DISTURBANCE OF A NONISOTHERMAL PLASMA BY A BODY MOVING WITH SUPERSONIC VELOCITY

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The dependence of the structure of the disturbed region on the degree of nonisothermy of a plasma is investigated for large distances from the rapidly moving body. If $T_e/T_i < 1.76$, the particle density behind the body is small. If $T_e/T_i > 0.23$ the rarefaction is maximal along the axis behind the body, whereas for $T_e/T_i < 0.23$ the rarefaction is maximal along the cone of maximal rarefaction. High and low particle densities are observed in the disturbed region if $T_e/T_i > 1.76$; if $1.76 < T_e/T_i < 2.43$, the maximal condensation occurs on the axis behind the body and for $T_e/T_i > 2.43$ the density is maximal on the maximal condensation cone enclosed by the maximal rarefaction cone.

1. INTRODUCTION

THE perturbation of a rarefied non-magnetized plasma by a rapidly moving body at large distances from the body was investigated in a number of papers.[1-7] It was assumed there that the velocity of the body V_0 satisfies the condition $v_i \ll V_0 \ll v_e$, where $v = (2\kappa T_i/M)^{1/2}$ and $v_e = (2\kappa T_e/m)^{1/2}$ are the thermal velocities of the body was assumed to be metallic, i.e., it was assumed that the surface absorbs the particles incident on it.

The authors of [1-7] confined themselves mainly to perturbations in an isothermal plasma, $T_e = T_i$. The maximum perturbation of the particle density at large distances from the body was reached on a cone analogous to the Mach cone in ordinary gas dynamics. In an isothermal plasma, ion-acoustic waves become strongly attenuated, leading to a smearing of the Mach cone. It is of definite interest to investigate the influence of the isothermal character of the plasma, characterized by the parameter $\eta = T_e/T_i$, on the structure of the perturbed region, since both laboratory and ionospheric plasmas can be essentially non-isothermal.

2. FOURIER COMPONENTS OF PERTURBATION OF PARTICLE DENSITY

Let us obtain expressions for the Fourier components of the perturbation of the particle density at large distances from a body moving in a non-isothermal plasma; we use for this purpose a method somewhat different from that described in [1], where the connection with the Cerenkov excitation of the ion-acoustic wave is more pronounced. The idea of this method is described in a paper by Pitaevskiĭ.[2]

We shall consider the perturbation of a plasma by a body in an immobile system of coordinates connected with the plasma. In this system, all the perturbations depend on the time, and in the case of stationary motion this dependence enters only in the form of a dependence on $\mathbf{r}' = \mathbf{r} - \mathbf{V}_0 t$. The distribution of particles of type α (α = e, i) is described by the collisionless kinetic equation

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{e_\alpha}{m_\alpha}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right\}\frac{\partial f_\alpha}{\partial \mathbf{v}} = \frac{\mathbf{r}'(\mathbf{v}+\mathbf{V}_0)}{r'}\delta(r'-R_0)H(-\mathbf{r}'(\mathbf{v}+\mathbf{V}_0))f_\alpha, \qquad (1)^*$$

where the term in the right-hand side takes into account the absorption of the particles by the surface of the body (the integral of the collisions of the particles with the body);[7] for simplicity, the body is assumed to be a sphere of radius R_0; H(x) is the Heaviside function: H(x) = 0 when x < 0 and H(x) = 1 when x ≥ 0. The self-consistent electric and magnetic fields should be determined from Maxwell's equations

$$\operatorname{div}\mathbf{E} = 4\pi\sum e_\alpha\int f_\alpha d\mathbf{v}, \qquad \operatorname{div}\mathbf{H} = 0,$$
$$\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\sum e_\alpha\int \mathbf{v}f_\alpha d\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}. \qquad (2)$$

At large distances from the body, the particle distribution function should, naturally, tend to the unperturbed Maxwellian distribution function

$$f_\alpha^0 = N_0\left(\frac{m_\alpha}{2\pi\varkappa T_\alpha}\right)^{3/2}\exp\left(-\frac{m_\alpha v^2}{2\varkappa T_\alpha}\right), \qquad (3)$$

and the fields vanish. We linearize the kinetic equation at large distances from the body, assuming the deviations from (3) to be small. The linearized kinetic equation for the Fourier components

$$f_\alpha(\mathbf{k}) = \int f_\alpha(\mathbf{r}')e^{-i\mathbf{kr}'}d\mathbf{r}' \qquad (4)$$

is of the form

$$i(\mathbf{kv}-\omega)f_\alpha + \frac{e_\alpha}{m_\alpha}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right\}\frac{\partial f_\alpha^0}{\partial \mathbf{v}} = I_\alpha(\mathbf{v}), \qquad (5)$$

where $I_\alpha(\mathbf{v})$ is the Fourier component of the integral of the collisions between the particles in the body, and by virtue of the dependence of all the perturbations on r′ it is necessary to put

$$\omega = \mathbf{kV}_0. \qquad (6)$$

From the kinetic equation (5) we see that the perturbed distribution functions consist of two terms, one

