2}}\left(\frac{m_e T_i}{m_i T_e}\right)^{1/2}\left(\ln\frac{\omega_{Le}}{\omega_{Li}}\right)^{-1/2}Z,$$

which are valid under conditions corresponding to the decay of a wave of frequency ω_0 and zero wave number into plasma and ion-acoustic waves, and in this case the growing oscillations are those having frequencies ω_s and $\omega_s - \omega_0$, where ω_s is the frequency of the ion sound.

In the opposite case of high collision frequencies, $\gamma_{ei} > \sqrt{2}\,\omega_{Li}\,[\ln(\omega_{Le}/\omega_{Li})]^{-1/2}$, the minimum threshold intensity of the external field is reached when $kr_{De} \approx [2\ln(\omega_{Le}/\nu_{ei})]^{-1/2}$, and is determined by the relations

$$\frac{E_0^2}{4\pi n_e T_e} = \frac{128}{15\sqrt{3}}\left|\frac{e}{e_i}\right|\frac{T_i}{T_e}\frac{\nu_{ii}\nu_{ei}^2}{\omega_{Le}\omega_{Li}^2}\ln\frac{\omega_{Le}}{\nu_{ei}},$$

$$\omega = \omega_s > \left[\frac{8}{15}\left|\frac{e}{e_i}\right|\frac{T_i}{T_e}\nu_{ei}\nu_{ii}\right]^{1/2};$$

$$\frac{E_0^2}{4\pi n_e T_e} = \frac{32}{3\sqrt{5}}\left[\left|\frac{e}{e_i}\right|\frac{T_i}{T_e}\frac{\nu_{ii}\nu_{ei}^3}{\omega_{Le}^2\omega_{Li}^2}\ln\frac{\omega_{Le}}{\nu_{ei}}\right]^{1/2}$$

$$\omega = \omega_s^{1/2}\left[\frac{8}{15}\left|\frac{e}{e_i}\right|\frac{T_i}{T_e}\nu_{ei}\nu_{ii}\right]^{1/4} > \omega_s.$$

In this case waves with frequencies ω, $\omega + \omega_0$, and $\omega - \omega_0$ are excited, and the amplitudes of the oscillations with frequencies $\omega \pm \omega_0$ are approximately equal.

1. In accordance with the usual premises of the theory of parametric resonance in a plasma, the spectrum of the longitudinal oscillations and the corresponding instability conditions for a fully ionized plasma situated in a homogeneous monochromatic RF field,

$$\mathbf{E}(t) = \mathbf{E}_0 \sin \omega_0 t, \tag{1.1}$$

are determined by the condition that a solution exist for the system of equations for the amplitudes of the electron and ion charge densities ρ_e and ρ_i in the following expansion of these densities in terms of the harmonics of the external-field frequency (a = i, e):

$$\rho_a(\mathbf{r}, t) = \exp\Big(-i\omega t + \gamma t + i\mathbf{kr} + i\frac{e_a \mathbf{E}_0 \mathbf{k}}{m_a \omega_0^2} \sin \omega_0 t\Big) \sum_{n=-\infty}^{\infty} e^{-in\omega_0 t} u_a^{(n)}. \tag{1.2}$$

Being interested in the weak-field case, when the velocity of the electron oscillations in the pump field is small compared with their thermal velocity, it suffices, for oscillations with wavelength exceeding the electron Debye radius, and with frequency ω and decrement γ that are small compared with the frequency of the pump field ω_0, which is close to the electron Langmuir frequency $\omega_{Le} = (4\pi e^2 n_e/m_e)^{1/2}$, to retain only to the zeroth and first-order amplitudes in the expansion (1.2). Accordingly, the expansion (1.2) for the electrons and ions can be written in the form

$$\rho_e(\mathbf{r}, t) = e^{i\mathbf{kr} - i\omega t + \gamma t}\{u_e^{(0)} - {}^1/_2 \mathbf{kr}_E [u_e^{(-1)} - u_e^{(+1)}] + [u_e^{(+1)} - {}^1/_2 \mathbf{kr}_E u_e^{(0)}] e^{-i\omega_0 t} + [u_e^{(-1)} + {}^1/_2 \mathbf{kr}_E u_e^{(0)}] e^{i\omega_0 t}\}, \tag{1.3}$$

$$\rho_i(\mathbf{r}, t) = e^{i\mathbf{kr} - i\omega t + \gamma t}\Big\{u_i^{(0)} - \frac{1}{2}\frac{e_i m_e}{e m_i}\mathbf{kr}_E [u_i^{(-1)} - u_i^{(+1)}] + \Big[u_i^{(+1)} - \frac{1}{2}\frac{e_i m_e}{e m_i}\mathbf{kr}_E u_i^{(0)}\Big] e^{-i\omega_0 t} + \Big[u_i^{(-1)} + \frac{1}{2}\frac{e_i m_e}{e m_i}\mathbf{kr}_E u_i^{(0)}\Big] e^{i\omega_0 t}\Big\}. \tag{1.4}$$

Here e, e_i, m_e, and m_i are the charges and masses of the electrons and ions, and $\mathbf{r}_E = e\mathbf{E}_0/m_e\omega_0^2$ is the amplitude of the electron oscillations in the RF electric field.

In the approximation of expansions (1.3) and (1.4), the dispersion equation of the longitudinal oscillations, accurate to terms quadratic in the RF field intensity E_0, can be written in the form[5]

$$\frac{\varepsilon(\omega + i\gamma, \mathbf{k})}{\delta\varepsilon_i(\omega + i\gamma, \mathbf{k})[1 + \delta\varepsilon_e(\omega + i\gamma, \mathbf{k})]} + \frac{(\mathbf{kr}_E)^2}{4} \times \Big[\frac{1}{\varepsilon(\omega + \omega_0 + i\gamma, \mathbf{k})} + \frac{1}{\varepsilon(\omega - \omega_0 + i\gamma, \mathbf{k})}\Big] = 0. \tag{1.5}$$

Here $\delta\epsilon_a(\omega, \mathbf{k})$ is the contribution of the particles of type a to the ordinary linear longitudinal dielectric constant

$$\varepsilon(\omega, \mathbf{k}) = 1 + \delta\varepsilon_e(\omega, \mathbf{k}) + \delta\varepsilon_i(\omega, \mathbf{k}). \tag{1.6}$$

As will be shown later, the frequencies of the excited oscillations turn out to be either much smaller or much larger than the increment (decrement) γ.

When $\gamma \ll \omega$, the following inequalities are satisfied for growing plasma oscillations (at a small amplitude E_0 of the RF field)

$$kv_{Ti} \ll |\omega + i\gamma| \ll kv_{Te}, \tag{1.7}$$

where $v_{Te} = (T_a/m_a)^{1/2}$ is the thermal velocity of the particle of type a. In addition, we shall assume that the wavelength of the excited oscillations is small compared with the electron mean free path:

$$kv_{Te} \gg \nu_{ei} = \frac{4}{3}\frac{\sqrt{2\pi}\, e^2 e_i^2 n_i L}{T_e^{3/2} m_e^{1/2}}, \tag{1.8}$$

and the frequency of the ion-ion collisions is also relatively small:

$$\frac{|\omega + i\gamma|^3}{k^2 v_{Ti}^2} \gg \nu_{ii} = \frac{4}{3}\frac{\sqrt{\pi}\, e_i^4 n_i L}{T_i^{3/2} m_i^{1/2}}. \tag{1.9}$$

Here L is the Coulomb logarithm.

