TRANSLATIONAL INVARIANCE IN UNIFORM FIELDS AND THE EQUATION FOR THE DENSITY MATRIX IN THE WIGNER REPRESENTATION

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It is shown that the density matrix for an electron which is in uniform electric and magnetic fields and interacts with long-wavelength equilibrium phonons in a homogeneous crystal is invariant with respect to translations which are applied together with appropriate gauge transformations of the potentials of the fields. It follows that in the Wigner representation the density matrix does not depend on the coordinate and the canonical momentum separately, but depends on them only through the kinetic momentum, i.e., through the velocity. The equation for the density matrix in velocity space is derived in the weak-coupling approximation.

INTRODUCTION

IN the treatment of galvanomagnetic effects in uniform fields $\mathbf{E}$ and $\mathbf{H}$ by means of the classical kinetic equation, the state of the electrons is described by a distribution function $f(\mathbf{k}, t)$, where $\mathbf{k} = m\mathbf{v}$ is the kinetic momentum of the electron. There are also situations in which the classical kinetic equation cannot be applied. This is the case, for example, in quantizing magnetic fields in which the quantization energy $\epsilon_H = \hbar eH/mc$ is comparable with the average energy of an electron. It is natural to suppose that the kinetic equation may not be applicable in electric fields of large amplitude or frequency, when the characteristic energies $\epsilon_E = (\hbar eE)^{2/3}/m^{1/3}$ and $\epsilon_\omega = \hbar\omega$ are comparable with ϵ. In all such cases one must use the equation for the density matrix, including the fields $\mathbf{E}$ and $\mathbf{H}$ in the Hamiltonian of the electron (cf., e.g.,[1,2]).

One has then, however, to deal with the following unpleasant circumstance. The magnetic field is described by a vector potential $\mathbf{A}$ which depends on the coordinates; therefore the Hamiltonian is not translationally invariant, even in uniform fields. The result is that the density matrix $(\mathbf{p}|\hat{\rho}(t)|\mathbf{p}')$ in the representation of the canonical momentum $\mathbf{p}$ is not diagonal with respect to $\mathbf{p}$. There are two aspects of this nondiagonal character. First, whereas the diagonal elements have a classical analog—the distribution function $f(\mathbf{k}, t)$—the nondiagonal elements have no such analog. Second, in the quantum case the function to be determined—$(\mathbf{p}|\hat{\rho}(t)|\mathbf{p}')$ instead of $f(\mathbf{k}, t)$—has twice as many arguments (not counting t).

On the other hand it is clear that in uniform fields different points of space are physically indistinguishable, since at all points the electron is acted on by the same force. Therefore the system is in some sense translationally symmetric. It is natural to expect that consideration of this symmetry can impose some restrictions on the dependence of the elements of the density matrix on $\mathbf{p}$ and $\mathbf{p}'$, and thus simplify the equation for $\hat{\rho}$.

In the present paper it is shown that if we use the Wigner representation for the density matrix, $w(\mathbf{r}, \mathbf{p}, t)$, then it follows from the translational invariance that w depends not on $\mathbf{r}$ and $\mathbf{p}$ separately, but only on the kinetic momentum $\mathbf{k}$:

$$w(\mathbf{r}, \mathbf{p}, t) = f(\mathbf{k}, t), \quad \mathbf{k} = \mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r}, t). \tag{1}$$

This fundamental property restores the velocity space to full status in the quantum case. As in the classical case, we can speak of the velocity distribution of the electrons (with the stipulation that the probability f can be negative). Owing to (1) the number of arguments on which the density matrix depends is the same as in the classical case. According to the known properties of the density w the quantum distribution goes over directly into the classical distribution for $\hbar \to 0$. It may be pointed out here that the convenience of using the Wigner representation in a magnetic field has also been noted in[3].

It is physically obvious that the property (1) is not destroyed when the electron is scattered by a system of long-wavelength phonons in thermodynamic equilibrium in a homogeneous crystal, since when only such photons are considered the crystal can be regarded as continuous, and then there are no spatially distinguished points.