*$[\mathbf{vH}] \equiv \mathbf{v}\times\mathbf{H}$.

of which describes the perturbation of the distribution function by the body itself, owing to the interaction between the particles on the surface of the body, while the other describes the polarization of the plasma by the produced field. The linear polarization of the plasma can be taken into account by introducing into Maxwell's equation the dielectric tensor,[8] and then all that remains of the current densities and charges in the right sides of Maxwell's equations are the "extraneous currents and charges," due to the perturbation of the plasma as a result of the interaction between the particles and the surface of the body, which play the role as field sources, i.e., we effect the resolution

$$n_\alpha = n_\alpha^{\text{extr}} + n_\alpha^{\text{polar}} = \int f_\alpha^{\text{extr}} d\mathbf{v} - i\frac{k_i \varkappa_{ij}^\alpha}{e_\alpha} E_j, \tag{7}$$

$$j_i^\alpha = j_i^{\alpha\ \text{extr}} + j_i^{\alpha\ \text{polar}} = \int v_i f_\alpha^{\text{extr}} d\mathbf{v} - i\omega\frac{\varkappa_{ij}^\alpha}{e_\alpha} E_j,$$

where f^{extr} satisfies the kinetic equation, but now without the perturbations of the electric and magnetic fields:

$$i(\mathbf{kv}-\omega) f_\alpha^{\text{extr}} = I_\alpha(\mathbf{v}). \tag{8}$$

Eliminating $\mathbf{H}$ from Maxwell's equations, we obtain

$$-[\mathbf{k}[\mathbf{kE}]]_i - \frac{\omega^2}{c^2}\varepsilon_{il}(\mathbf{k},\omega)E_l = i\frac{\omega}{c^2}4\pi\sum e_\alpha j_i^{\alpha\ \text{extr}}. \tag{9}$$

The dielectric tensor is expressed in terms of the polarizability tensor:

$$\varepsilon_{il}(\mathbf{k},\omega) = \delta_{il} + 4\pi\varkappa_{il}(\mathbf{k},\omega), \quad \varkappa_{il}(\mathbf{k},\omega) = \sum_\alpha \varkappa_{il}^\alpha(\mathbf{k},\omega). \tag{10}$$

We shall not stop to discuss their calculation, since this question is discussed in detail,[8] and we present only the information that will be needed in what follows. The dielectric constant of a homogeneous isotropic plasma can be represented in the form

$$\varepsilon_{ij}(\mathbf{k},\omega) = \Big(\delta_{ij} - \frac{k_i k_j}{k^2}\Big)\varepsilon_t(k,\omega) + \frac{k_i k_j}{k^2}\varepsilon_l(k,\omega), \tag{11}$$

where

$$\varepsilon_l(k,\omega) = 1 + \sum_\alpha \frac{1}{k^2 d_\alpha^2}[1 + i\sqrt{\pi}z_\alpha W(z_\alpha)],$$
$$\varepsilon_t(k,\omega) = 1 + i\sqrt{\pi}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} z_\alpha W(z_\alpha). \tag{12}$$

The notation is standard: $\omega_{p\alpha} = (4\pi e^2 N_0/m_\alpha)^{1/2}$ is the plasma frequency, $d_\alpha = v_\alpha/\omega_{p\alpha}$ is the Debye radius, and $v_\alpha = (2\kappa T_\alpha/m_\alpha)^{1/2}$ is the thermal velocity of the α-component of the plasma, and $z_\alpha = \omega/kv_\alpha$. The function W(z) in (12) is the Kramp function:[9]

$$W(z) = e^{-z^2}\Big\{1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{u^2}du\Big\}. \tag{13}$$

The polarizability tensor $\varkappa_{ij}(\mathbf{k},\omega)$ has a representation similar to (12). For example, the longitudinal polarizability of the α component, which we shall need subsequently, can be obtained from the formula

$$\varkappa_l^\alpha(k,\omega) = \frac{1}{4\pi k^2 d_\alpha^2}[1 + i\sqrt{\pi}z_\alpha W(z_\alpha)]. \tag{14}$$

