THE INFLUENCE OF A STRONG MAGNETIC FIELD ON SMALL SIZE SUPERCONDUCTORS WHEN DIFFUSE REFLECTION TAKES PLACE AT THE WALLS

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We find the density of states and the magnetic field dependence of the ordering parameter Δ for thin films in the regions $eHld \gg 1$ and $eHld \ll 1$ (d is the thickness of the film and l the electron mean free path). We find the magnetic moment and the parameter Δ in any field for a sphere with diffuse reflection from the walls.

THE properties of small size superconductors depend in an essential way upon the mean free path. In the "very dirty" limit $l \ll d$ (d is the characteristic size of the superconductor) the magnetic field dependence of the characteristics of the superconductor are for an appropriate choice of gauge for the vector potential determined by the expression $\langle(e\mathbf{A})^2\rangle\tau_{tr}$.[1] However, a much more complicated situation occurs in the region $l \gg d$. Thompson[2] considered thin films in a magnetic field satisfying the condition $eHld \gg 1$ using the Born approximation to take scattering by impurities into account. He found an equation for the ordering parameter Δ and showed that there is a gap in the excitation spectrum. We shall show in the following that the conclusion about the gap is valid only in zeroth approximation in the small parameter d/ξ_0. Taking terms of order d/ξ_0 and d/l into account in the Green function leads to the fact that the density of states in the field range $eHld \gg 1$ is always finite and non-zero. Taking the impurity scattering amplitude exactly into account leads for $l \gtrsim \xi_0$ to the appearance in the density of states to an additional peak which is connected with the occurrence of "bound pair states" at impurities in a strong magnetic field. A similar phenomenon occurs in superconductors with paramagnetic impurities.[3]

We consider also the case of a pure superconducting sphere in a magnetic field. We find expressions for Δ and for the magnetic moment in an arbitrary field. The expression for the critical field is not the same as the one found in [4]. This is connected with the fact that the law for the angular distribution of trajectories was not taken into account in the evaluation of averages in [4].

1. THIN FILM IN A STRONG MAGNETIC FIELD ($eH\xi_0 d \gg 1$, $eHld \gg 1$)

The set of equations for the Green function

$$G_{\mathbf{p}}(\mathbf{r}) = \frac{i}{\pi}\int G_{\mathbf{p}}(\mathbf{r}, \xi)\, d\xi$$

has for any form of impurity scattering the form[5]

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right) G_{\mathbf{p}}(\mathbf{r}) + G_{\mathbf{p}}(\mathbf{r})\hat{\omega} - \hat{\omega} G_{\mathbf{p}}(\mathbf{r}) = 0,$$
$$\hat{\omega} = \omega\tau_z - ie(\mathbf{vA})\tau_z - i\hat{\Delta} + in\Sigma_{\mathbf{pp}}(\mathbf{r}), \tag{1}$$
$$\Sigma_{\mathbf{pp}'}(\mathbf{r}) = \chi_{\mathbf{pp}'} - \frac{i\vartheta}{4}\int \chi_{\mathbf{pp}_1} G_{\mathbf{p}_1}(\mathbf{r})\Sigma_{\mathbf{p}_1\mathbf{p}'}(\mathbf{r})\, d\Omega_{\mathbf{p}_1},$$
$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix}, \quad G_{\mathbf{p}}^2(\mathbf{r}) = 1, \quad \mathrm{Tr}\, G_{\mathbf{p}}(\mathbf{r}) = 0,$$

where n is the impurity concentration, $\vartheta = mP_0/2\pi^2$ the density of states at the Fermi surface, v the electron velocity on the Fermi surface, while $\chi_{\mathbf{pp}'}$ is connected with the scattering amplitude through the relation

$$f_{\mathbf{pp}'} = \chi_{\mathbf{pp}'} - \frac{i\vartheta}{4}\int \chi_{\mathbf{pp}_1} f_{\mathbf{p}_1\mathbf{p}'}\, d\Omega_{\mathbf{p}_1}.$$

We shall henceforth assume for the sake of simplicity that the impurity scattering is isotropic. In that case the total scattering cross-section σ can be expressed in terms of χ through the formula

$$\sigma = m^2\chi^2/\pi[1 + (\pi\vartheta\chi)^2].$$

We choose our coordinate system in such a way that the y axis is at right angles to the film while at the boundaries of the film $y = \pm d/2$, while the x axis is along the magnetic field H. The vector potential **A** can then be chosen such that it is parallel to the z axis and

$$A_z = Hy. \tag{2}$$

In this gauge for the vector potential the order parameter Δ is real and depends on y only. We look for the Green function $G_{\mathbf{p}}(\mathbf{r})$ in the form

$$G_{\mathbf{p}}(\mathbf{r}) = f_1\tau_z + \frac{1}{\sqrt{2}} f_2(\tau_y + i\tau_x) + \frac{1}{\sqrt{2}} f_3(\tau_y - i\tau_x). \tag{3}$$

The functions f_i satisfy the condition

$$f_i(\mathbf{v}, y) \equiv f_i(\theta, \varphi, y) = f_i(\pi - \theta, \pi + \varphi, -y), \tag{4}$$

where

$$\mathbf{v} = v\{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\}.$$

We shall therefore restrict our considerations in the following to only the angular interval $0 < \varphi < \pi$.