We shall consider "one" electron in a uniform and stationary field $\mathbf{H}$ and a uniform (and perhaps alternating) field $\mathbf{E}(t)$, and interacting weakly with a phonon thermostat. That the interaction is weak means that $\epsilon \gg \hbar/\tau$, where τ is the lifetime of the electron in a state, associated with the electron-phonon interaction. By decoupling the hierarchic chain of electron-phonon correlators, we can obtain the equation for $\hat{\rho}$ and then transform it into an equation for $f(\mathbf{k}, t)$. This procedure has been carried out for the lowest level of decoupling, but it is clear that since (1) follows from a symmetry, an equation for $f(\mathbf{k}, t)$ can be obtained with decoupling of the chain at any level. Since only one electron is considered, effects of degeneracy and of electron-electron interaction are not taken into account; if desired, these effects can be included in the same scheme.

1. THE EQUATION FOR THE DENSITY MATRIX OF THE ELECTRON

We consider an electron with Hamiltonian $\hat{H}$, interacting with a system of oscillators. Let s denote a type of oscillator, Ω_s its frequency, and $\hat{b}_s$ the creation operator (sic). Then the Hamiltonian of the oscillator is

$$\hat{W}_s = \hbar\Omega_s(\hat{b}_s^+\hat{b}_s + {}^1/_2), \tag{1.1}$$

and its interaction with the electron is

$$\hat{V}_s = c_s\hat{b}_s\hat{\chi}_s + \text{Herm. adj.} \tag{1.2}$$

where c_s is the appropriate matrix element and $\hat{\chi}_s$ is the electron operator. It is assumed that the interaction of the electron with the oscillator is weak; the small parameter is taken to be included in c_s.

The density matrix $\hat{\eta}$ of the entire system is governed by the equation

$$\frac{\partial}{\partial t}\hat{\eta} = -\frac{i}{\hbar}\Big[\hat{H} + \sum_s \hat{W}_s + \sum_s \hat{V}_s, \hat{\eta}\Big]. \tag{1.3}$$

Let us introduce the following notations for traces: sp is the trace with respect to the electron variables, tr_s, with respect to the variables of oscillator s, Tr, with respect to those of all the oscillators, Tr_s, those of all the oscillators except s, and $\mathrm{Tr}_{ss'}$, those of all oscillators except s and s′, with $s \neq s'$.

Our ultimate interest is in the equation for the density matrix of the electron,

$$\hat{\rho} = \mathrm{Tr}\hat{\eta}. \tag{1.4}$$

It will be derived by the well known method of decoupling a hierarchic chain,[4,5] exactly as this was done in[2]. Here it is assumed that the system of oscillators forms a thermostat for the electron, or in other words that the density matrix of each oscillator

$$\hat{\sigma}_s = \mathrm{sp}\,\mathrm{Tr}_s\hat{\eta} \tag{1.5}$$

has its equilibrium form, i.e.,

$$\hat{\sigma}_s = Z_s^{-1}\exp\{-\beta\hat{W}_s\}, \quad \beta = 1/T, \quad Z_s = N_s + 1, \quad N_s = (\exp\{\beta\hbar\Omega_s\} - 1)^{-1}. \tag{1.6}$$

The derivation of the equation for $\hat{\rho}$ also involves the quantities

$$\mathrm{Tr}_s\hat{\eta} \equiv \hat{\eta}_s = \hat{\rho}\hat{\sigma}_s + \hat{\varkappa}_s, \quad \mathrm{Tr}_{ss'}\hat{\eta} \equiv \hat{\eta}_{ss'} = \hat{\rho}\hat{\sigma}_s\hat{\sigma}_{s'} + \hat{\varkappa}_{ss'} \quad (s \neq s'). \tag{1.7}$$

Because c_s is small the correlators κ_s and $\kappa_{ss'}$ are also small.