Resolving in (9) the field and the extraneous current into components that are longitudinal and transverse relative to the vector $\mathbf{k}$, we get

$$\mathbf{E}_\parallel = -\frac{4\pi i}{\omega\varepsilon_l}\sum e_\alpha \mathbf{j}_\parallel^{\alpha\ \text{extr}}, \quad \mathbf{E}_\perp = -\frac{4\pi i\omega}{(\omega^2\varepsilon_t - k^2c^2)}\sum e_\alpha \mathbf{j}_\perp^{\alpha\ \text{extr}}. \tag{15}$$

Thus, the calculation of the Fourier components of the perturbations reduces to a determination of the extraneous currents, which can be readily carried out in our case. Let us find the ion contribution to the extraneous current, after first simplifying the integral of collisions between the ions and the body. By virtue of the condition $V_0 \gg v_i$, the ions are incident on the body from the forward end, and since the body moving in the plasma acquires a negative floating potential φ_0, equal approximately to $(3-4)\kappa T_e/e$,[7] we can disregard in the ion-body collision integral, when $V_0 \gg (\kappa T_e/M)^{1/2}$, the influence of the electric field on the ions, replace f_i by f_0^i in the right side of (1), and neglect v compared with V_0. The Fourier component of the ion-body collision integral then reduces to the form

$$I_i(\mathbf{v}) = -\pi R_0^2 V_0 f_0^i(\mathbf{v}) G(\mathbf{k}). \tag{16}$$

The form factor $G(\mathbf{k})$ in (16) is best calculated in spherical coordinates with the axis along the vector $\mathbf{V}_0$

$$\mathbf{r}' = r'\{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\}, \mathbf{k} = k\{\sin\chi\cos\varphi_1, \sin\chi\sin\varphi_1, \cos\chi\}. \tag{17}$$

We have

$$G(\mathbf{k}) = \frac{-1}{\pi R_0^2}\int\frac{\mathbf{rV}_0}{rV_0}\delta(r-R_0)H(-\mathbf{rV}_0)e^{-i\mathbf{kr}}d\mathbf{r}$$
$$= -2\int_0^{\pi/2}\cos\theta\sin\theta j_0(kR_0\sin\theta\sin\chi)\exp(-ikR_0\cos\theta\cos\chi)d\theta. \tag{18}$$

where small k, such that $kR_0 \ll 1$, as will be assumed beforehand, the form factor $G(\mathbf{k}) = 1$. This fact is obvious physically—at distances $r \gg R_0$ the body can be regarded as an exact absorber of particles, absorbing $N_0 V_0 \pi R_0^2$ ions (and the same number of electrons) per unit time. If we consider a body of nonspherical form, then πR_0^2 in (16) should be replaced by S—the area of the projection of the body on the plane perpendicular to the direction of motion. As a result

$$\mathbf{j}_i^{\text{extr}} = iSV_0\int\frac{\mathbf{v}f_0^i(\mathbf{v})}{\mathbf{kv}-\omega}d\mathbf{v} = iSN_0V_0\frac{\mathbf{k}}{k^2}[1 + i\sqrt{\pi}z_i W(z_i)]. \tag{19}$$

The singularity in the integration in (19) is gotten around in accordance with the Landau rule, i.e., by replacing $\omega \to \omega + i\nu$, $\nu \to 0$.[10]

The electron contribution to the extraneous current is easiest to determine from the continuity equation for the electron current:

$$i\mathbf{kj}_e^{\text{extr}} = -N_0V_0S. \tag{20}$$

Since the velocity of the body V_0 is small compared with the electron thermal velocity, the electron current can be directed only among the vector $\mathbf{k}$, and consequently we get from (20)

$$\mathbf{j}_e^{\text{extr}} = iN_0V_0S\mathbf{k}/k^2. \tag{21}$$

We note that the electron current (21) can be determined formally from (19) by replacing z_i by $z_e = \mathbf{k} \times \mathbf{V}_0/kv_e \ll 1$.

The total extraneous current entering in (15) is determined by the formula

$$\sum_\alpha e_\alpha \mathbf{j}_\alpha^{\text{extr}} = -eN_0V_0S\sqrt{\pi}z_i W(z_i)\mathbf{k}/k^2. \tag{22}$$

Since the extraneous current is longitudinal, the electric field (15) is also longitudinal; in the coordinate system fixed in the plasma, consequently, there will be no perturbation of the magnetic field. In a coordinate system moving with the body, there will exist a weak magnetic field owing to the coordinate transformation:

$$\mathbf{H} = -c^{-1}[\mathbf{V}_0\mathbf{E}]. \quad (23)$$

The potential of the electric field ($\mathbf{E} = -i\mathbf{k}\varphi$) will be determined, in accordance with (15), where the equation

$$\varphi = \frac{4\pi}{\omega\,\varepsilon_l(k,\omega)}\frac{\mathbf{k}}{k^2}\sum_\alpha e_\alpha \mathbf{j}_\alpha^{\text{extr}}. \quad (24)$$