Substituting Eq. (3) for the Green function $G_{\mathbf{p}}(\mathbf{r})$ into the Eqs. (1) we get a set of equations for the functions f_i which we can write in the form of a formal solution:

$$(y_1) = B_1 + \frac{1}{vx}\int_0^{y_1}\Big\{\sqrt{2}\Delta(y)(f_2(y) - f_3(y)) + a(y)\Big[f_2(y)\Big(\int f_3(\mathbf{v}_1, y)\, d\Omega_{\mathbf{v}_1}\Big)$$
$$- f_3(y)\Big(\int f_2(\mathbf{v}_1, y)\, d\Omega_{\mathbf{v}_1}\Big)\Big]\Big\}\, dy,$$
$$f_2(y_1) = e^{-W(y_1)}\Big\{B_2 + \frac{1}{vx}\int_{-d/2}^{y_1} e^{W(y)}\Big[\sqrt{2}\Delta(y) f_1(y)$$
$$\times a(y) f_1(y)\Big(\int f_2(\mathbf{v}_1, y)\, d\Omega_{\mathbf{v}_1}\Big)\Big]\, dy\Big\}, \tag{5}$$
$$f_3(y_1) = e^{W(y_1)}\Big\{B_3 + \frac{1}{vx}\int_{y_1}^{d/2} e^{-W(y)}\Big[\sqrt{2}\Delta(y) f_1(y)$$

$$+ a(y) f_1(y)\left(\int f_3(\mathbf{v}_1, y)\, d\Omega_{\mathbf{v}_1}\right)\Big] dy\Big\},$$
$$0 < \omega, \quad 0 < \varphi < \pi,$$

where

$$a(y) = \frac{\vartheta n \chi^2}{2}\Big\{1 + \frac{\vartheta^2\chi^2}{16}\Big[\Big(\int f_1(\mathbf{v}, y)\, d\Omega_{\mathbf{v}}\Big)^2 \times 2\Big(\int f_2(\mathbf{v}, y)\, d\Omega_{\mathbf{v}}\Big)\Big(\int f_3(\mathbf{v}, y)\, d\Omega_{\mathbf{v}}\Big)\Big]\Big\}^{-1},$$
$$z(y) = \frac{\sqrt{2}(1+i)}{x}\Big(\frac{\omega}{v} - ieHy\cos\theta\Big)\Big(\Big|\frac{eH\cos\theta}{x}\Big|\Big)^{-1/2} \times \exp\Big[-\frac{i\pi}{4}(2 - \operatorname{sign}\cos\theta - \operatorname{sign}\sin\varphi)\Big], \tag{6}$$
$$x = \sin\theta\sin\varphi, \quad W(y) = \frac{z^2(y)}{4} + \frac{1}{vx}\int_0^y a(y_1)\Big(\int f_1(\mathbf{v}, y_1)\, d\Omega_{\mathbf{v}}\Big) dy_1.$$

We have shown[6] that for diffuse scattering of the electrons one can write the boundary condition for the set (1) in the form

$$G_{\mathbf{p}}(\mathbf{r})| = iC_1(\mathbf{n}_1\tau) + \mathbf{n}_0\tau + C_2(\mathbf{n}_2\tau),$$
$$p_0(C_1 + C_2)_{\mathbf{pn}>0} = \frac{1}{\pi}\int_{\mathbf{pn}<0}(C_1 + C_2)(\mathbf{pn})\, d\Omega_{\mathbf{p}} = 0,$$
$$p_0(C_1 - C_2)_{\mathbf{pn}<0} = \frac{1}{\pi}\int_{\mathbf{pn}>0}(C_1 - C_2)(\mathbf{pn})\, d\Omega_{\mathbf{p}} = 0, \tag{7}$$

where $\mathbf{n}$ is the inward normal to the surface while the vectors $\mathbf{n}_0$, $\mathbf{n}_1$, and $\mathbf{n}_2$ depend only on the coordinate of the point on the surface and satisfy the conditions

$$\mathbf{n}_i\mathbf{n}_k = \delta_{ik}, \quad i = 0, 1, 2,$$
$$(\mathbf{n}_1\tau)(\mathbf{n}_0\tau) = -i(\mathbf{n}_2\tau). \tag{8}$$

In the chosen gauge of the vector potential $\mathbf{A}$, the matrix is $\mathbf{n}_1 \cdot \tau = \tau_x$ while we can choose the matrices $\mathbf{n}_0 \cdot \tau$ and $\mathbf{n}_2 \cdot \tau$ in the form

$$\mathbf{n}_0\tau = \alpha\tau_z + \beta\tau_y, \quad \mathbf{n}_2\tau = -\beta\tau_z + \alpha\tau_y, \tag{9}$$

where α and β are constants,

$$\alpha^2 + \beta^2 = 1. \tag{10}$$

It follows from Eqs. (3), (7), and (9) that the boundary condition for the functions f_i ($i = 1, 2, 3$) can be written in the form

$$f_i(-d/2) = Q_i + R_i\Big[\beta(f_1(-d/2) - f_1(d/2)) \times \frac{1-\alpha}{\sqrt{2}}(f_2(-d/2) - f_2(d/2)) - \frac{1+\alpha}{\sqrt{2}}(f_3(-d/2) - f_3(d/2))\Big]$$
$$0 < \varphi < \pi, \tag{11}$$
$$Q_1 = \alpha, \quad Q_2 = Q_3 = \frac{\beta}{\sqrt{2}}, \quad R_1 = \frac{\beta}{2}, \quad R_2 = \frac{1-\alpha}{2\sqrt{2}}, \quad R_3 = -\frac{1+\alpha}{2\sqrt{2}}.$$
$$\int_0^\pi d\varphi \int_0^\pi d\theta \sin^2\theta \sin\varphi\, [f_2(-d/2) - f_3(-d/2)] = 0. \tag{12}$$

We now consider the mean free path $l \gg (\xi_0 d)^{1/2}$. In that case there exists a large range of magnetic fields for which

$$eHd \gg l^{-1}, \xi_0^{-1}. \tag{13}$$

When (13) is valid we can find the solution of the set (5) with boundary condition (11) by simple iteration. We find from the set (5) and the boundary condition (11) up to terms d/ξ_0, d/l the coefficients B_i:

$$B_i = \alpha\delta_{1i} + \Gamma_i + S_i[(1+\alpha)e^{\tilde{t}/2} + (1-\alpha)e^{-\tilde{t}/2}]^{-1},$$
$$S_1 = \beta^2[e^{\tilde{t}/2} - e^{-\tilde{t}/2}], \quad S_2 = \beta\cdot\sqrt{2}e^{Z_1}, \quad S_3 = \beta\cdot\sqrt{2}e^{-Z_1}, \tag{14}$$