In accordance with inequalities (1.7)–(1.9), and also bearing in mind that $|\omega \pm \omega_0| \gg kv_{Te}$, we can write the following expressions for the partial contributions to the dielectric constant:

$$\delta\varepsilon_e(\omega, \mathbf{k}) = \frac{1}{k^2 r_{De}^2}\Big(1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{kv_{Te}}\Big), \tag{1.10}$$

$$\delta\varepsilon_e(\omega \pm \omega_0, \mathbf{k}) = -\frac{\omega_{Le}^2}{(\omega \pm \omega_0)^2}\Big[1 + 3\frac{k^2 v_{Te}^2}{(\omega \pm \omega_0)^2} - i\sqrt{\frac{\pi}{2}}\frac{(\omega \pm \omega_0)^3}{k^3 v_{Te}^3} \times \exp\Big\{-\frac{(\omega \pm \omega_0)^2}{2k^2 v_{Te}^2}\Big\} - i\frac{\nu_{ei}}{\omega \pm \omega_0}\Big], \tag{1.11}$$

$$\delta\varepsilon_i(\omega, \mathbf{k}) = -\frac{\omega_{Li}^2}{\omega^2}\Big[1 - i\sqrt{\frac{\pi}{2}}\frac{\omega^3}{k^3 v_{Ti}^3} \times \exp\Big(-\frac{\omega^2}{2k^2 v_{Ti}^2}\Big) - i\frac{8}{5}\frac{k^2 v_{Ti}^2 \nu_{ii}}{\omega^3}\Big]. \tag{1.12}$$

Under conditions when γ turns out to be much larger than the frequency ω, it is necessary to consider excitations for which, besides (1.7), there is also satisfied the inequality

$$|\omega + i\gamma| \ll kv_{Ti}. \tag{1.13}$$

When (1.13) is satisfied, we can use for the partial permittivities the expression

$$\delta\varepsilon_a(\omega + i\gamma, \mathbf{k}) = 1/k^2 r_{Da}^2. \tag{1.14}$$

Simultaneously with the dispersion equation (1.5), we can write the following relations for the amplitudes $u_a^{(n)}$:

$$\frac{u_e^{(0)} + u_i^{(0)}}{u_i^{(0)}} = [1 + \delta\varepsilon_e(\omega + i\gamma, \mathbf{k})]^{-1} \sim k^2 r_{De}^2 \ll 1, \tag{1.15}$$

$$\frac{\omega_{Le}^2}{\omega_{Li}^2} u_i^{(\mp 1)} = u_e^{(\pm 1)} = \mp u_e^{(0)} \frac{\mathbf{kr}_E}{2\varepsilon(\omega + i\gamma \pm \omega_0, \mathbf{k})}. \tag{1.16}$$

Here $\omega + i\gamma$ is the solution of the dispersion equation (1.5). Relation (1.15) reflects the quasineutrality property of the low-frequency long-wave oscillations. Formulas (1.16) and (1.3) make it possible to determine the relative magnitude of the amplitudes corresponding to the harmonics $\omega \pm \omega_0$. Thus, for example, if $\epsilon(\omega + i\gamma - \omega_0, \mathbf{k}) \ll \epsilon(\omega + i\gamma + \omega_0, \mathbf{k})$, then the spectrum of the electron-density oscillations (and accordingly of the oscillations of the electric field in the plasma) contains principally the frequencies ω and $\omega - \omega_0$. To the contrary, if $\epsilon(\omega - \omega_0 + i\gamma) \approx \epsilon(\omega + \omega_0 + i\gamma)$, then the oscillations with frequencies $\omega - \omega_0$ and $\omega + \omega_0$ have approximately equal amplitudes.

It turns out that the dispersion equation (1.5) has both periodic ($\gamma \ll \omega$) and aperiodic ($\gamma \gg \omega$) solutions. Let us consider first the relatively simple case of aperiodic instability.

2. The hydrodynamic theory of parametric resonance in a plasma[1,2] has predicted the existence of aperiodic instability at $\omega_0 < \omega_{Le}$. Similarly, our dispersion equation (1.5), which takes into account a large group of kinetic effects, also leads to such an instability, provided the difference between the frequency of the external field and the frequency of the longitudinal plasma wave

is negative in this case

$$\Delta\omega_0 \equiv \omega_0 - (\omega_{Le}^2 + \omega_{Li}^2 + 3k^2 r_{De}^2 \omega_{Le}^2)^{1/2}. \qquad (2.1)$$

For the growth increment of the perturbations we have

$$\gamma = -\tilde{\gamma} + \left[-(\Delta\omega_0)^2 - \frac{1}{4}\omega_{Le}\Delta\omega_0 \frac{(\mathbf{kr}_E)^2}{k^2 r_{De}^2 + k^2 r_{Di}^2} \right]^{1/2} \quad \text{for} \quad \gamma < k v_{Ti}; \qquad (2.2)$$

$$\gamma = -\tilde{\gamma} + \left\{ \frac{1}{2}\left[((\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2)^2 - \omega_s^2 \omega_{Le} \Delta\omega_0 \frac{(\mathbf{kr}_E)^2}{k^2 r_{De}^2} \right]^{1/2} + \tilde{\gamma}^2 - \frac{1}{2}((\Delta\omega_0)^2 + \tilde{\gamma}^2 + \omega_s^2) \right\}^{1/2} \quad \text{for} \quad v_{Te} > \frac{\gamma}{k} > v_{Ti}. \qquad (2.3)$$

Here $\omega_s = \omega_{Li} k r_{De}$ is the ordinary frequency of the ion-acoustic oscillations, and

$$\tilde{\gamma} = \sqrt{\frac{\pi}{8}} \frac{\omega_{Le}}{k^3 r_{De}^3} \exp\left(-\frac{\omega_0^2}{2k^2 v_{Te}^2} \right) + \frac{1}{2}\nu_{ei}. \qquad (2.4)$$

Equating the increment (2.1) to zero, we obtain an equation for the limit of the aperiodic instability:

$$\frac{(\mathbf{kr}_E)^2_{\lim}}{k^2 r_{De}^2 + k^2 r_{Di}^2} = 4\frac{(\Delta\omega_0)^2 + \tilde{\gamma}^2}{\omega_{Le}|\Delta\omega_0|} \quad (\Delta\omega_0 < 0). \qquad (2.5)$$

This relation differs from the hydrodynamic result of Nishikawa[2] in that the thermal motion of the ion is consistently taken into account. We note that in a real plasma the threshold value of the wave vector may turn out to be the decisive plasma dimension or the characteristic inhomogeneity scale (l) of the electric pump field (1.1). It is then necessary to substitute in the right side of (2) $1/l$ in place of k, and this determines directly the threshold value of the pump-field intensity.

For wavelengths shorter than the characteristic dimension of the inhomogeneity, an analysis of formula (2.5) is somewhat more complicated. The threshold value of the field (1.1) and the value of the wave vector at which the instability becomes possible are determined by the frequency difference between the external field ω_0 and the plasma frequency $\omega_p = (\omega_{Le}^2 + \omega_{Li}^2)^{1/2}$.

Let us discuss certain consequences that follow from (2.1)–(2.5) for a transparent plasma ($\omega_0 > \omega_p$). We denote by k_{st} the value of the wave vector at which both terms in the right side of (2.4) are equal to each other. Then, at sufficiently long wavelengths ($k < k_{st}$) of the growing perturbations, and at sufficiently small frequency deviations,

$$\omega_0 - \omega_p < {}^3/_2 \omega_p k_{st}^2 r_{De}^2,$$

we obtain from (2.5) the following simple formulas for the threshold values of the wave vector and the pump field intensity (see [6]):

$$k_{\text{thr}}^2 r_{De}^2 = \frac{\nu_{ei} + 2(\omega_0 - \omega_p)}{3\omega_p}, \qquad \frac{r_{E,\,\text{thr}}^2}{r_{De}^2 + r_{Di}^2} = 4\frac{\nu_{ei}}{\omega_{Le}}. \qquad (2.6)$$