Substituting here the expression given above for $\epsilon_l(k, \omega)$ and for the extraneous current, we obtain when $k^2 d_e^2 \ll 1$:

$$\varphi = -\frac{\varkappa T_e}{e}\frac{a_0 S}{k}\sqrt{\pi}\,W(z_i)\left[1+\frac{T_e}{T_i}\left(1+i\sqrt{\pi}z_i W(z_i)\right)\right]^{-1}. \quad (25)$$

In simplifying the electron contribution to $\epsilon_l(k, \omega)$, we took into account the fact that $z_e \ll 1$, and introduced the symbol $a_0 = V_0/v_i$. Calculating

$$n_i^{\text{extr}} = iSV_0\int\frac{f_0^i(\mathbf{v})}{\mathbf{kv}-\omega}d\mathbf{v} = -SN_0 a_0\frac{1}{k}\sqrt{\pi}\,W(z_i) \quad (26)$$

and substituting (26), the polarizability tensor of the ion component (14), and the field (24) in the right side of (7), we obtain, for $kd_e \ll 1$ and $z_e \ll 1$, a final expression for the Fourier component of the perturbation of the ion density:

$$N_\mathbf{k} = -N_0\frac{a_0 S}{k}\sqrt{\pi}\,W(z_i)\left[1+\frac{T_e}{T_i}\left(1+i\sqrt{\pi}z_i W(z_i)\right)\right]^{-1}. \quad (27)$$

A perfectly analogous expression is obtained also for the perturbation of the electron density, since the plasma is quasineutral when $kd_e \ll 1$. We note that when $T_e/T_i = 0$, Eq. (26) follows from (27); this simply denotes that the electric field can be neglected when $T_e \to 0$.

The equation $\epsilon_l(k, \omega) = 0$ is the dispersion equation for the longitudinal plasma waves. In the case of real k, this equation has in the general case, as solutions, the complex quantities $\omega(k) = \omega'(k) - i\gamma(k)$, where $\gamma(k) > 0$, corresponding to damped waves. The presence of such singularities in the Fourier components should lead to the appearance of extrema in the spatial distribution of φ or δN. Thus, for example, the approach of the poles to the real axis ($\gamma(k) \to 0$) leads to the appearance of a discontinuous conical wave[11] (for example, the Mach cone in hydrodynamics) if the velocity of the body V_0 exceeds the phase velocity of the waves radiated by the body $\omega'(k)/k$.

In an isotropic plasma, weakly-damped ion-acoustic waves exist when $T_e/T_i \gg 1$, and in this case one can speak of Cerenkov excitation of ion-acoustic waves by a body moving in a plasma. This situation will be studied in greater detail in the next section.

3. PARTICLE-DENSITY PERTURBATION AT LARGE DISTANCES FROM THE BODY

The particle-density perturbation is obtained from the known Fourier component by using the inverse Fourier transformation:

$$\delta N(\mathbf{r}) = \frac{1}{(2\pi)^3}\int N_\mathbf{k} e^{i\mathbf{kr}}\,d\mathbf{k}. \quad (28)$$

The distance from the body $\mathbf{r} = \mathbf{r} - \mathbf{V}_0 t$ will be denoted in this section by $\mathbf{r}$. Since the Fourier component at small k has a singularity $1/k$, it is obvious without calculations that the law governing the decrease of the density perturbation at large distances from the body is r^{-2}. The angular dependence of the density perturbation at large values of r can be calculated in explicit form. Introducing a spherical coordinate system with axis along the vector $\mathbf{V}_0$, in accordance with formulas (17), we transform the expression for δN, after integrating with respect to the angle φ_l, into

$$\delta N(r,0) = -\frac{SN_0 a_0}{(2\pi)^2}\int_0^\infty k\,dk\int_0^\pi \sin\chi\,d\chi\,B(z)\,e^{ikr\cos\theta\cos\chi}J_0(kr\sin\theta\sin\chi), \quad (29)$$

where

$$B(z) = \sqrt{\pi}\,W(z)\left[1+\frac{T_e}{T_i}\left(1+i\sqrt{\pi}zW(z)\right)\right]^{-1}, \quad z = a_0\cos\chi. \quad (30)$$