where δ_{ij} is the Kronecker symbol, $Z_1 = \frac{1}{4}z^2(-d/2) + t/2$,

$$t = \frac{2\omega d}{vx}, \quad \tilde{t} = \frac{2}{vx}\Big[\omega d + \int_0^{d/2} a(y)\Big(\int f_1(\mathbf{v}, y)\, d\Omega_{\mathbf{v}}\Big) dy\Big],$$
$$\Gamma_1 = -\frac{\alpha\beta}{2vx}\int_{-d/2}^{d/2} b(y)[e^{iT(y)} + e^{-iT(y)}]\, dy,$$
$$\Gamma_2\exp\Big(-Z_1 + \frac{\tilde{t}}{2}\Big) = \frac{1}{2\sqrt{2}vx}\int_{-d/2}^{d/2} b(y)[(1-\alpha)^2\exp(iT(y)) - \beta^2\exp(-iT(y))]\, dy, \tag{15}$$
$$\Gamma_3\exp\Big(Z_1 - \frac{\tilde{t}}{2}\Big) = \frac{1}{2\sqrt{2}vx}\int_{-d/2}^{d/2} b(y)[(1-\alpha)^2\exp(-iT(y)) - \beta^2\exp(iT(y))]\, dy,$$
$$T(y) = eH\frac{\cos\theta}{x}\Big(\frac{d^2}{4} - y^2\Big), \quad b(y) = \Delta(y) + \frac{\pi\beta\vartheta n\chi^2\mathscr{E}(y)}{1 + (\pi\vartheta\chi)^2[\alpha^2 + \beta^2\mathscr{E}^2(y)]};$$
$$\mathscr{E}(y) = \exp[-eH(d^2/4 - y^2)]. \tag{16}$$

Equation (14) for the coefficients B_i, together with condition (12), enables us to find the constants α and β up to terms of order d/ξ_0 and d/l. This accuracy is required to find the density of states. However, to find an expression for $\Delta(y)$ it is sufficient to restrict ourselves to the zeroth approximation. In that case we find from (12) and (14)

$$\frac{\alpha}{d}\int_{-d/2}^{d/2}\Delta(y)\mathscr{E}(y)\, dy = \beta\omega + \alpha\beta\frac{\pi\vartheta n\chi^2}{d}\int_{-d/2}^{d/2}\frac{1 - \mathscr{E}^2(y)}{1 + (\pi\vartheta\chi)^2[\alpha^2 + \beta^2\mathscr{E}^2(y)]}\, dy. \tag{17}$$

The ordering parameter can be expressed in terms of the Green function $G_{\mathbf{p}}(\mathbf{r})$ through the formula

$$\Delta(y) = -\frac{i|\lambda|mp_0}{8\pi^2}T\sum_\omega \operatorname{Sp}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\int G_{\mathbf{p}}(\mathbf{r})\, d\Omega_{\mathbf{p}} = \frac{|\lambda|mp_0}{2\sqrt{2}\pi^2}T\sum_\omega\int_0^\pi d\varphi\int_0^\pi d\theta\sin\theta\, f_2(\theta, \varphi, y). \tag{18}$$

Substituting here the expression for f_2 from Eqs. (5) and (14) we get in zeroth approximation in d/ξ_0, d/l

$$\Delta(y) = \Delta_1\mathscr{E}(y), \quad \Delta_1 = \frac{|\lambda|mp_0}{2\pi}T\sum_\omega\beta(\omega). \tag{19}$$

Substituting this expression into Eq. (17) we find

$$\alpha\Delta_1\Phi - \beta\omega = \alpha\beta\frac{\pi\vartheta n\chi^2}{d}\int_{-d/2}^{d/2}\frac{1 - \mathscr{E}^2(y)}{1 + (\pi\vartheta\chi)^2[\alpha^2 + \beta^2\mathscr{E}^2(y)]}\, dy, \tag{20}$$

where $\Phi = \Phi(1, \frac{3}{2}, -\frac{1}{2}eHd^2)$ is the confluent hypergeometric function while the quantity $\mathscr{E}(y)$ was defined in (16). From (20) we get in the Born approximation

$$\alpha\Delta_1\Phi - \beta\omega = \alpha\beta\frac{1 - \Phi}{2\tau}, \tag{21}$$

which is the same as the corresponding expression from [2].

The density of states can be expressed in terms of the Green function through the formula

$$\rho(\omega) = \frac{1}{2}\rho_0\operatorname{Im}\frac{i}{4\pi d}\int_{-d/2}^{d/2} dy\int d\Omega_{\mathbf{p}}\operatorname{Sp}\tau_z G_{-i\omega}, \tag{22}$$

where ρ_0 is the density of states in the normal metal and $G_{-i\omega}$ the analytical continuation of the function G_ω with values $\omega = \pi T(2n+1)$ ($n \geq 0$) onto the imaginary axis. In zeroth approximation we get from Eqs. (14) and (22)

$$\rho(\omega) = \rho_0\operatorname{Im}[i\alpha(-i\omega)]. \tag{23}$$

It follows from Eqs. (21) and (23) that there is a gap in the spectrum in the Born approximation[2]

$$\omega_0=\left[(\Delta_1\Phi)^{2/3}-\left(\frac{1-\Phi}{2\tau}\right)^{2/3}\right]^{3/2}. \qquad (24)$$

Taking terms of order d/ξ_0 and d/l into account in Eq. (22) leads to the appearance of a density of states for $\omega<\omega_0$. Up to terms in d/ξ_0, d/l we find from Eqs. (3), (14), and (22)

$$\rho(\omega)=\rho_0\,\mathrm{Im}\frac{i}{2\pi}\int_0^\pi d\varphi\int_0^\pi d\theta\sin\theta B_1(-i\omega). \qquad (25)$$

Substituting the expression for B_1 from (14) into Eq. (25) and retaining terms of order d/ξ_0, d/l, we get

$$\rho(\omega)=\rho_0\,\mathrm{Im}\left\{i\alpha+i\beta^2\int_0^1 dx\,[\alpha+\mathrm{cth}(\tilde{t}_0/2)]^{-1}\right\}, \qquad (26)$$

where

$$\tilde{t}_0=-\frac{2i}{x}\varkappa=-\frac{2i}{x}\left[\frac{\omega d}{v}+\frac{i\alpha nm^2\chi^2}{\pi}\int_0^{d/2}\frac{dy}{1+(\pi\vartheta\chi)^2(\alpha^2+\beta^2\mathscr{E}^2(y))}\right]$$

$$\alpha=\alpha(-i\omega),\quad \beta=\beta(-i\omega). \qquad (27)$$

Integrating over x in Eq. (26) we get

$$\rho(\omega)=\rho_0\,\mathrm{Im}\,\{i\alpha(-i\omega)+iT_0\}, \qquad (28)$$

where

$$T_0=\varkappa\left[\pi\beta^2-\frac{1}{\pi}\psi'\left(\frac{1}{2}-\frac{i}{2\pi}\ln\frac{1-\alpha}{1+\alpha}\right)\right], \qquad (29)$$

while $\psi'(x)$ is the derivative of the psi-function.