If the pump field is only slightly higher than threshold, when $\gamma \ll \gamma \ll \tilde{\gamma}$, we have for the oscillation amplitudes $u_e^{(1)} = i u_e^{(-1)}$, thus indicating a phase shift of $\pi/4$ for the excited RF oscillations of frequency ω_0, relative to the phase of the pump field. We present also an expression for the maximum increment, if this increment is described by formula (2.2), as well as the corresponding value of the wave vector

$$\gamma_{max} = \frac{1}{8}\omega_{Le}\frac{r_E^2 - r_{E,\,\text{thr}}^2}{r_{De}^2 + r_{Di}^2},$$

$$k_{max}^2 r_{De}^2 = \frac{2}{3}\frac{\omega_0 - \omega_p}{\omega_p} + \frac{r_E^2}{12(r_{De}^2 + r_{Di}^2)}. \qquad (2.7)$$

It is similarly easy to obtain an expression for the maximum increment in the case of formula (2.3). Thus, when

$$\omega_0 - \omega_p \ll \omega_p (\omega_{Li}/\omega_{Le})^2$$

we get

$$\gamma_{max} \approx \omega_{Li} r_E / \sqrt{6} r_{De}.$$

At shorter wavelengths ($k > k_{st}$) we can neglect the collisions. This means simultaneously that one should speak of large detunings, when

$$\omega_p \gg \omega_0 - \omega_p > {}^3/_2 \omega_p k_{st}^2 r_{De}^2.$$

For the threshold values of the pump field and of the wave vector we then obtain the following formulas:

$$\frac{r_{E,\,\text{thr}}^2}{r_{De}^2 + r_{Di}^2} = \frac{\pi}{12}\frac{1}{k_{\text{thr}}^{10} r_{De}^{10}} \exp\left(-\frac{\omega_0^2}{2k_{\text{thr}}^2 v_{Te}^2} \right), \qquad (2.8)$$

$$k_{\text{thr}}^2 r_{De}^2 = \frac{1}{2} - \left[\frac{1}{4} - \frac{2}{3}\frac{\omega_0 - \omega_p}{\omega_p} \right]^{1/2}. \qquad (2.9)$$

It follows therefore that the threshold pump-field intensity increases exponentially with increasing frequency deviation. Finally, for pump fields exceeding (2.8) we have $u_e^{(1)} \approx u_e^{(-1)}$, meaning that the phase of the RF oscillations is shifted by $\pi/2$ relative to the phase of the pump field.

In concluding this section, let us touch upon the case when the pump-field frequency ω_0 is smaller than the plasma frequency. If at the same time $\omega_p - \omega_0 < \nu_{ei}/2$, then obviously expressions (2.6) and (2.7) hold for the threshold and for the maximum increment. With further decrease of the frequency, the field threshold value (1.1), as follows from (2.5), increases in proportion to the square root of the frequency deviation:

$$E_{0,\,\text{thr}} = 4[\pi(n_e T_e + n_i T_i)(\omega_p - \omega_0)/\omega_p]^{1/2}.$$

3. Let us consider almost-periodic oscillations whose frequency ω greatly exceeds the increment γ. As will be shown here, such oscillations can grow only in the transparency region of the plasma, when the frequency ω_0 exceeds the plasma frequency ω_p. Assuming inequalities (1.7)–(1.9) to be satisfied and substituting in the dispersion equation (1.5) the expressions (1.10)–(1.12) for the partial permittivities, we obtain a system of two biquadratic equations for the frequency and for the increment. Solutions of this system of equations are expressions for the spectrum of two branches of periodic oscillations, one of which, as will be shown later, corresponds (in the limit of low intensity E_0) to the Langmuir RF oscillations:

$$\omega_+^2 = (\Delta\omega_0)^2 + \tilde{\gamma}^2 - {}^1/_2(1 - f_+)[(\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2], \qquad (3.1)$$

$$\gamma_+ = [\gamma_s(1 - f_+) - \tilde{\gamma}(1 + f_+)]/2f_+, \qquad (3.2)$$

whereas the other branch corresponds to the low-frequency ion-acoustic oscillations:

$$\omega_-^2 = \omega_s^2 + {}^1/_2(1 - f_-)[(\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2], \qquad (3.3)$$

$$\gamma_- = [\tilde{\gamma}(1 - f_-) - \gamma_s(1 + f_-)]/2f_-. \qquad (3.4)$$

The function $f_\pm$ in expressions (3.1)–(3.4) for the spectrum of the periodic oscillations can be written in the form

$$f_\pm = \frac{1}{\sqrt{2}}\left\{1 - \frac{(\alpha+\beta_\pm)x}{(1-x)^2} + \left|\left[1 - \frac{(\alpha+\beta_\pm)x}{(1-x)^2}\right]^2 + \frac{4\beta_\pm x}{(1-x)^2}\right|^{1/2}\right\}^{1/2} > 0, \quad (3.5)$$

where

$$x = \frac{\omega_s^2}{(\Delta\omega_0)^2 + \tilde{\gamma}^2}, \quad \alpha = \left(\frac{\mathbf{kr}_E}{kr_{De}}\right)^2 \frac{\omega_0\Delta\omega_0}{(\Delta\omega_0)^2+\tilde{\gamma}^2},$$

$$\beta_\pm = 4\left(\frac{\gamma_s - \tilde{\gamma}}{\omega_s}\right)^2\left[1 - \frac{1-x}{2}(1 \mp f_\pm)\right].$$

The low-frequency (γ_S) and the high-frequency ($\tilde{\gamma}$) damping decrements are then given by

$$\gamma_s = \sqrt{\frac{\pi}{8}}\frac{\omega_{Li}}{\omega_{Le}}\omega_s\left[1 + \frac{\omega^4 r_{De}^2 v_{Te}}{\omega_s^4 r_{Di}^2 v_{Ti}}\exp\left(-\frac{\omega^2}{2k^2v_{Ti}^2}\right)\right] + \frac{4}{5}\frac{k^2v_{Ti}^2\nu_{ii}}{\omega^2}, \quad (3.6)$$

$$\tilde{\gamma} = \sqrt{\frac{\pi}{8}}\frac{\omega_{Le}}{k^3r_{De}^3}\exp\left[-\frac{(\omega_0 - |\omega|)^2}{2k^2v_{Te}^2}\right] + \frac{1}{2}\nu_{ei}. \quad (3.7)$$

The increments (3.2) and (3.4) are positive, i.e., they correspond to growing oscillations only when $f_\pm < 1$, which calls for $\Delta\omega_0 > 0$. This leads to a limitation on the region of admissible wave vectors of the growing oscillations $k^2r_{De}^2 < \frac{2}{3}(\omega_0 - \omega_p)/\omega_p$. Further, $\gamma_\pm > 0$ only in that wavelength region where $\gamma_S > \tilde{\gamma}$. To the contrary, $\gamma_- > 0$ if $\tilde{\gamma} > \gamma_S$.

In the near-threshold region of instability, we can write relatively simple general asymptotic formulas for the limit of the instability region. Considering the branch ω_-, we assume that $\tilde{\gamma} \gg \gamma_S$. It is then obvious that $1 - f_- \ll 1$ near the threshold for the buildup of such oscillations. Accordingly, expanding (3.5) in powers of r_E, we obtain

$$\omega_-^2 = \omega_s^2 + A[(\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2]\left(\frac{\mathbf{kr}_E}{kr_{De}}\right)^2 \quad (3.8)$$

$$A = \frac{1}{4}\frac{\omega_s^2\omega_0\Delta\omega_0}{[(\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2]^2 + 4\omega_s^2\tilde{\gamma}^2},$$

$$\gamma_- = \left(\frac{\mathbf{kr}_E}{kr_{De}}\right)^2 A\tilde{\gamma} - \gamma_s. \quad (3.9)$$

Equating γ_- to zero, we obtain in place of these two equations the following equations for the limit of the instability region:

$$\left(\frac{\mathbf{kr}_E}{kr_{De}}\right)^2_{\lim} = \frac{\gamma_s}{\tilde{\gamma}}A^{-1}, \quad (3.10)$$

$$(\omega_-^2)_{\lim} = \omega_s^2 + \gamma_s\tilde{\gamma} + (\Delta\omega_0)^2\frac{\gamma_s}{\tilde{\gamma}}. \quad (3.11)$$

Equations (3.10) and (3.11) define in the (E_0, k) plane the limit of the instability region. The minimum (threshold) value $E_{0\,thr}$ will be realized at a wave vector value k_{thr} which we shall call the threshold value.