The divergence in (29) at large values of k in the case of integration with respect to k is fictitious, and is connected with the fact that formula (27) for the Fourier component is valid only when $kR_0 \ll 1$. Actually the integrand decreases at large values of k in such a way that the integral converges. In order to eliminate the divergence, we introduce an effective cutoff at large k, making the following substitution in the argument of the exponential[4]

$$\cos\theta\cos\chi \to \cos\theta\cos\chi + i\delta, \quad \delta \to +0.$$

The integration with respect to k is then carried out by using the Lipschitz integral[12]

$$\int_0^\infty e^{-at}J_0(bt)\,t\,dt = \frac{a}{(a^2+b^2)^{3/2}} \quad (31)$$

with Re $a > 0$, and the value of the root $\sqrt{a^2+b^2}$ is chosen such as to satisfy the condition

$$|a+\sqrt{a^2+b^2}| > |b|. \quad (32)$$

After simple transformations we reduce (29) to the form

$$\delta N(r,\alpha) = \frac{SN_0 a_0\cos\alpha}{(2\pi)^2 r^2}\int_{-1}^{1} B(a_0 t)\frac{t\,dt}{(t^2-\sin^2\alpha-2i\delta t\,\mathrm{sign}\cos\alpha)^{3/2}}. \quad (33)$$

The angle α is the angle between the vector $\mathbf{r}$ and the axis $-\mathbf{V}_0$, so that $\cos\alpha > 0$ behind the body and $\cos\alpha < 0$ in front of the body. It is necessary to choose the negative branch of the root in (33) when $t < -|\sin\alpha|$, and the positive branch when $t > |\sin\alpha|$. The integral in (33) was tabulated numerically for the case of an isothermal plasma and for different values of a_0 in [4,5].

Taking into account the connection between the denominator of B(z) and the dispersion equation for ion-acoustic waves, we can develop a simple approximate method for finding the perturbation of the particle density, as was done in [6] also for the case of an isothermal plasma. We apply this method to the investigation of the influence of the non-isothermal character of the plasma on the structure of the perburbed region at large distances from the body. Making the substitution $a_0 t = z$ and taking into account the choice of the branches, we can extend the integration contour in (33) to $\pm\infty$ when $a_0 \gg 1$, accurate to exponentially small terms (of the order of $\sqrt{\pi}\,a_0^{-3}\cos^{-3}\alpha\exp(-a_0^2)$), reducing (33) to the form

$$\delta N(r,\alpha) = -N_0\frac{Sa_0^2}{\pi r^2}\cos\alpha F(a_0\sin\alpha), \tag{34}$$

where

$$F(a) = -\frac{1}{4\pi}\int_L \frac{B(z)z}{(z^2-a^2)^{3/2}}dz, \tag{35}$$

and the integration contour L for $\cos\alpha > 0$ is shown in Fig. 1.

The function F(a) is universal (apart from exponentially small terms) for the parameters of the problem, and does not depend on a_0, but only on $a_0 \sin\alpha$. The positive values of F(a) correspond to a decrease in the plasma density, and negative to condensation (when $\cos\alpha > 0$, i.e., behind the body). The universal function should satisfy a sui generis conservation law for the number of absorbed particles, which follows from the decrease of the particle density perturbation function in proportion to r^{-2}. Integrating over a remote sphere at whose center the body is located, we get

$$\oint \delta N\, dS = \text{const} = -SN_0.$$

Substituting (34), we find that the function F(a) has the property

$$\int_0^{a_0} aF(a)\,da = \frac{1}{2}. \tag{36}$$

Were we to consider the region in front of the body, $\cos\alpha < 0$, then the integration contour L would go around the branch points $\pm a$ from above, and could therefore be closed in the upper half-plane of z, where the integrand has no singularities; that is to say, in front of the body, at large distances from it, the plasma is not perturbed, $\delta N = 0$. When $\cos\alpha > 0$, such a closing of the integration contour leads to an integral over the edges of the cut joining the branch point $\pm a$, which is difficult to calculate analytically.