We now find an expression for $\alpha(-i\omega)$ up to terms in d/ξ_0 and d/l. Using the substitution $\omega\to -i\omega$ we get from Eqs. (5), (12), (14), and (19)

$$\frac{v\beta}{d}\int_0^1 x\left[\alpha+\mathrm{cth}\frac{\tilde{t}}{2}\right]^{-1}dx-\alpha\Delta_1\Phi$$
$$=\alpha\beta\frac{\pi\vartheta n\chi^2}{d}\int_{-d/2}^{d/2}\frac{\mathscr{E}^2(y)}{1+(\pi\vartheta\chi)^2(\alpha^2+\beta^2\mathscr{E}^2(y)+2\alpha T_0)}dy, \qquad (30)$$

where

$$\tilde{t}=-\frac{2i}{x}D=-\frac{2i}{x}\left[\frac{\omega d}{v}+\frac{inm^2\chi^2}{\pi}\right.$$
$$\left.\times\int_0^{d/2}\frac{\alpha+T_0}{1+(\pi\vartheta\chi)^2(\alpha^2+\beta^2\mathscr{E}^2(y)+2\alpha T_0)}dy\right].$$

Integrating over x in Eq. (30), we get

$$\alpha\Delta_1\Phi+\alpha\beta\frac{\pi\vartheta n\chi^2}{d}\int_{-d/2}^{d/2}\frac{\mathscr{E}^2(y)}{1+(\pi\vartheta\chi)^2(\alpha^2+\beta^2\mathscr{E}^2(y)+2\alpha T_0)}dy$$
$$=\beta\left(\frac{v}{d}\right)\left[-iD+i\pi\alpha D^2-\frac{D^2}{2\pi^2\beta^2}\psi''\left(\frac{1}{2}-\frac{i}{2\pi}\ln\frac{1-\alpha}{1+\alpha}\right)\right]$$
$$\times\frac{\alpha\beta}{2}\left(\frac{v}{d}\right)D^2\ln\frac{1}{-\alpha^2D^2}. \qquad (31)$$

The last term in Eq. (31) is written down with logarithmic accuracy. When $\omega<\omega_0$ β is real in the zeroth approximation and α purely imaginary. It therefore does not contribute to the density of states but turns out to be important near ω_0.

Equations (28), (29), and (31) determine completely the density of states for any ω and fields satisfying the conditions $eHld\gg 1$, $eH\xi_0 d\gg 1$. However, the general expression for $\rho(\omega)$ is rather complicated. We consider therefore in detail some particular cases.

A. Pure film, $l=\infty$. We have

$$\rho(\omega)=\rho_0\frac{\omega}{[\omega^2-(\Delta_1\Phi)^2]^{1/2}},\quad \omega-\Delta_1\Phi\gg\Delta_1\Phi\left(\frac{d}{\xi_0}\right)^{1/2}. \qquad (32)$$

In the region where $\Delta_1\Phi\gg|\omega-\Delta_1\Phi|$ the constant $|\alpha|\gg 1$ and we get from (31) the following equation for α:

$$\alpha^3\left(\frac{\omega d}{v}\right)\left[\pi-\frac{i}{2}\ln\frac{1}{-\alpha^2(\omega d/v)^2}\right]+\alpha^2\frac{\Delta_1\Phi-\omega}{\omega}+\frac{1}{2}=0. \qquad (33)$$

The density of states is in that case given by Eq. (23). The maximum value of ρ is with logarithmic accuracy reached in the point $\omega=\Delta_1\Phi$ and in that point

$$\rho=\rho_0\frac{\sqrt{3}}{2}\left(\frac{v}{\Delta_1\Phi d}\right)^{1/3}\left[\ln\left(\frac{v}{\Delta_1\Phi d}\right)^{4/3}\right]^{-1/3}. \qquad (34)$$

In the range $\Delta_1\Phi-\omega\gg\Delta_1\Phi(d/\xi_0)^{1/2}$ we find for the density of states

$$\rho(\omega)=\rho_0\left(\frac{\omega d}{v}\right)\left[\frac{\pi(\Delta_1\Phi)^4}{[(\Delta_1\Phi)^2-\omega^2]^2}-\frac{1}{\pi}\psi'(x_0)-\frac{\omega}{2\pi^2[(\Delta_1\Phi)^2-\omega^2]^{1/2}}\psi''(x_0)\right], \qquad (35)$$

where

$$x_0=\frac{1}{2}-\frac{i}{\pi}\ln\frac{[(\Delta_1\Phi)^2-\omega^2]^{1/2}+i\omega}{\Delta_1\Phi}.$$

When $\omega\ll\Delta_1\Phi$ we find from Eq. (34)[6]

$$\rho(\omega)=\rho_0\left(\frac{\omega d}{v}\right)\left[\frac{\pi}{2}+\frac{21}{\pi^2}\frac{\omega}{\Delta_1\Phi}\zeta(3)\right].$$

B. Born approximation. The expression obtained for $\rho(\omega)$ at arbitrary ω is awkward and we shall not give it. In the simplest case of small ω we find

$$\rho(\omega)=\rho_0\frac{\pi}{2}\frac{\omega d}{v}\Phi^2\left(\Phi-\frac{1}{1+2\tau\Delta_1}\right)^{-2},\quad \Phi>\frac{1}{1+2\tau\Delta_1}.$$