For the branch ω_+, we assume $\tilde{\gamma} \ll \gamma_S$, which calls for the inequality $1 - f_+ \ll 1$ in the near-threshold region. Accordingly, using the expansion of (3.5) in powers of r_E, we obtain

$$\omega_+^2 = (\Delta\omega_0)^2 + \tilde{\gamma}^2 - B[(\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2]\left(\frac{\mathbf{kr}_E}{kr_{De}}\right)^2, \quad (3.12)$$

$$\gamma_+ = \left(\frac{\mathbf{kr}_E}{kr_{De}}\right)^2 B\gamma_s - \tilde{\gamma}, \quad B = \frac{\frac{1}{4}\omega_s^2\omega_0\Delta\omega_0}{[(\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2]^2 + 4\gamma_s^2[(\Delta\omega_0)^2 + \tilde{\gamma}^2]}. \quad (3.13)$$

The limit of the instability region is then determined by the equations

$$\left(\frac{\mathbf{kr}_E}{kr_{De}}\right)^2_{\lim} = \frac{\tilde{\gamma}}{\gamma_s}B^{-1}, \quad (3.14)$$

$$(\omega_+^2)_{\lim} = (\Delta\omega_0)^2 + \tilde{\gamma}^2 + \omega_s^2\tilde{\gamma}/\gamma_s. \quad (3.15)$$

In the particular case $\Delta\omega_0 = \omega_S \gg \tilde{\gamma}$, formulas (3.10) and (3.14) coincide with each other; when the ion-ion collisions are neglected, they coincide also with the results of [7]. Formula (3.10) with $\tilde{\gamma} \gg \omega_S$ is similar to that obtained by Nishikawa[6] for a partly ionized plasma.

The main content of the exposition that follows is connected with a detailed analysis of the instability region, and also with the study of the near-threshold region. It should be noted, however, that formulas (3.1)–(3.5) make it possible to move quite far away from this region. For example, in a sufficiently strong field, when $\beta_+ < \alpha < 1$ and $\alpha x > (1-x)^2$, we get (3.1)–(3.2)

$$\omega_+^2 = \tfrac{1}{2}[(\Delta\omega_0)^2 + \tilde{\gamma}^2 + \omega_s^2], \quad (3.16)$$

$$\gamma_+ = -\frac{\gamma_s + \tilde{\gamma}}{2} + \frac{\omega_s}{2\sqrt{2}} \times \left\{\frac{(\mathbf{kr}_E/kr_{De})^2\omega_0\omega_s^2\Delta\omega_0 - [(\Delta\omega_0)^2 + \tilde{\gamma}^2 - \omega_s^2]^2}{\omega_s^2[(\Delta\omega_0)^2 + \tilde{\gamma}^2 + \omega_s^2]}\right\}^{1/2}. \quad (3.17)$$

Finally, with increasing field, the increment increases and the condition on which the derivation of formulas (3.1) and (3.5) is based no longer holds.

It is seen even from (3.17) that γ_- does not depend on $\tilde{\gamma}$ or γ_S at a sufficiently large value of $\Delta\omega_0$ and when the field greatly exceeds the threshold. This result can be obtained, obviously, from the dispersion equation (1.5) by neglecting completely the dissipative effects corresponding to the so-called hydrodynamic approximation. The general solution of (1.5) in such an approximation yields

$$\omega^2 = \tfrac{1}{4}[(\Delta\omega_0)^2 + \omega_s^2 + \omega_s\Delta\omega_0(4+\alpha_0)^{1/2}],$$

$$\gamma = \tfrac{1}{2}[\omega_s\Delta\omega_0(4+\alpha_0)^{1/2} - \omega_s^2 - (\Delta\omega_0)^2]^{1/2}, \quad (3.18)$$

where $\alpha_0 = (\mathbf{k}\cdot\mathbf{r}_E/kr_{De})^2(\omega_0/\Delta\omega_0)$. It follows from this, in particular, that at small α_0 the instability takes place near wave vectors satisfying the condition $\Delta\omega_0 = \omega_S$, which corresponds to the decay instability.[8,9] With increasing α_0, the picture changes. Thus, when $\alpha_0 > 16$ and $\Delta\omega_0 = \omega_S$ we get $\omega \sim \gamma \sim \omega_S\alpha_0^{1/4}$. Obviously, under such conditions it is meaningless already to speak of decay instability.

It follows in particular from (3.18) that

$$\gamma_{max} = \sqrt{\frac{3}{5}}\,\omega(k_{max}) = \frac{3^{1/6}}{4}[\omega_{Li}^2(\omega_0 - \omega_p)]^{1/3}\left(\frac{E_0^2}{4\pi n_eT_e}\right)^{1/3}$$

$$k_{max}^2 r_{De}^2 \approx \frac{2}{3}\frac{\omega_0 - \omega_p}{\omega_p},$$

when

$$\frac{1}{\sqrt{2}}\left(\frac{16}{3}\right)^2\frac{\omega_{Li}}{\omega_{Le}}\left(\frac{\omega_0-\omega_p}{\omega_p}\right)^{1/2} < \frac{E_0^2}{4\pi n_eT_e} < 24\left(\frac{\omega_0-\omega_p}{\omega_{Li}}\right)^2, \quad \frac{\omega_p}{\omega_0-\omega_p}$$

and

$$\gamma_{max} \approx \omega(k_{max}) = 2^{-5/4}\cdot 3^{-1/4}[\omega_{Li}(\omega_0-\omega_p)]^{1/2}\left(\frac{E_0^2}{4\pi n_eT_e}\right)^{1/4}$$

$$k_{max}^2 r_{De}^2 \approx \frac{1}{3}\frac{\omega_0-\omega_p}{\omega_p},$$

when

$$\frac{32}{3}\frac{\omega_{Li}^2}{\omega_{Le}^2}, \quad 24\left(\frac{\omega_0-\omega_p}{\omega_{Li}}\right)^2 < \frac{E_0^2}{4\pi n_eT_e} < \frac{\omega_p}{\omega_0-\omega_p}.$$

4. Let us consider in detail the parametric buildup of oscillations with spectrum ω_-. Assuming that the temperature of the electrons is not much higher than the temperature of the ions, $e_i^2 T_e m_e \ll e^2 T_i m_i$, we can immediately obtain from the condition $\tilde{\gamma} > \gamma_s$ an inequality whose satisfaction means that the plasma instability can be connected only with the buildup of this branch:

$$\nu_{ei} > \sqrt{\frac{\pi}{2}} k_{st} r_{De} \frac{\omega_{Li}^2}{\omega_{Le}} Y_-. \tag{4.1}$$

Here

$$Y_\pm = 1 + \frac{\omega_\pm^4}{\omega_s^4} \frac{r_{De}^2}{r_{Di}^2} \frac{v_{Te}}{v_{Ti}} \exp\left(-\frac{\omega_\pm^2}{2k^2 v_{Ti}^2}\right),$$

and

$$k_{st} r_{De} \sim [2\ln(\omega_{Le}/\nu_{ei})]^{-1/2}$$

is the wave vector at which the two terms in $\tilde{\gamma}$ [Eq. (3.7)], due to the Cerenkov effect and to electron-ion collisions, become equal. If the inequality (4.1) is not satisfied, then in the wavelength region

$$r_{De}^{-1} \sqrt{\frac{2}{\pi}} \frac{\nu_{ei}\omega_{Le}}{\omega_{Li}^2} Y_+^{-1} < k < k_1 \tag{4.2}$$

(where

$$k_1 \sim r_{De}^{-1}[2\ln(\omega_{Le}/\omega_{Li})]^{-1/2}$$

is the wave number at which decrement γ_s [Eq. (3.6)] becomes comparable with the contribution made to $\tilde{\gamma}$ by the Cerenkov effect) the instability can be connected only with the buildup of the branch ω_+, and outside this interval (where $\tilde{\gamma} > \gamma_s$), it can be connected only with the buildup of the branch ω_-. We have already pointed out a limitation on the oscillation wave vector, imposed by the deviation from resonance $\frac{3}{2}\omega_p k^2 r_{De}^2 < \omega_0 - \omega_p$. Therefore, discussing the instability for different wavelengths, we consider by the same token different values of the frequency deviation.