In the lower half-plane, the function B(z) has poles when its denominator vanishes, i.e.,

$$1 + \frac{T_e}{T_i}(1 + i\sqrt{\pi}zW(z)) = 0. \tag{37}$$

This condition is a dispersion equation for the ion-acoustic waves when $kd_e \ll 1$, provided $z = w/kv_i$. Since $W^*(z) = W(-z^*)$, the poles of the function B(z) are located symmetrically relative to the imaginary axis and we can confine ourselves to consideration of the quadrant $x > 0$, $y < 0$. Asymptotically, at large $|z|$, the poles should lie near the line $\varphi = -\pi/4$. Since the Kramp function has in the lower half-plane the asymptotic form

$$W(z) = 2e^{-z^2} + \frac{i}{\sqrt{\pi}z}\left[1 + \frac{1}{2z^2} + \ldots\right], \tag{38}$$

it follows that, by substituting in (37) $z = \rho\exp[i(-\pi/4 + \alpha)]$ and expanding in powers of α, it is easy to obtain an asymptotic expression for the poles:

$$\rho_n^2 = \frac{3\pi}{4} + 2\pi n, \quad \alpha_n = \frac{\ln 4\pi\rho_n^2\eta^2}{4\rho_n^2}. \tag{39}$$

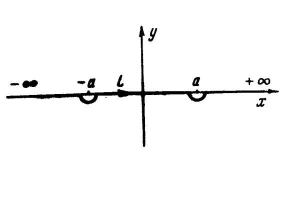

FIG. 1. The integration contour L.

Of course, only the first branches of the ion-acoustic oscillations of the plasma which have the smallest damping, have a physical meaning; these are the branches that should make the main contribution to the formation of the perturbed region behind the body. We present the values of the first three poles of B(z), obtained by numerically solving Eq. (36) for two values of $\eta = T_e/T_i$:

	z_0	z_1	z_2
$\eta = 1$:	$1{,}48 - i\ 0{,}58$	$2{,}36 - i\ 1{,}85$	$3{,}00 - i\ 2{,}49$
$\eta = 4$:	$2{,}05 - i\ 0{,}21$	$2{,}57 - i\ 1{,}56$	$3{,}26 - i\ 2{,}13$

The poles are numbered in increasing order of their moduli. When η increases, the strongest decrease of the relative damping γ/ω' occurs for the first branch of the ion-acoustic oscillations (we have $\gamma'/\omega \sim 0.1$ already for $\eta = 4$), and therefore the relative contribution of the first pole to the formation of the perturbed region should increase with increasing η.

By finding the residues of B(z) at its poles and using the symmetry in the arrangement of the poles, we represent the universal function by means of the following sum over the poles that lie only in the lower right quadrant:

$$F(a) = -\text{Re}\sum_{n=0}^{\infty}\frac{z_n(1+1/\eta)}{[2z_n^2-(1+\eta)]}(z_n^2-a^2)^{-3/2}; \tag{40}$$

$\sqrt{z^2-a^2}$ is calculated in the right quadrant, so that its real part is positive and the imaginary part negative. At large n, the sum (40) over n converges in absolute value like n^{-2}.

Figure 2 shows the universal functions for $\eta = 1$ and $\eta = 4$, constructed using the first three poles presented above. The contribution of the first pole to F(a) is represented by the dashed line. Even in the case of an isothermal plasma, the single-pole approximation makes it possible to describe quite satisfactorily the angular dependence of the density perturbation. Figure 3 shows the angular dependences of the density perturbation $\Phi(a_0, \alpha) = \cos\alpha\ F(a_0 \sin\alpha)$, constructed in accordance with these universal functions for different values of a_0.

An isothermal plasma is always rarefied behind the body; the maximum rarefaction is reached at $a = 1.4$, i.e., on a cone with angle $\alpha \approx \sin^{-1}(1.4/a_0)$, analogous to the Mach cone. When η increases, a transition takes

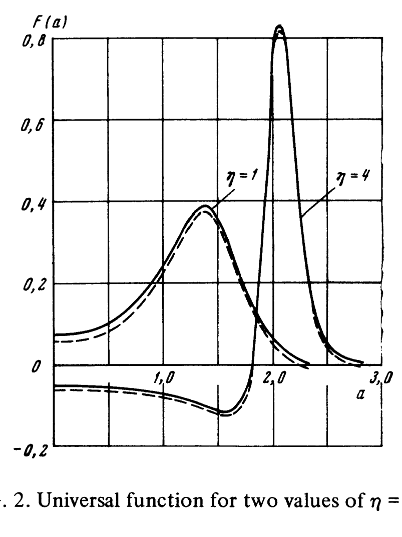

FIG. 2. Universal function for two values of $\eta = T_e/T_i$.