C. Non-Born approximation neglecting terms of order d/ξ_0 and d/l. Taking electron-impurity scattering in a magnetic field exactly into account leads to the appearance of an additional peak in the density of states for $\omega<\omega_0$. Neglecting terms of order d/ξ_0 and d/l, the density of states is given by Eq. (23) and $\alpha(-i\omega)$ is determined by Eq. (20) with the substitution $\omega\to -i\omega$. When $l\gg\xi_0$ one can easily find the real part of α and in the frequency range

$$\Delta_1\Phi\left(\frac{1+(\pi\vartheta\chi)^2\exp(-eHd^2/2)}{1+(\pi\vartheta\chi)^2}\right)^{1/2}<\omega<\Delta_1\Phi$$

we get for the density of states

$$\rho(\omega)=\rho_0\frac{n\omega}{\vartheta eHd^2}\frac{1+(\pi\vartheta\chi)^2}{\omega^2[1+(\pi\vartheta\chi)^2]-(\Delta_1\Phi)^2}\left[1\right.$$
$$\left.+\frac{2}{eHd^2}\ln\frac{\omega^2[1+(\pi\vartheta\chi)^2]-(\Delta_1\Phi)^2}{(\pi\vartheta\chi)^2(\Delta_1\Phi)^2}\right]^{-1/2}.$$

Decreasing the mean free path leads to a smearing-out of the peak.

2. THIN FILM IN A MAGNETIC FIELD ($\xi_0^{-1}\ll eHd\ll l^{-1}$)

We now consider the range of magnetic fields

$$\xi_0^{-1}\ll eHd\ll l^{-1}. \qquad (36)$$

When (36) is valid the Gree function $G_p(r)$ changes little over the thickness of the film and in that case the assumption of an isotropic character of the impurity scattering does not lead to an essential simplification of the problem. In the following we shall therefore at once consider the case of an arbitrary impurity scattering amplitude. We look for the Green function in the form

$$G_{\mathbf{p}}(\mathbf{r}) = (\mathbf{n}_0\tau) + i(\mathbf{n}_1\tau)(B_2 - B_3) - (\mathbf{n}_2\tau)(B_2 + B_3) + B_1(\mathbf{n}_0\tau), \quad (37)$$

where the vectors $\mathbf{n}_i$ are defined by Eqs. (8), (9), and (10). Substituting Eq. (37) into the set (1) we get a set of equations for the coefficients B_i:

$$\begin{aligned}\left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)B_1 &= 2ie\beta(\mathbf{v}\mathbf{A})(B_2 - B_3) + nv\Big[(B_2 + B_3)\int\sigma_{\mathbf{pp}_1}(B_2(\mathbf{p}_1) \\ &\quad - B_3(\mathbf{p}_1))\,d\Omega_{\mathbf{p}_1} - (B_2 - B_3)\int\sigma_{\mathbf{pp}_1}(B_2(\mathbf{p}_1) + B_3(\mathbf{p}_1))\,d\Omega_{\mathbf{p}_1}\Big], \\ \left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)B_2 &= nv\left(\sigma B_2 - \int\sigma_{\mathbf{pp}_1}B_2(\mathbf{p}_1)\,d\Omega_{\mathbf{p}_1}\right) + 2B_2[\alpha\omega + \beta\Delta \\ &\quad - i\alpha e(\mathbf{v}\mathbf{A})] + 2[\alpha\Delta - \beta\omega + ie\beta(\mathbf{v}\mathbf{A})], \\ \left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)B_3 &= -nv\left(\sigma B_3 - \int\sigma_{\mathbf{pp}_1}B_3(\mathbf{p}_1)\,d\Omega_{\mathbf{p}_1}\right) - 2B_3[\alpha\omega + \beta\Delta \\ &\quad - i\alpha e(\mathbf{v}\mathbf{A})] - 2[\alpha\Delta - \beta\omega + ie\beta(\mathbf{v}\mathbf{A})].\end{aligned} \quad (38)$$

We choose our coordinate system and the gauge of the vector potential to be the same as in Sec. 1. The boundary conditions (8) can then be written in the form

$$B_1(\pm d/2) = 0, \quad B_2(x > 0, d/2) = B_3(x > 0, -d/2) = 0, \quad (39)$$
$$\int_0^{2\pi} d\varphi' \int_0^1 xB_2(x, -d/2)\,dx = 0,$$

where

$$\mathbf{v} = v\{\sqrt{1 - x^2}\sin\varphi',\ x,\ \sqrt{1 - x^2}\cos\varphi'\}.$$

To obtain expressions for Δ and the density of states it is only necessary to find the function B_2. Expanding B_2 in a series in $\cos n\varphi'$:

$$B_2 = \chi_0(x, y) + \chi_1(x, y)(1 - x^2)^{1/2}\cos\varphi' + \ldots,$$

and substituting this expansion into the second equation of the set (38) we get

$$\frac{\partial\chi_1}{\partial y} = \frac{1}{x}\left[\frac{1}{l}\chi_1 - \frac{3}{4l_1}\int_{-1}^{1}(1 - x_1^2)\chi_1(x_1, y)\,dx_1\right] + \frac{2i\beta eHy}{x}, \quad (40)$$
$$\frac{\partial\chi_0}{\partial y} = \frac{1}{xl}\left[\chi_0 - \frac{1}{2}\int_{-1}^{1}\chi_0(x_1, y)\,dx_1\right] - i\alpha eHy\frac{1 - x^2}{x}\chi_1 + \frac{2(\alpha\Delta - \beta\omega)}{vx},$$

where

$$\sigma_{\mathbf{pp}'} = \frac{1}{4\pi}[\sigma + 3\sigma_1\cos\theta + \ldots], \quad n\sigma = l^{-1}, \quad n\sigma_1 = l_1^{-1}.$$