Applying the operation Tr to (1.3), we find

$$\frac{\partial}{\partial t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{i}{\hbar}\sum_s \mathrm{tr}_s[\hat{V}_s, \hat{\eta}_s]. \tag{1.8}$$

This equation becomes closed if we neglect the second-order terms $\hat{V}_s\hat{\varkappa}_s$. Then, however, the interaction drops out altogether. In fact, what remains is

$$\sum_s \mathrm{tr}_s[\hat{V}_s, \hat{\rho}\hat{\sigma}_s] = \sum_s [\mathrm{tr}_s\hat{V}_s\hat{\sigma}_s, \hat{\rho}]; \tag{1.9}$$

and then $\mathrm{tr}_s\hat{V}_s\hat{\sigma}_s$ reduces to $\mathrm{tr}_s\hat{b}_s\hat{\sigma}_s$ and $\mathrm{tr}_s b_s^+\hat{\sigma}_s$, and these last quantities vanish, since $\hat{\sigma}_s$ is diagonal in the occupation numbers of the oscillators. Therefore we have to consider the correlation functions of a larger number of particles. Applying the operation Tr_s to (1.3), we find

$$\frac{\partial}{\partial t}\hat{\eta}_s = -\frac{i}{\hbar}[\hat{H} + \hat{W}_s + \hat{V}_s, \hat{\eta}_s] - \frac{i}{\hbar}\sum_{s'}{}' \mathrm{tr}_{s'}[\hat{V}_{s'}, \hat{\eta}_{ss'}]. \tag{1.10}$$

Here we can now neglect the second-order terms $\hat{V}_{s'}\hat{\varkappa}_{ss'}$. We then have

$$\frac{\partial}{\partial t}\hat{\varkappa}_s = -\frac{i}{\hbar}[\hat{H} + \hat{W}_s, \hat{\varkappa}_s] - \frac{i}{\hbar}[\hat{V}_s, \hat{\rho}\hat{\sigma}_s]. \tag{1.11}$$

The system of Eqs. (1.8) and (1.11) for $\hat{\varkappa}_s$ and $\hat{\rho}$ is closed. Elimination of $\hat{\varkappa}_s$ from this system gives the required equation for $\hat{\rho}$. It can be seen from (1.11) that $\hat{\varkappa}_s$ does not depend on the other quantities $\hat{\varkappa}_{s'}$ with $s' \neq s$.

On the other hand, in (1.8) the terms for interaction with different oscillators add without interference. Therefore in the derivation of the equation for $\hat{\rho}$ we can keep one oscillator (dropping the index s), and then in the final result we can write the sum over all the oscillators.

To solve (1.11) we introduce the matrix for the motion of an electron not interacting with the oscillators:

$$\hat{S}(t, t_0) = P\exp\Big\{-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,\hat{H}(t')\Big\} \tag{1.12}$$

and the interaction representation for the phonon operators:

$$\hat{V}(t) = c\hat{b}(t)\hat{\chi} + \text{Herm. adj.}, \quad \hat{b}(t) = \exp\Big\{+\frac{i}{\hbar}\hat{W}t\Big\}\hat{b}\exp\Big\{-\frac{i}{\hbar}\hat{W}t\Big\}. \tag{1.13}$$

Then for a causal initial condition[4] corresponding to adiabatic turning-on of the interaction,

$$\hat{\varkappa}(-\infty) = 0, \tag{1.14}$$

we find

$$\hat{\varkappa}(t) = -\frac{i}{\hbar}\int_{-\infty}^{t} dt_0\hat{S}(t, t_0)[\hat{V}(t_0 - t), \hat{\rho}(t_0)\hat{\sigma}]\hat{S}(t, t_0)^+, \tag{1.15}$$

where the integral is to be understood as having a convergence factor $\exp\{\lambda t_0\}$ ($\lambda > 0$), with $\lambda \to 0$. Substituting (1.15) in (1.8), we find the equation for the density matrix of the electron,

$$\frac{\partial}{\partial t}\hat{\rho} + \frac{i}{\hbar}[\hat{H}, \hat{\rho}] = I(\hat{\rho}\,|\,t), \tag{1.16}$$

where

$$I(\hat{\rho}\,|\,t) = -\frac{1}{\hbar^2}\int_{-\infty}^{t} dt_0\,\mathrm{tr}[\hat{V}, \hat{S}(t, t_0)[\hat{V}(t_0 - t), \hat{\rho}(t_0)\hat{\sigma}]\hat{S}(t, t_0)^+]. \tag{1.17}$$