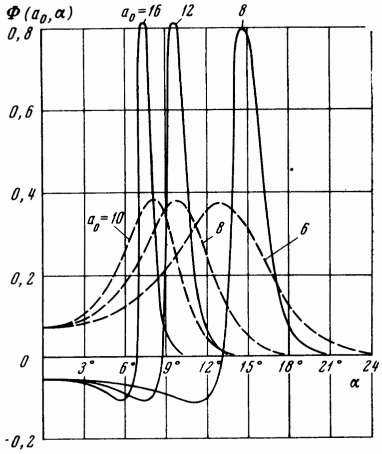

FIG. 3. Angle factors for different values of a_0 and two values of η: solid curves – $\eta = 4$, dashed – $\eta = 1$.

place from rarefaction on the axis to condensation. This transition can be investigated in greater detail. We note that when $a = 0$ the branch points $\pm a$ in the integrand of $F(a)$ merge into a pole, which is circuited from below. By closing the integration contour L in the upper half-plane we can obtain the exact value of the universal function and of its derivatives at $a = 0$. For example,

$$F(0) = -\frac{i}{2}B'(z)\Big|_{z=0} = \frac{(1+\eta)-\pi\eta/2}{(1+\eta)^2},$$
$$F''(0) = -\frac{i}{4}B'''(z)\Big|_{z=0}. \quad (41)$$

The transition on the axis $(a = 0)$ from rarefaction to condensation occurs consequently at $\eta = 1.76$; with further increase of η, the region of condensation broadens and the maximal condensation at $\eta > 2.43$ begins to be reached already not on the axis, but on the maximum-condensation cone, which is imbedded in the Mach cone. When $\eta = 4$, the angle of maximum condensation is determined by the relation $\alpha \simeq \sin^{-1}(1.6/a_0)$, and the angle of maximum rarefaction by $\alpha \approx \sin^1(2.08/a_0)$.

If we decrease η, starting with $\eta = 1$, then the rarefaction on the Mach cone decreases, the Mach cone becomes less pronounced because of the increased attenuation of the ion-acoustic waves, and when $\eta < 0.23$ the maximum rarefaction is reached already on the axis behind the body and not on the Mach cone. The values $\eta = 0.23$ and $\eta = 2.43$ are the roots of the cubic equation (in η) obtained by equating $F''(0)$ to zero, the third root being negative and having no physical meaning. When η tends to zero, the influence of the electric field on the ions decreases and when $\eta = 0$ the universal function should go over into $F(a) \approx \exp(-a^2)$, obtained in [7] without allowance for the influence of the self-consistent field on the ions.

Particular interest attaches to the case of a strongly isothermal plasma, $T_e/T_i \gg 1$, in connection with the possibility of its hydrodynamic description.[13] For ion-acoustic waves in a strongly non-isothermal plasma, the collisionless Landau damping due to resonant interaction between the wave and the ions becomes exponentially small, and the principal role is assumed by collisionless damping on electrons, and therefore it is necessary to retain in the denominator of B(z) the small electronic term that takes this damping into account:

$$B(z) = \sqrt{\pi}W(z)\left[1 + \frac{T_e}{T_i}(1 + i\sqrt{\pi}zW(z)) + i\sqrt{\pi}z\sqrt{\frac{T_i}{T_e}\frac{m}{M}}\right]^{-1}. \quad (42)$$

The function B(z) has the pole at

$$z_0 = \sqrt{\frac{T_e}{2T_i}} - i\sqrt{\frac{\pi}{16}\frac{m}{M}\frac{T_e}{T_i}}, \quad (43)$$

corresponding to ion-acoustic waves with $kd_e \ll 1$. If the velocity of the body is supersonic, $a_0 \gg \sqrt{T_e/2T_i}$, then this pole will lie inside the integration contour L and make the main contribution to the universal function. Using the asymptotic form of the Kramp function, we obtain the residue of B(z):

$$\operatorname{Res} B(z_0) \approx i/2. \quad (44)$$

Consequently

$$F(a) = -\operatorname{Re}[z_0/2(z_0^2 - a^2)^{3/2}]. \quad (45)$$

Inside the Mach cone, when $a^2 \ll x_0^2$, we obtain in accordance with (45) a condensation:

$$F(a) = -x_0/2(x_0^2 - a^2)^{3/2}. \quad (46)$$

The maximum value $F(a) \approx \frac{1}{8}x_0^{-1/2}y_0^{-3/2}$, which determines the maximum rarefaction on the Mach cone, is reached at $a \approx x_0 + 0.325\,y_0$, the maximum condensation is reached at $a_3 \approx x_0 - 1.376\,y_0$, and the transition from condensation to rarefaction occurs at $a_2 \approx 0.577\,y_0$; behind the Mach cone, the perturbation is proportional to y_0 and decreases rapidly. The angular width of the spreading of the Mach cone (the maximum-rarefaction cone) is consequently determined by the damping of the ion-acoustic waves, $\Delta\alpha = \Delta a/a_0 \approx (a_1 - a_2)/a_0 = 0.9p_0/a_0$. A plot of $F(a)$ for $T_e/T_i \gg 1$ is shown symbolically in Fig. 4.