From the set (40) and the boundary condition (39) we get the following expressions for χ_0 and χ_1:

$$\begin{aligned}\chi_1(x, y) &= \frac{1}{x}\int_y^{d/2} Y\left[-2i\beta eHy_1 + \frac{3}{4l_1}\Phi_1(y_1)\right]dy_1, \quad x > 0, \\ \chi_0(x, y) &= \frac{1}{x}\int_y^{d/2} dy_1\, Y\left[\frac{1}{2l}\Phi(y_1) + i\alpha eH(1 - x^2)y_1\chi_1(x, y_1) - \frac{2(\alpha\Delta - \beta\omega)}{v}\right] \\ &\qquad x > 0, \\ \chi_0(x, y) &= \chi_0(-x, -y), \quad \chi_1(x, y) = -\chi_1(-x, -y) \\ Y &= \exp\left[-\frac{|y - y_1|}{xl}\right],\end{aligned} \quad (41)$$

where the functions $\Phi_1(y)$ and $\Phi(y)$ are solutions of the integral equations

$$\begin{aligned}\Phi_1(y) &= \frac{3}{4l_1}\int_0^1 dx\frac{1 - x^2}{x}\int_{-d/2}^{d/2} Y\Phi_1(y_1)\,dy_1 - 2i\beta eH\int_0^1 dx\frac{1 - x^2}{x}\int_{-d/2}^{d/2} y_1Y\,dy, \\ \Phi(y) &= \frac{1}{2l}\int_0^1\frac{dx}{x}\int_{-d/2}^{d/2} Y\Phi(y_1)\,dy_1 - 2\int_0^1\frac{dx}{x}\int_{-d/2}^{d/2} Y\frac{\alpha\Delta - \beta\omega}{v}\,dy_1 \\ &\times i\alpha eH\int_0^1 dx\frac{1 - x^2}{x}\left[\int_y^{d/2} Yy_1\chi_1(x, y_1)\,dy_1 - \int_{-d/2}^{y} Yy_1\chi_1(x, -y_1)\,dy_1\right].\end{aligned} \quad (42)$$

From the condition (39) and Eq. (41) for χ_0 we find an equation for the constants α and β:

$$i\alpha eH\int_0^1 dx(1 - x^2)\int_{-d/2}^{d/2} y_1\chi_1(x, y_1)\,dy_1 = 2\int_{-d/2}^{d/2}\frac{\alpha\Delta - \beta\omega}{v}\,dy. \quad (43)$$

The ordering parameter Δ is in the principal approximation independent of the coordinates and can be expressed in terms of $\beta(\omega)$ through the formula

$$\Delta = \frac{|\lambda| mp_0}{2\pi}T\sum_\omega\beta(\omega). \quad (44)$$

Equations (41)–(43) enable us to find the function $\Delta(H)$ when (36) is valid. If $l \ll (\xi_0 d)^{1/2}$ this condition is satisfied for all fields up to the critical one. In the main approximation the density of states can be expressed in terms of $\alpha(-i\omega)$ through Eq. (23). Taking small corrections into account leads, in contrast to the case $eHld \gg 1$, to no important change in the excitation spectrum.

When the scattering is isotropic we obtain at once from Eqs. (41) and (43) an algebraic equation for the constants α and β in the whole range $eHld \ll 1$:

$$\begin{aligned}\alpha\Delta - \beta\omega &= \alpha\beta(eH)^2 v\frac{l}{d}\Big\{\frac{d^3}{18} - l\left(\frac{d^2}{16} - \frac{l^2}{12}\right) \\ &\quad - l\int_0^1 x(1 - x^2)\left(\frac{d}{2} + xl\right)^2 e^{-d/xl}dx\Big\}.\end{aligned} \quad (45)$$

When the scattering is anisotropic it is necessary to solve the integral equation (42) for the function $\Phi_1(y)$ to obtain an equation for α and β. We consider two limiting cases.

A. Maki's case, $l \ll d$. Then

$$\Phi_1(y) = -\frac{8i}{3}\beta evHy\tau_{tr}, \quad \chi_1 = -2i\beta evHy\tau_{tr}.$$

Substituting the expression for χ_1 into Eq. (43) we get an equation for α and β:[1]

$$\alpha\Delta - \beta\omega = {}^1/_{18}\alpha\beta(evHd)^2\tau_{tr}.$$

The integral equation (42) for $\Phi(y)$ reduces to a differential equation:

$$\frac{\partial^2\Phi}{\partial y^2} = \frac{4}{l}\left[\frac{3(\alpha\Delta - \beta\omega)}{v} - 2\alpha\beta(eH)^2 y^2 v\tau_{tr}\right] \quad (46)$$

with the boundary condition $\Phi(y = \pm\frac{1}{2}d) = 0$. Solving Eq. (46) we find

$$\Phi(y) = -\frac{2}{3}\alpha\beta(eH)^2\frac{v\tau_{tr}}{l}\left(\frac{d^2}{4} - y^2\right)^2.$$

B. In the limiting case $d \ll l \ll (\xi_0 d)^{1/2}$ the integral equations (42) can easily be solved and we find for the functions χ_1 and $\Phi(y)$:

$$\chi_1 = -\frac{2i\beta eH}{x}\int_y^{d/2} Yy_1\,dy_1, \quad \Phi(y) = \frac{\alpha\beta(eH)^2 ld^3}{12}.$$

Substituting the expression for χ_1 into Eq. (43) we find an equation for α and β:

$$\alpha\Delta - \beta\omega = {}^1/_{32}\alpha\beta(eH)^2 d^3 v. \quad (47)$$

Equation (47) for the constants α and β has the same form as in the "very dirty" case but with a different coefficient of H^2. From Eqs. (44) and (47) we easily find the critical field[7]

$$\ln\frac{T_c}{T} = \psi\left(\frac{1}{2} + \frac{e^2H^2vd^3}{64\pi T}\right) - \psi\left(\frac{1}{2}\right).$$