Substituting (1.6) in this, we can calculate the trace over the oscillator variables by means of the easily verified relations

$$\mathrm{tr}\{\hat{b}\hat{b}(t)\hat{\sigma}\} = \mathrm{tr}\{\hat{b}^+\hat{b}^+(t)\hat{\sigma}\} = 0, \quad \mathrm{tr}\{\hat{b}\hat{b}^+(t)\hat{\sigma}\} = \exp\{+i\Omega t\}(N+1), \quad \mathrm{tr}\{\hat{b}^+\hat{b}(t)\hat{\sigma}\} = \exp\{-i\Omega t\}(N). \tag{1.18}$$

After this the right member of (1.17) takes the following form:

$$I(\hat{\rho}\,|\,t) = I^{(+)}(\hat{\rho}\,|\,t) + I^{(-)}(\hat{\rho}\,|\,t). \tag{1.19}$$

Here the first term is due to the emission of oscillator quanta,

$$I^{(+)}(\hat{\rho}\,|\,t) = (N+1)\frac{|c|^2}{\hbar^2}\int_{-\infty}^{t} dt_0\exp\{-i\Omega(t - t_0)\} \times [\hat{S}(t, t_0)\hat{\chi}^+\hat{\rho}(t_0)\hat{S}(t, t_0)^+, \hat{\chi}] + \text{Herm. adj.} \tag{1.20}$$

The second term is due to the absorption of quanta and is obtained from the first by replacing (N + 1) by N, $+\Omega$ by $-\Omega$, and $\hat{\chi}$ by $\hat{\chi}^+$. The expression (1.19) (summed over all the oscillators) is the quantum analog of the collision term in the kinetic equation. As will be shown, in the quasiclassical limit the two terms of the commutator in (1.20) correspond to the entering and leaving terms in the Boltzmann form of the collision term. Therefore we write

$$I^{(+)}(\hat{\rho}\,|\,t) = -A^{(+)}(\hat{\rho}\,|\,t) + B^{(+)}(\hat{\rho}\,|\,t), \tag{1.21}$$

where the analog of the leaving term (owing to emissions) is

$$A^{(+)}(\hat{\rho}\,|\,t) \tag{1.22}$$
$$= (N+1)\frac{|c|^2}{\hbar^2}\int_{-\infty}^{t} dt_0 \exp\{-i\Omega(t-t_0)\}$$
$$\times\ \hat{\chi}\hat{S}(t, t_0)\hat{\chi}^+\hat{\rho}(t_0)\,\hat{S}(t, t_0)^+ + \text{Herm. adj.}$$

and the analog of the entering term (owing to emissions) is

$$B^{(+)}(\hat{\rho}\,|\,t) \tag{1.23}$$
$$= (N+1)\frac{|c|^2}{\hbar^2}\int_{-\infty}^{t} dt_0 \exp\{-i\Omega(t-t_0)\}$$
$$\times \hat{S}(t, t_0)\,\hat{\chi}^+\hat{\rho}(t_0)\,\hat{S}(t, t_0)^+\,\hat{\chi} + \text{Herm. adj.}$$

The terms leaving and for entering owing to absorptions, $A^{(-)}$ and $B^{(-)}$, can be written in analogous forms (with the replacements indicated above).