We begin our discussion with the case of sufficiently long waves, $k < k_{st}$, when the contribution to $\tilde{\gamma}$ by the Cerenkov effect can be neglected. In such a relatively broad region, we first consider the subregion of wave-vector values for which

$$\omega_0 - \omega_p(1 + \tfrac{3}{2}k^2 r_{De}^2) \ll \nu_{ei}. \tag{4.3}$$

Formulas (3.10) and (3.11) then take the form

$$\left(\frac{\mathbf{k}\mathbf{r}_E}{k r_{De}}\right)^2_{\lim} = \frac{1}{2}\frac{\gamma_s(\nu_{ei}^2 + 4\omega_s^2)^2}{\nu_{ei}\omega_s^2\omega_0\cdot\Delta\omega_0}, \qquad (\omega_-^2)_{\lim} = \omega_s^2 + \tfrac{1}{2}\nu_{ei}\gamma_s. \tag{4.4}$$

These formulas become particularly simple in the limit

$$8 r_{Di}^2 \nu_{ei}\nu_{ii} \gg 5 r_{De}^2 \omega_s^2,$$

when

$$(\omega_-^4)_{\lim} = \tfrac{2}{5}k^2 v_{Ti}^2 \nu_{ei}\nu_{ii}:$$

namely:

$$\left(\frac{\mathbf{k}\mathbf{r}_E}{k r_{De}}\right)^2_{\lim} = \sqrt{\frac{2}{5}} \frac{\nu_{ei}^{5/2}\nu_{ii}^{1/2}}{\omega_s\omega_0\cdot\Delta\omega_0}\frac{r_{Di}}{r_{De}} \times\left\{1 + \frac{1}{4}\sqrt{\frac{\pi}{5}}\frac{r_{Di}}{r_{De}}\frac{\nu_{ii}^{1/2}\nu_{ei}^{3/2}}{k^2 v_{Ti}^2}\exp\left[-\left(\frac{\nu_{ei}\nu_{ii}}{10k^2 v_{Ti}^2}\right)^{1/2}\right]\right\}, \tag{4.5}$$

and in the opposite limiting case

$$8 r_{Di}^2 \nu_{ei}\nu_{ii} \ll 5 r_{De}^2 \omega_s^2,$$

when $(\omega_-)_{\lim} = \omega_s$:

$$\left(\frac{\mathbf{k}\mathbf{r}_E}{k r_{De}}\right)^2_{\lim} = \frac{\nu_{ei}^2 + 4\omega_s^2}{\nu_{ei}\omega_s^2\omega_0\cdot\Delta\omega_0}\left\{\frac{2}{5}\frac{r_{Di}^2}{r_{De}^2}\nu_{ii} + \sqrt{\frac{\pi}{32}}\frac{\omega_{Li}\omega_s}{\omega_{Le}} Z\right\},$$
$$Z = 1 + \frac{r_{De}^2 v_{Te}}{r_{Di}^2 v_{Ti}}\exp\left(-\frac{r_{De}^2}{2r_{Di}^2}\right). \tag{4.6}$$

We emphasize that for ion-acoustic frequency to occur it is necessary, strictly speaking, to satisfy the non-isothermal condition $|e_i| T_e \gg |e| T_i$. Relations (4.5) and (4.6) make it easy to find the threshold values of the pump field, when the contribution made to γ_s by the Cerenkov effect on the ions can be neglected (this is possible, in any case, when $|e_i| T_e/|e| T_i > \ln(e_i^2 T_e^3 m_i / e^2 T_i^3 m_e)$). Thus, the minimum of $E_{0,\lim}$, defined by (4.5), occurs when $k_{thr} = r_{De}^{-1}[2(\omega_0 - \omega_p)/9\omega_p]^{1/2}$, and is given by the expression

$$\left(\frac{r_E}{r_{De}}\right)^2_{thr} = \frac{9}{2}\left[\frac{1}{5}\frac{r_{Di}^2\nu_{ii}\nu_{ei}^5}{r_{De}^2\omega_{Le}\omega_{Li}^2(\omega_0 - \omega_p)^3}\right]^{1/2},$$
$$\omega_0 - \omega_p < \nu_{ei}, \quad \frac{6}{5}\left(\frac{r_{Di}}{r_{De}}\right)^2 \frac{\nu_{ei}\nu_{ii}\omega_{Le}}{\omega_{Li}^2}. \tag{4.7}$$

Similarly, we get from (4.6)

$$\left(\frac{r_E}{r_{De}}\right)^2_{thr} = \frac{32}{5}\frac{r_{Di}^2\nu_{ei}\nu_{ii}}{r_{De}^2(\omega_0 - \omega_p)^2}\left[\frac{\omega_0 - \omega_p}{\omega_p} + \frac{3}{8}\frac{\nu_{ei}^2}{\omega_{Li}^2}\right],$$
$$\frac{6}{5}\left(\frac{r_{Di}}{r_{De}}\right)^2\frac{\nu_{ei}\nu_{ii}\omega_{Le}}{\omega_{Li}^2} < \omega_0 - \omega_p < \nu_{ei}. \tag{4.8}$$

We present an expression for the maximum increment in the near-threshold region, when expression (3.9) can be used, i.e., for fields at which $\gamma_{max} < \omega$, $\nu_{ei}/8$. For frequency deviations that lead to the threshold (4.8), the maximum increment is reached at $k_{max} = k_{thr}$ and equals

$$\gamma_{max} = \frac{1}{8}\frac{r_E^2 - r_{E,\,thr}^2}{r_{De}^2}\frac{(\omega_0 - \omega_p)^2}{\nu_{ei}}\left[\frac{\omega_0 - \omega_p}{\omega_p} + \frac{3}{8}\frac{\nu_{ei}^2}{\omega_{Li}^2}\right]^{-1}. \tag{4.9}$$

For frequency deviations corresponding to the threshold (4.7), we can use the same expression, but in fields greatly exceeding the threshold value, recognizing that

$$\omega_0 - \omega_p \ll 3\nu_{ei}^2\omega_{Le}/8\omega_{Li}^2 \text{ и } k_{max} \approx r_{De}^{-1}[(\omega_0 - \omega_p)/3\omega_p]^{1/2}.$$

With the aid of (1.7) we can easily determine the ratio of the amplitudes $u_e^{(\pm 1)}$ in the expansion (1.3). Bearing in mind that in the case under consideration we have $\Delta\omega_0 \ll \tilde{\gamma}$, we get

$$\frac{u_e^{(+1)}}{u_e^{(-1)}} = -\frac{\Delta\omega_0 - \omega_- - i(\gamma_- + \tilde{\gamma})}{\Delta\omega_0 + \omega_- + i(\gamma_- + \tilde{\gamma})} \approx 1. \tag{4.10}$$

This ratio means that at pump-field values exceeding the threshold the spectrum of the excited oscillations contains (besides the frequency ω_-) also harmonics $\omega_- \pm \omega_0$, with equal amplitudes.