3. SMALL SIZE SPHERE ($R \ll \xi_0$) IN A MAGNETIC FIELD

Larkin[8] has considered the properties of a small size superconducting sphere in a magnetic field when there is specular reflection from the walls. He found the field dependence of the parameter Δ and of the magnetic moment and also the magnitude of the gap in the excitation spectrum. We solve here the analogous problem for diffuse reflection from the walls. We choose the vector potential in the form $\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{r}$. The parameter Δ is real in this gauge. We restrict our considerations to the case of a pure sphere. As in section 1 we look for the Green function in the form (3). Similar to Eq. (5) we find for the f_i

$$f_1 = B_1(\tilde{\mathbf{r}}) + \frac{\sqrt{2}}{v^2}\int_0^{\mathbf{rv}} \Delta(f_2 - f_3)\, d(\mathbf{r}_1, \mathbf{v}),$$

$$f_2 = e^{-W(\mathbf{v},\mathbf{r})}\left\{B_2(\tilde{\mathbf{r}}) + \frac{\sqrt{2}}{v^2}\int_{\tilde{\mathbf{r}}\mathbf{v}}^{\mathbf{rv}} \Delta f_1 e^{W(\mathbf{v},\mathbf{r}_1)}\, d(\mathbf{r}_1, \mathbf{v})\right\}, \tag{48}$$

$$f_3 = e^{W(\mathbf{v},\mathbf{r})}\left\{B_3(\tilde{\mathbf{r}}) + \frac{\sqrt{2}}{v^2}\int_{\mathbf{rv}}^{\tilde{\mathbf{R}}\mathbf{v}} \Delta f_1 e^{-W(\mathbf{v},\mathbf{r}_1)}\, d(\mathbf{r}_1, \mathbf{v})\right\},$$

where

$$W(\mathbf{v},\mathbf{r}) = \frac{2(\mathbf{rv})}{v^2}\left(\omega - \frac{ie}{2}(\mathbf{v}[\mathbf{Hr}])\right),$$

$$\tilde{\mathbf{r}} = \mathbf{r} - \frac{\mathbf{v}(\mathbf{rv})}{v^2} - \frac{\mathbf{v}}{v}\left(R^2 - r^2 + \frac{(\mathbf{vr})^2}{v^2}\right)^{1/2}, \tag{48'*}$$

$$\tilde{\mathbf{R}} = \mathbf{r} - \frac{\mathbf{v}(\mathbf{rv})}{v^2} + \frac{\mathbf{v}}{v}\left(R^2 - r^2 + \frac{(\mathbf{vr})^2}{v^2}\right)^{1/2}.$$

For our choice of gauge of the vector potential the matrices $(\mathbf{n}_i \cdot \tau)$ occurring in the boundary condition (7) can be chosen in the same form (8) and (9) as for a thin film. However, the coefficients α and β will now already depend on the coordinates of the points on the surface. We can by analogy to Eq. (11) write the boundary condition in the form

$$f_i(\mathbf{v}, \mathbf{r}_1) = Q_i(\mathbf{r}_1) + R_i(\mathbf{r}_1)\tilde{\psi}(\mathbf{v}, \mathbf{r}_1), \tag{49}$$

$$\int_{\mathbf{vn}>0} (\mathbf{vn})(f_2 - f_3)\, d\Omega_{\mathbf{v}} = 0, \tag{50}$$

where $\mathbf{r}_1$ is a point on the surface $\mathbf{v} \cdot \mathbf{r}_1 < 0$,

$$\tilde{\psi}(\mathbf{v},\mathbf{r}_1) = (\beta f_1 - \tilde{\beta}\tilde{f}_1) + \left(\frac{1-\alpha}{\sqrt{2}} f_2 - \frac{1-\tilde{\alpha}}{\sqrt{2}}\tilde{f}_2\right) - \left(\frac{1+\alpha}{\sqrt{2}} f_3 - \frac{1+\tilde{\alpha}}{\sqrt{2}}\tilde{f}_3\right),$$

$$\beta = \beta(\mathbf{r}_1), \quad \tilde{\beta} = \beta\left(\mathbf{r}_1 - \frac{2\mathbf{v}(\mathbf{r}_1\mathbf{v})}{v^2}\right), \quad f_i = f_i(\mathbf{v}, \mathbf{r}_1), \tag{51}$$

$$\tilde{f}_i = f_i\left(\mathbf{v}, \mathbf{r}_1 - \frac{2\mathbf{v}(\mathbf{r}_1\mathbf{v})}{v^2}\right),$$

while the quantities Q_i and R_i are defined in (11). The functions f_i satisfy the conditions $f_i(\mathbf{v}, \mathbf{r}) = f_i(-\mathbf{v}, -\mathbf{r})$.

We can solve the set of equations (48) together with the boundary conditions (49) by simple iteration. One can show that the parameter $eHR^2 \ll 1$ where R is the radius of the sphere. Therefore, Δ can be considered to be constant and the coefficients α and β have the form

$$\alpha = \alpha(\cos\theta) = \alpha_0 + \alpha_1\cos^2\theta,$$
$$\beta = \beta(\cos\theta) = \beta_0 + \beta_1\cos^2\theta, \tag{52}$$

where

$$\cos\theta = \frac{\mathbf{r}_1\mathbf{H}}{RH}, \quad |\alpha_1| \ll 1, \quad |\beta_1| \ll 1.$$

* $[\mathbf{Hr}] \equiv \mathbf{H} \times \mathbf{r}$.

From the condition $\alpha^2 + \beta^2 = 1$ it follows that $\alpha_0^2 + \beta_0^2 = 1$.

Using (52) we get from Eqs. (48) and (49)

$$\tilde{\psi}(\mathbf{v},\mathbf{r}_1) = (\beta\tilde{\alpha} - \alpha\tilde{\beta}) - \frac{4\beta}{1+\alpha\tilde{\alpha}+\beta\tilde{\beta}}\left[\frac{2(\mathbf{r}_1\mathbf{v})}{v^2}\left(\omega - \frac{ie}{2}(\mathbf{v}[\mathbf{Hr}_1])\right)\right] \times 2\alpha_0\beta_0\frac{(\mathbf{r}_1\mathbf{v})^2}{v^4}e^2(\mathbf{v}[\mathbf{Hr}_1])^2 + \frac{4\alpha_0(\mathbf{r}_1\mathbf{v})}{v^2}\Delta. \tag{53}$$

Substituting the expression for $\tilde{\psi}$ from Eq. (53) into condition (50) we find

$$\alpha_0\Delta - \beta_0\omega = \alpha_0\beta_0\frac{e^2H^2vR^3}{24},$$

$$\alpha_1 = -\alpha_0\beta_0^2\frac{(eHR^2)^2}{6}, \quad \beta_1 = \beta_0\alpha_0^2\frac{(eHR^2)^2}{6}. \tag{54}$$

The ordering parameter Δ can be expressed in terms of $\beta(\omega)$ through Eq. (44).