2. TRANSLATIONAL PROPERTIES OF THE DENSITY MATRIX

For the study of the conductivity in a uniform external field $\mathbf{E}(t)$, with the crystal placed in a stationary uniform magnetic field $\mathbf{H}$, we write the Hamiltonian of the electron in the form

$$\hat{H} = \varepsilon(\hat{\mathbf{k}}) + e\varphi(\mathbf{x}, t), \quad \varepsilon(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}, \quad \hat{\mathbf{k}} = \hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t). \tag{2.1}$$

where $\epsilon(\mathbf{k})$ is the kinetic energy of the electron in the effective-mass approximation. The oscillators are phonons, so that s denotes the branch, the polarization, and the momentum $\mathbf{q}$, and

$$\chi_s = \exp\left\{+\frac{i}{\hbar}\mathbf{qx}\right\}. \tag{2.2}$$

In solving Eq. (1.16) for the density matrix $\hat{\rho}$ it is natural to use the properties of $\hat{\rho}$ and $\hat{S}$ which follow from the homogeneity of the system. If $\mathbf{H} = 0$, then we can choose a gauge for the potentials in which $\varphi = 0$ and $\mathbf{A}$ is independent of $\mathbf{x}$, as has been done in[2], for example. With such a gauge the Hamiltonian $\hat{H}$ does not contain $\mathbf{x}$ and is invariant with respect to translation by any vector $\mathbf{a}$:

$$\hat{H} = \hat{T}^+\hat{H}\hat{T}, \quad \hat{T} = \exp\left\{-\frac{i}{\hbar}\hat{\mathbf{p}}\mathbf{a}\right\}. \tag{2.3}$$

For spatially uniform current states it is to be expected that $\hat{\rho}$ is also invariant with respect to the translation, i.e.,

$$\hat{\rho}(t) = \hat{T}^+\hat{\rho}(t)\hat{T}. \tag{2.4}$$

From this it follows, in particular, that $\hat{\rho}$ is diagonal in the momentum representation.

If, however, $\mathbf{H} \neq 0$, we cannot eliminate $\mathbf{x}$ from the Hamiltonian. In what follows we shall choose a gauge in which the constant part of the electric field $\mathbf{E}$ is described by a scalar potential and the alternating part $\tilde{\mathbf{E}}(t)$, by a vector potential, i.e.,

$$\varphi(x) = -\bar{\mathbf{E}}\mathbf{x}, \quad \mathbf{A}(\mathbf{x}, t) = \bar{\mathbf{A}}(\mathbf{x}) + \tilde{\mathbf{A}}(t), \tag{2.5}*$$
$$\bar{A}(x) = \frac{1}{2}[\mathbf{Hx}], \quad -\frac{1}{c}\frac{\partial}{\partial t}\tilde{\mathbf{A}}(t) = \tilde{\mathbf{E}}(t), \quad \mathbf{E}(t) = \bar{\mathbf{E}} + \tilde{\mathbf{E}}(t).$$

Then $\hat{T}^+\hat{H}\hat{T} = \hat{H}' \neq \hat{H}$, and it would seem that the system does not possess translational invariance. It is obvious, however, that in uniform fields different points of space are physically indistinguishable, and that in fact there must be a translational invariance of some sort. A formal expression of this is the fact that $\hat{H}'$ is obtained from $\hat{H}$ by translation of the "unphysical" potentials

$$\bar{\mathbf{A}}(\mathbf{x}) \to \bar{\mathbf{A}}'(\mathbf{x}) = \bar{\mathbf{A}}(\mathbf{x}+\mathbf{a}), \quad \varphi(\mathbf{x}) \to \varphi'(\mathbf{x}) = \varphi(\mathbf{x}+\mathbf{a}), \tag{2.6}$$

and this translation, in turn, is equivalent to a gauge transformation

$$\mathbf{A}(\mathbf{x}+\mathbf{a}) = \bar{\mathbf{A}}(\mathbf{x}) + \nabla F(\mathbf{x}, t), \quad \varphi(\mathbf{x}+\mathbf{a}) = \varphi(\mathbf{x}) - \frac{1}{c}\frac{\partial}{\partial t}F(\mathbf{x}, t) \tag{2.7}$$

with the function

$$F(\mathbf{x}, t) = \bar{\mathbf{E}}\mathbf{a}ct + {}^1\!/_2[\mathbf{Ha}]\mathbf{x} = -\varphi(\mathbf{a})ct + \bar{\mathbf{A}}(\mathbf{a})\mathbf{x}. \tag{2.8}$$