Going over into the subregion of waves that are shorter than (4.3) (but assuming, as before, that $k < k_{st}$), we must consider larger frequency deviations:

$$\nu_{ei} < \omega_0 - \omega_p < \tfrac{3}{2}\omega_p k_{st}^2 r_{De}^2. \tag{4.11}$$

We turn first to the case when at threshold values of the fields the acoustic frequency is small compared with the decrement $\tilde{\gamma}$ and the Cerenkov effect on the ions can be neglected in γ_s. With the aid of (3.10) and (3.11) we obtain the following threshold characteristics for different values of the frequency deviation:

$$\left(\frac{r_E}{r_{De}}\right)^2_{\text{thr}} = \frac{16}{5}\left[\frac{3}{5}\frac{r_{Di}^2\nu_{ii}\nu_{ei}^3}{r_{De}^2\omega_{Le}\omega_{Li}^2(\omega_0-\omega_p)}\right]^{1/2},$$
$$\omega_-^4(k_{\text{thr}}) = {}^8/_{15}\omega_s^2(k_{\text{thr}})\nu_{ei}\nu_{ii}r_{Di}^2/r_{De}^2 \qquad (4.12)$$

when

$$\frac{\nu_{ei}}{\omega_{Le}} < \frac{\omega_0-\omega_p}{\omega_p} < \frac{6}{5}\frac{r_{Di}^2\nu_{ii}\nu_{ei}}{r_{De}^2\omega_{Li}^2}, \quad \frac{3}{2}k_{st}^2r_{De}^2$$

and

$$\left(\frac{r_E}{r_{De}}\right)^2_{\text{thr}} = \frac{32}{5\sqrt{3}}\frac{r_{Di}^2\nu_{ii}\nu_{ei}^2}{r_{De}^2\omega_{Li}^2(\omega_0-\omega_p)}, \quad \omega_-(k_{\text{thr}}) = \omega_s(k_{\text{thr}}) \qquad (4.13)$$

when

$$\frac{\nu_{ei}}{\omega_{Le}}, \ \frac{6}{5}\frac{r_{Di}^2\nu_{ii}\nu_{ei}}{r_{De}^2\omega_{Li}^2} < \frac{\omega_0-\omega_p}{\omega_p} < \frac{3}{8}\frac{\nu_{ei}^2}{\omega_{Li}^2}, \ \frac{3}{2}k_{st}^2r_{De}^2.$$

These two thresholds are reached for a wave number satisfying the condition $\Delta\omega_0 = \nu_{ei}/2\sqrt{3}$ at $\omega_0 - \omega_p \gg \nu_{ei}$. For fields leading to the increment $\gamma_{max} < \omega$, $\nu_{ei}/8$, we get in this case from (3.9)

$$\gamma_{max} = \frac{\sqrt{3}}{8}\frac{r_E^2 - r_{E,\text{thr}}^2}{r_{De}^2}\frac{(\omega_0-\omega_p)\omega_{Li}^2}{\nu_{ei}^2}, \qquad (4.14)$$

where $r_{E,\text{thr}}$ is determined by (4.12) or (4.13), depending on the frequency deviation, and $k_{max} \approx k_{\text{thr}}$. Since in this case $\omega_- \ll \tilde{\gamma}$, the harmonics with frequencies $\omega_- \pm \omega_0$ have, in accord with (4.2), equal amplitudes. We note that both the formulas (4.7)–(4.9) and their continuations (4.12)–(4.14) into the region of larger frequency deviations (when $\omega_0 - \omega_p < 3\nu_{ei}^2\omega_{Le}/8\omega_{Li}^2$) show that the threshold for the buildup of oscillations decreases and the maximum increment increases with increasing frequency deviation.

Let us consider now a case opposite to the one just considered, when the decrement $\tilde{\gamma} = \nu_{ei}/2$ is small compared with ω_s at the threshold value of the field. It is then seen directly from (3.10) that the threshold is reached under decay conditions, i.e., for $k_{\text{thr}} = k_0$ defined by the equation

$$\omega_s(k_0) = \Delta\omega_0(k_0) \equiv \omega_0 - \omega_p[1 + {}^3/_2k_0^2r_{De}^2] \qquad (4.15)$$

It is then obvious from (3.8) that $\omega_- = \omega_s$ and, depending on the magnitude of the frequency deviation that determines k_0, we obtain the following expression for the threshold:

$$\left(\frac{r_E}{r_{De}}\right)^2_{\text{thr}} = \frac{32}{5}\frac{r_{Di}^2\nu_{ii}\nu_{ei}}{r_{De}^2\omega_0\omega_s(k_0)},$$
$$\frac{\nu_{ei}}{2\omega_{Li}} < k_0r_{De} < \frac{8}{5}\sqrt{\frac{2}{\pi}}\frac{r_{Di}^2\nu_{ii}\omega_{Le}}{r_{De}^2\omega_{Li}^2}Z^{-1}, \qquad (4.16)$$

which is an extension of formula (4.8) into the region of large frequency deviations when $\omega_0 - \omega_p > 3\nu_{ei}^2\omega_{Le}/8\omega_{Li}^2$, and

$$\left(\frac{r_E}{r_{De}}\right)^2_{\text{thr}} = \sqrt{8\pi}\frac{\nu_{ei}\omega_{Li}}{\omega_{Le}^2}Z,$$
$$\frac{\nu_{ei}}{2\omega_{Li}}, \ \frac{8}{5}\sqrt{\frac{2}{\pi}}\frac{r_{Di}^2\nu_{ii}\omega_{Le}}{r_{De}^2\omega_{Li}^2Z} < k_0r_{De} < \sqrt{\frac{2}{\pi}}\frac{\nu_{ei}\omega_{Le}}{\omega_{Li}^2Z}, \ k_{st}r_{De}. \qquad (4.17)$$

This threshold, as will be shown subsequently, is the smallest for the buildup of periodic oscillations, but the region of its applicability exists only at not too high collision frequencies.

The maximum increment in the near-threshold region (4.16), (4.17) is reached also under the decay conditions (4.15) and equals

$$\gamma_{max} = \frac{1}{8}\frac{r_E^2 - r_{E,\text{thr}}^2}{r_{De}^2}\frac{\omega_0\omega_s(k_0)}{\nu_{ei}} < \frac{1}{8}\nu_{ei}. \qquad (4.18)$$

On the other hand, if the pump field intensity is still higher, then we get from (3.17)

$$\gamma_{max} = \frac{1}{4}\frac{r_E}{r_{De}}[\omega_0\omega_s(k_0)]^{1/2}, \quad \tilde{\gamma} < \gamma_{max} < \omega_s(k_0). \qquad (4.19)$$

From (1.16) it follows that under decay conditions $\Delta\omega_0 = \omega_s = \omega_-$ the following relations hold for the amplitudes:

$$\left|\frac{u_e^{(+1)}}{u_e^{(-1)}}\right|^2 = \frac{[\gamma_- + \tilde{\gamma}(k_0)]^2}{4\omega_s^2(k_0)} \ll 1, \qquad (4.20)$$

$$k_0^2r_{De}^2\left|\frac{u_e^{(0)}}{u_e^{(-1)}}\right|^2 = \frac{\tilde{\gamma}(k_0)}{\gamma_- + \gamma_s(k_0)}\frac{\omega_s(k_0)}{\omega_{Le}}, \quad \frac{r_E^2}{r_{De}^2} < 4\frac{\tilde{\gamma}^2(k_0)}{\omega_{Le}\omega_s(k_0)}, \qquad (4.21)$$

$$k_0^2r_{De}^2\left|\frac{u_e^{(0)}}{u_e^{(-1)}}\right|^2 = \frac{\omega_s(k_0)}{\omega_{Le}}, \quad 16\frac{\tilde{\gamma}^2(k_0)}{\omega_{Le}\omega_s(k_0)} < \frac{r_E^2}{r_{De}^2} < 16\frac{\omega_s(k_0)}{\omega_{Le}}. \qquad (4.22)$$

Let us discuss these relations, using the following expression for the energy of the excited oscillations:

$$W_{\mathbf{k}} = \frac{1}{4\pi}|\mathbf{E}_{\mathbf{k}}|^2\frac{\partial}{\partial\omega}[\omega\,\mathrm{Re}\,\varepsilon(\omega,\mathbf{k})].$$