From Eqs. (44) and (54) we find the critical field

$$\ln\frac{T_c}{T} = \psi\left(\frac{1}{2} + \frac{e^2H^2vR^3}{48\pi T}\right) - \psi\left(\frac{1}{2}\right). \tag{55}$$

This expression differs from the corresponding equation for the critical field found in [4]. This discrepancy is connected with the fact that in evaluating the average $\langle(\int \mathbf{A}\,d\mathbf{l})^2\rangle$ and the average flight time the angular distribution law for the trajectories was not taken into account in [4]. When the appropriate corrections are taken into account an expression is obtained which is the same as Eq. (55).

We now find the magnetic moment of the sphere. The current density can be expressed in terms of f_1 and is equal to

$$\mathbf{j} = -\frac{iep_0}{4\pi^2}T\sum_\omega \int \mathbf{p}f_1(\mathbf{v},\mathbf{r})\, d\Omega_{\mathbf{v}} \tag{56}$$

From Eqs. (48) and (53) it follows that

$$f_1(\mathbf{v},\mathbf{r}) = \alpha(\tilde{\mathbf{r}}) - \frac{ie}{v}\beta_0^2(\mathbf{v}[\mathbf{Hr}])\left(R^2 - r^2 + \frac{(\mathbf{rv})^2}{v^2}\right)^{1/2}, \tag{57}$$

where $\tilde{\mathbf{r}}$ is given by Eq. (48'). Using Eqs. (56) and (57) and the definition of the magnetic moment,

$$\mathbf{M} = \frac{1}{2}\int [\mathbf{rj}]\, d\mathbf{r},$$

we find easily

$$\mathbf{M} = -\frac{e^2p_0^2}{18}R^6\mathbf{H}T\sum_\omega \beta_0^2.$$

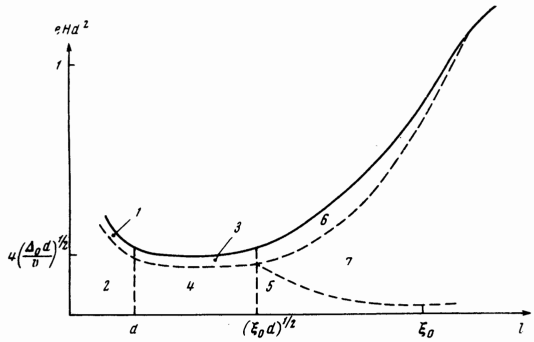
In a weak field

$$\beta_0 = \frac{\Delta}{(\omega^2+\Delta^2)^{1/2}}, \quad \mathbf{M} = -\frac{e^2p_0^2}{36}R^6\mathbf{H}\left(\Delta\,\mathrm{th}\frac{\Delta}{2T}\right),$$

which is the same as the well-known expression.[9]

4. CONCLUSION

The behavior of thin films in a magnetic field depends in an essential way on the parameter $eHld$. In the region $eHld \gg 1$ there is no gap in the excitation spectrum, but the density of states is proportional to the small parameter $(d/\xi_0, d/l)$ up to some threshold value ω_0. Even in the limit as $l \to \infty$ the density of states nowhere becomes infinite and reaches its maximum value of order $\rho_0(\xi_0/d)^{1/3}$ near the point $\omega = \Delta_1\Phi$. Decreasing the mean free path initially leads to an increase in the density of states for small ω. However, when we go over into the range $eHld \ll 1$ the usual threshold situation arises where the density of states vanishes for ω less than a



well-defined value ω_0 and only in very strong fields, just as in the "very dirty" limit, there occurs again gapless superconductivity. Both in the range $l \gg d$ and in the range $d \ll l \ll (\xi_0 d)^{1/2}$ at T = 0 the field at which the gap in the spectrum vanishes can be expressed in terms of the critical field through the formula $H^2 = 0.91\, H_{cr}^2$. In the range $eHld \gg 1$ there appears an additional peak in the density of states in the non-Born approximation, the position of which depends both on the magnitude of the field and on the parameter σp_0^2. Decreasing the mean free path leads to a smearing-out of this peak. The maximum area under the peak is reached when $l \sim \xi_0$. In the figure we have indicated the mean free path dependence of the critical field at T = 0. The region under the curve is divided into a number of subregions in each of which there is a particular $\Delta(H)$ dependence. In the regions 1, 2, 3, 4, 5 the parameter $eHld \ll 1$, in the regions 6 and 7 $eHld \gg 1$. In the regions 1,3, and 6 there are gapless excitations in the density of states of order ρ_0 while in region 7 there are gapless excitations with a small density of states. We found the $\Delta(H)$ dependence both in the region $eHld \gg 1$ and in the region $eHld \ll 1$. In the region $eHld \ll 1$ there occurs a linear integral equation the solution of which determines the $\Delta(H)$ dependence. For $l \ll d$ and $l \gg d$ this equation can easily be solved and in the region $l \ll d$ Maki's well-known result[1] is obtained.

We considered also a sphere with diffuse reflection at the walls in a magnetic field. The $\Delta(H)$ dependence and the field depencence of the magnetic moment are found.

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[1] K. Maki, Progr. Theoret. Phys. 29, 603 (1963); 31, 731 (1964).

[2] R. S. Thompson, Zh. Eksp. Teor. Fiz. 53, 759 (1967) [Sov. Phys.-JETP 26, 470 (1968)].

[3] A. I. Rusinov, Zh. Eksp. Teor. Fiz. 56, 2047 (1969) [Sov. Phys.-JETP 29, 1101 (1969)].

[4] P. G. de Gennes and M. Tinkham, Physics 1, 107 (1964).

[5] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 55, 2262 (1968) [Sov. Phys.-JETP 28, 1200 (1969)].

[6] Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 56, 1590 (1969) [Sov. Phys.-JETP 29, 853 (1969)].

[7] E. A. Shapoval, Zh. Eksp. Teor. Fiz. 49, 930 (1965) [Sov. Phys.-JETP 22, 647 (1966)].

[8] A. I. Larkin, Zh. Eksp. Teor. Fiz. 48, 232 (1965) [Sov. Phys.-JETP 21, 153 (1965)].

[9] A. A. Abrikosov and I. M. Khalatnikov, Usp. Fiz. Nauk 65, 551 (1958) [English translation published by Office of Naval Research].

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