Gurevich and Pitaevskiĭ[3] investigated the perturbation of a strongly non-isothermal plasma in the case of supersonic motion of the body on the basis of the hydrodynamic equations. Taking the limit $|z| \gg 1$ in formula (27) for the Fourier component of the density perturbation, and neglecting the damping, we obtain an expression for the Fourier component of the density perturbation in the hydrodynamic approximation:

$$N_{\mathbf{k}} = -iSN_0V_0\frac{\mathbf{kV}_0}{(\mathbf{kV}_0)^2 - k^2c_s^2} \quad (47)$$

($c_s = (\kappa T_e/M)^{1/2}$ is the velocity of the ion sound in a strongly non-isothermal plasma). When taking the in-

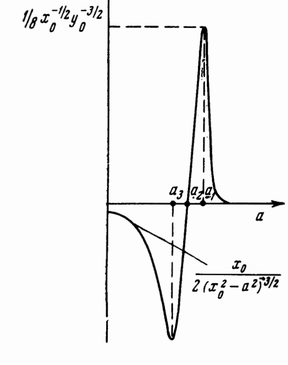

FIG. 4. Universal function for $\eta \gg 1$.

verse Fourier transform, expression (47) must be regarded as the limiting case of an infinitesimally small damping, i.e., $\mathbf{k}\cdot\mathbf{V}_0 \to \mathbf{k}\cdot\mathbf{V}_0 + i\nu$, $\nu \to +0$. We then get

$$\delta N = \frac{SN_0}{2\pi r^2}\cdot \tag{48}$$

$$\cdot\begin{cases} (c_s/V_0)\cos\alpha\,[(c_s/V_0)^2 - \sin^2\alpha]^{-3/2}, & \sin\alpha < c_s/V_0 \\ -\infty & \sin\alpha = c_s/V_0. \\ 0, & \sin\alpha > c_s/V_0 \end{cases}$$

A similar expression can be obtained directly from formulas (34) and (45) by letting the damping y_0 tend to zero in formula (45).

We note now that in [3], in the investigation of the perturbation of a plasma with $T_e/T_i \gg 1$ in the hydrodynamic approximation, an error has crept in and distorted the physical meaning of the results obtained there: first, there is an error in sign in the formula for the Fourier component of the density perturbation, and second, this error is repeated in the formula analogous to (48) for $\sin\alpha < c_s/V_0$, as a result of which they have erroneously obtained inside the Mach cone rarefaction rather than the required condensation; the singularity of δN on the Mach cone was not investigated in [3]. The result (48) agrees with Landau's conclusion[14] that in hydrodynamics, at large distances from the body, there is produced on the Mach cone a double shock wave (one on the condensation cone and another on the rarefaction cone). The intensity of the shock wave in hydrodynamics decreases with distance from the body like $r^{-4/3}$, which leads to a divergence on the Mach cone in the asymptotic terms of the density perturbation proportional to r^{-2}.

The picture of supersonic hydrodynamic flow around an absorbing body admits of a simple interpretation.[14] The rapidly moving body cuts in the plasma a semi-infinite cylinder with axis along V_0. The spreading out of this perturbation at large distances from the body by cylindrical ion-acoustic waves is described by the wave equation

$$\Delta\delta N - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\delta N = 0 \tag{49}$$

with initial conditions

$$\delta N(\rho, 0) = -SN_0\delta(\rho), \quad \partial\delta N/\partial t|_{t=0} = 0. \tag{50}$$

(The problem is analogous to the spreading of an initial concentrated perturbation for an infinite membrane.) Using the Laplace transformation in time and the two-dimensional Fourier transformation in the coordinates, we obtain

$$\delta N = -\frac{SN_0}{2\pi}\,\mathrm{Re}\,\frac{\delta - ic_s t}{[(\delta - ic_s t)^2 + \rho^2]^{3/2}} \quad \delta \to +0. \tag{51}$$

Substituting here $t = -z/V_0 = -(r/V_0)\cos\alpha$ and $\rho = r\sin\alpha$, we obtain for δN at $V_0/c_s \gg 1$ an expression that coincides with (48).

It is interesting to note that in the case of flow around an infinite cylinder (the cylinder axis is perpendicular to $\mathbf{V}_0$), the spreading-out of the perturbation occurs by means of one-dimensional ion-acoustic waves (in analogy with the spreading of the initial concentrated perturbation on a string). In this case, as the damping tends to zero, a δ-function singularity is produced on the Mach cone, and the body excites only a rarefaction shock wave.

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