Here $\epsilon(\omega, \mathbf{k})$ is the linear longitudinal dielectric constant, and the amplitudes of the excited oscillations $\mathbf{E_k}$ are determined by Maxwell's equations in terms of the electron and ion charge densities.

$$i\mathbf{k}\mathbf{E}_{\mathbf{k}}^{(0)} = 4\pi\left\{u_e^{(0)} + u_i^{(0)} + \frac{\mathbf{k}\mathbf{r}_E}{2}[u_e^{(+1)} - u_e^{(-1)}]\right\} = \frac{4\pi u_e^{(0)}}{\delta\varepsilon_i(\omega + i\gamma)}$$
$$i\mathbf{k}\mathbf{E}_{\mathbf{k}}^{(\pm1)} \cong 4\pi u_e^{(\pm1)}.$$

It is easy to see that the left side of (4.20) is equal to the energy ratio $W_k^{(1)}/W_k^{(-1)}$ of the oscillations with frequencies $\omega_- + \omega_0$ and $\omega_- - \omega_0$, and expressions (4.21) and (4.22) give the ratios $W_k^{(0)}/W_k^{(-1)}$ of the energies of the oscillations with frequencies ω_- and $\omega_- - \omega_0$. Owing to the smallness of the frequency ω_- compared with ω_0, the inequality (4.20) denotes that the number of excited quanta with frequency $\omega_- + \omega_0$ is small in comparison with those for the frequency $\omega_- - \omega_0$. At the same time, relation (4.22) shows that under decay conditions, when the dissipative processes are negligible, the numbers of the excited quanta with frequencies ω_- and $\omega_- - \omega_0$ are equal.

Let us consider, finally, the region of shorter waves, $k > k_{st}$, and of the corresponding frequency deviations $\omega_0 - \omega_p > {}^3/_2\omega_{Le}k_{st}^2r_{De}^2$. Depending on the collision frequency, we can encounter here three cases.

If the collision frequency is sufficiently high, $\nu_{ei} > 2\omega_{Li}k_{st}r_{De}$, then the decrement $\tilde{\gamma}$ always exceeds ω_s. It is then easily seen from (3.10) that the threshold is reached at a wave number $k_{\text{thr}} \sim k_\gamma$, defined by the equation $\Delta\omega_0(k_\gamma) = \tilde{\gamma}(k_\gamma)$, and we can neglect in $\tilde{\gamma}$ the contribution due to the collisions. For in the threshold value of the pump field we have the following expression:

$$\left(\frac{r_E}{r_{De}}\right)^2_{\text{thr}} \approx 16\frac{\gamma_s(k_\gamma)}{\omega_0}\left[\frac{\Delta\omega_0(k_\gamma)}{\omega_s(k_\gamma)}\right]^2. \qquad (4.23)$$

From (3.9) we find that the maxima of the increment also takes place at a wave number $k_{max} \sim k_\gamma$, and equals

$$\gamma_{max} \approx \frac{1}{16} \frac{r_E^2 - r_{E,\,thr}^2}{r_{De}^2} \omega_0 \left[\frac{\omega_s(k_\gamma)}{\Delta\omega_0(k_\gamma)} \right]^2. \quad (4.24)$$

Relation (4.10) shows in this case (since $\omega_- \ll \tilde{\gamma}$) that the harmonics with frequencies $\omega_- \pm \omega_0$ have equal amplitudes.

If the collision frequency is smaller than the one just considered ($\nu_{ei} < 2\omega_{Li} k_{st} r_{De}$) but satisfies the inequality (4.1), then the acoustic frequency exceeds $\tilde{\gamma}$ in the wavelength region $k_{st} < k < k_2$ (k_2—wave number at which the acoustic frequency becomes comparable with the contribution made to $\tilde{\gamma}$ by the Cerenkov effect). In this case $\omega_- = \omega_s$, and the threshold and maximum of the increment are reached apparently under the decay conditions (4.15) and are determined by the expressions

$$\left(\frac{r_E}{r_{De}}\right)^2_{thr} = 16 \cdot \frac{\gamma_s(k_0)\tilde{\gamma}(k_0)}{\omega_0 \cdot \omega_s(k_0)},$$

$$\frac{3}{2} k_{st}^2 r_{De}^2 < \frac{\omega_0 - \omega_p}{\omega_p} < \frac{3}{2} k_2^2 r_{De}^2; \quad (4.25)$$

$$\gamma_{max} = \frac{1}{16} \frac{r_E^2 - r_{E,\,thr}^2}{r_{De}^2} \frac{\omega_0 \cdot \omega_s(k_0)}{\tilde{\gamma}(k_0)} < \frac{1}{4} \tilde{\gamma}(k_0). \quad (4.26)$$

At large pump-field intensities we get Eq. (4.19) for the maximum of increment. The relations for the amplitudes then take the form (4.20)–(4.22). On the other hand, in the wavelength region $k > k_2$, i.e., for $\omega_0 - \omega_p > {}^3\!/_2 \omega_p k_2^2 r_{De}^2$, we have $\tilde{\gamma} > \omega_s$ and consequently formulas (4.23) and (4.24) hold.

If finally the collision frequency is so small that it satisfies an inequality opposite to (4.21), then only oscillations with the spectrum ω_+ can build up in the wavelength region (4.2). In the wave-number region $k_1 < k < k_2$, i.e., in the case

$${}^3\!/_2 \omega_p k_1^2 r_{De}^2 < \omega_0 - \omega_p < {}^3\!/_2 \omega_p k_2^2 r_{De}^2,$$

we can use the expressions (4.25), (4.26), and (4.27), and in the case $\omega_0 - \omega_p > {}^3\!/_2 \omega_p k_2^2 r_{De}^2$ formulas (4.23) and (4.24) hold. In all the indicated expressions we can neglect the contribution made by the collisions to γ_s. We note, however, that in this case the expressions (4.23) and (4.25) determine the minima of the limiting value of the pump field only in the region $k > k_1$. In the wavelength region $k < k_1$ there is a minimum of the limiting value for the buildup of the branch ω_+ (see below).

5. It was shown in Sec. 3 that the buildup of the branch ω_+ is possible only at wave numbers from which $\gamma_s > \tilde{\gamma}$. Such a region exists, as can be readily seen from (3.6) and (3.7), at low electron-ion collision frequencies, when an inequality opposite to (4.1) is satisfied, and in accordance with the requirement $\Delta\omega_0 > 0$, at sufficiently large frequency deviations, i.e.,

$$\omega_0 - \omega_p > (3/\pi)\,\omega_p (\nu_{ei}\omega_{Le} / \omega_{Li}^2 Y_+)^2.$$

We then obtain from (3.13) for the threshold intensity E_0 the expression (4.17), which does not depend on the frequency deviation, with $k_{thr} \approx k_0$, if

$$\sqrt{\frac{2}{\pi}} \frac{\nu_{ei}}{\omega_{Li}} Y_+^{-1}, \; \frac{3}{\pi}\left(\frac{\nu_{ei}\omega_{Le}}{\omega_{Li}^2 Y_+}\right)^2 < \frac{\omega_0 - \omega_p}{\omega_p} < \frac{3}{2} k_{st}^2 r_{De}^2 \quad (5.1)$$

When E_0 exceeds (4.17), but the increment is smaller than $\gamma_s(k_0)/4$, the maximum of γ_+ is reached under the decay conditions $\Delta\omega_0 = \omega_s$, with $k_{max} = k_0$, and

$$\gamma_{max} = \frac{1}{16} \frac{r_E^2 - r_{E,\,thr}^2}{r_{De}^2} \frac{\omega_{Le}\omega_s(k_0)}{\gamma_s(k_0)}, \quad \frac{r_E^2}{r_{De}^2} < 4 \frac{\gamma_s^2(k_0)}{\omega_{Le}\omega_s(k_0)}. \quad (5.2)$$

For the frequency deviations (5.1) we can neglect here the contributions to the decrement $\gamma_s(k_0)$ from the ion-ion collisions and to the decrement $\tilde{\gamma}(k_0)$ from the Cerenkov effect on the electrons. If the field E_0 is so strong that $\omega_s(k_0) > \gamma_{max} > \gamma_s$, then we get from (3.18) expression (4.19) for the increment γ_{max}. Both in this case and under the condition (5.2), the frequency ω_+ differs little from ω_s, as can be seen from (3.12). Therefore, as follows from (4.16), the amplitude ratio $|u_e^{(+1)}/u_e^{(-1)}|$ is approximately equal to

$$(\gamma_+ + \tilde{\gamma}(k_0))\,/\,2\omega_s(k_0) \ll 1,$$

and for the amplitudes $u_e^{(0)}$ and $u_e^{(-1)}$ we have

$$(k_0 r_{De})^2 \left|\frac{u_e^{(0)}}{u_e^{(-1)}}\right|^2 = \frac{\gamma_+ + \tilde{\gamma}(k_0)}{\gamma_s(k_0)} \frac{\omega_s(k_0)}{\omega_{Le}}, \quad \frac{r_E^2}{r_{De}^2} < 4 \frac{\gamma_s^2(k_0)}{\omega_{Le}\omega_s(k_0)},$$

$$(k_0 r_{De})^2 \left|\frac{u_e^{(0)}}{u_e^{(-1)}}\right|^2 = \frac{\omega_s(k_0)}{\omega_{Le}}, \quad 16 \frac{\gamma_s^2(k_0)}{\omega_{Le}\omega_s(k_0)} < \frac{r_E^2}{r_{De}^2} < 16 \frac{\omega_s(k_0)}{\omega_{Le}},$$

i.e., relations analogous to (4.21) and (4.22). We note that (5.2) differs from the corresponding expression (4.18) for γ_{max} in that γ_s is replaced by $\tilde{\gamma}$, and the expressions for the increments coincide in the region of E_0 where $\gamma > \max\{\gamma_s(k_0), \tilde{\gamma}(k_0)\}$. Thus, if the intensity E_0 does not exceed greatly the threshold value, so that the increment is smaller than the larger of the decrements γ_s and γ, then γ is proportional to E_0^2 and to Γ^-, where $\Gamma = \max\{\gamma_s(k_0), \tilde{\gamma}(k_0)\}$. On the other hand, if $\Gamma < \gamma < \omega_s(k_0)$, then the growth increment of the periodic oscillations under decay conditions is proportional to E_0 and does not depend on the decrements. Such a dependence on E_0 and Γ was obtained in [8,9] for the increment of the decay instability of the plasma wave into a plasma wave and an ion-acoustic wave. It should be noted, however, that the theory developed in our paper is not applicable to the description of the decay instability of the plasma wave, for in this case the wave numbers of the three interacting waves are approximately equal, and the condition for the homogeneity of the "external field" is not satisfied.

If the frequency deviation is larger than (5.2), so that

$$\frac{3}{2} k_{st}^2 r_{De}^2 < \frac{\omega_0 - \omega_p}{\omega_p} < \frac{3}{2} k_1^2 r_{De}^2, \quad (5.3)$$

then, as can be seen from (3.14), the limit of the instability of the branch ω_+ has two minima, one of which—the decay minimum—coincides with that described above, except that in this case the contribution to the decrement $\tilde{\gamma}$ from the Cerenkov effect on the electrons is appreciable. The other minimum is connected with the exponential decrease of the limiting value of E_0, which is proportional to $\tilde{\gamma}$, at small wave numbers, and is determined by a relation that holds under non-decay conditions:

$$\left(\frac{r_E}{r_{De}}\right)^2_{min} \approx 4 \frac{\nu_{ei}(\omega_0 - \omega_p)^3}{\omega_{Le}\omega_s^2(k_{min})\gamma_s(k_{min})} \quad k_{min} \sim k_{st}. \quad (5.4)$$

The threshold for the buildup of the branch ω_+ is determined by the smaller of the values (5.4) or (4.25). If E_0

exceeds the value of (5.4) but $\gamma_{max} < \gamma_s/4$, then we obtain from (3.13) for the maximum increment

$$\gamma_{max} \approx -\nu_{ei} + \frac{1}{4}\frac{r_E^2}{r_{De}^2}\frac{\omega_{Le}\omega_s^2(k_{max})\gamma_s(k_{max})}{(\omega_0-\omega_p)^3},$$

$$\frac{r_E^2}{r_{De}^2} < \frac{4(\omega_0-\omega_p)^3}{\omega_s^2(k_{max})\,\omega_{Li}}.$$

The wave number k_{max} is determined here by (3.14) at a specified value of E_0 and coincides near the threshold for the buildup of such oscillations with k_{min} (5.4). At this value of E_0, as can be seen from (3.12), we have $\omega_+(k_{max}) \approx \Delta\omega_0(k_{max})$, and with the aid of relation (1.16) for the amplitude we get

$$|u_e^{(+1)}/u_e^{(-1)}| \approx (\gamma_+ + \tilde{\gamma})/2(\omega_0-\omega_p) \ll 1,$$

$$k_{max}^2 r_{De}^2 \left|\frac{u_e^{(0)}}{u_e^{(-1)}}\right|^2 \approx 4\,\frac{\gamma_s(k_{max})\,\omega_s^2(k_{max})\,(\gamma_+ + \tilde{\gamma})}{\omega_{Le}(\omega_0-\omega_p)^3} \ll 1.$$

On the other hand, if the detuning exceeds (5.3), the limit of the periodic instability has two minima, one of which is reached in the region of instability of the branch ω_- at large wave numbers (see (4.25) and (4.23) above), and the other coincides with (5.4). The threshold intensity E_0 is then determined by the smallest of the values (5.4), (4.25), or (4.23).

CONCLUSIONS

As shown above, in a plasma situated in a relatively weak RF electric field, an instability is produced against the buildup of oscillations with frequencies equal to zero and ω_0, for which the minimal threshold intensity of the external field (2.6) is determined only by the frequency of the electron-ion collisions. We note that the oscillations with frequency ω_0 can be distinguished from the external field by the relative phase shift near the threshold, which equals $\pi/4$.

An instability connected with the growth of almost periodic oscillations is possible when $\omega_0 > \omega_p$. Here, as shown above, depending on the collision frequency and on the frequency deviation, two possibilities arise. First, under the decay conditions $\Delta\omega_0 = \omega_s$, the oscillations with frequencies ω and $\omega - \omega_0$ increase, with $\omega \approx \omega_0$. Second, a situation is possible in which the oscillations with frequencies $\omega + \omega_0$ and $\omega - \omega_0$ are excited, and the amplitudes of the last two are approximately equal, while the frequency ω can greatly exceed the ion-acoustic frequency even at the threshold.

The here-investigated oscillation buildup in a plasma, can lead to the appearance of a turbulent state, and consequently to a nonlinear absorption (and reflection) of the external field. Such phenomena can be observed at relatively small and easily attainable external microwave field intensities. An experimental investigation of the reflection from the plasma of electromagnetic waves with frequency $\omega_0 \sim \omega_p \approx 2\times10^{10}$ sec^{-1} was carried out in [10]. In a plasma with parameters $n_e \approx 10^{11}$ cm^{-3}, $T_e \approx 4$ eV, and $T_i \approx 0.1$–0.2 eV, a sharp decrease of the reflection coefficient was observed at $E_0 \sim (0.1$–$0.2)(4\pi n_e T_e)$.[1] This value is larger by approximately one order of magnitude than the threshold electric field intensity given by (4.17), and consequently does not contradict the theory of plasma instability developed in this article. It is too early, however, to speak of an actual comparison of theory with experiment.

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