Therefore it is natural to expect that when there are static fields one must use instead of $\hat{T}$ a translation accompanied by a gauge transformation, i.e.,

$$\hat{T}_{EH} = G\hat{T}, \quad G = \exp\left\{+\frac{ie}{\hbar c}F\right\}. \tag{2.9}$$

With the chosen gauge for the constant part of the vector potential in (2.5) the factors G and $\hat{T}$ commute, as can be easily verified directly; in addition, we can write

$$\hat{T}_{EH}(t) = T_E(t)\,\hat{T}_H, \tag{2.10}$$
$$T_E(t) = \exp\left\{+\frac{i}{\hbar}e\bar{\mathbf{E}}\mathbf{a}t\right\},$$
$$\hat{T}_H = \exp\left\{-\frac{i}{\hbar}\left(\hat{\mathbf{p}} + \frac{e}{c}\bar{\mathbf{A}}\right)\mathbf{a}\right\} = \exp\left\{-\frac{ie}{\hbar c}\bar{\mathbf{A}}\mathbf{a}\right\}\hat{T},$$

where all the factors commute. We note that the "translation" $\hat{T}_H$ has already been used in[6].

When there are static fields one must have instead of (2.4) for spatially uniform current states the equation

$$\hat{\rho}(t) = \hat{T}_{EH}(t)^+\,\hat{\rho}(t)\,\hat{T}_{EH}(t) = \hat{T}_H^+\hat{\rho}(t)\hat{T}_H \equiv \hat{\rho}''(t). \tag{2.11}$$

Its correctness can be easily verified by means of the following consideration. Let the fields $\mathbf{E}$ and $\mathbf{H}$ be turned on adiabatically, beginning at $t = -\infty$, when the distribution is at equilibrium. Since at $t = -\infty$ there are no fields, $\hat{T}_{EH}$ is the same as $\hat{T}$, and since $\hat{\rho}(-\infty)$ commutes with $\hat{T}$, $\hat{\rho}''(t)$ and $\hat{\rho}(t)$ are equal at $t = -\infty$. We now calculate the rate of change of $\hat{\rho}''(t)$, noting that

$$\hat{H}'' \equiv \hat{T}_H^+\hat{H}\hat{T}_H = \hat{H} - e\bar{\mathbf{E}}\mathbf{a}, \tag{2.12}$$

and neglecting the time derivatives of $\mathbf{H}$, since it is turned on adiabatically. Then we find

$$\frac{\partial}{\partial t}\hat{\rho}''(t) = \hat{T}_H^+\left(-\frac{i}{\hbar}[\hat{H}, \hat{\rho}]\right)\hat{T}_H = -\frac{i}{\hbar}[\hat{H}'', \hat{\rho}'] = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}''], \tag{2.13}$$

*$[\mathbf{Hx}] \equiv \mathbf{H} \times \mathbf{x}$.

From this it can be seen that $\hat{\rho}''(t)$ is governed by the same Hamiltonian as $\hat{\rho}(t)$, and since these quantities are equal at $t = -\infty$, they are also equal for all t, which proves (2.11). This relation (2.11) is the desired restriction on the form of the density matrix, which is imposed by the translational invariance of the system.

The properties which the matrix of motion (1.12) has as a result of translational invariance can be found in an analogous way. Here we must remember that, unlike the density matrix, which is a "one-time" quantity

$$\hat{\rho}(t) = \mathrm{Av}|\psi(t))(\psi(t)|, \tag{2.14}$$

the matrix of motion is a "two-time" quantity

$$\hat{S}(t, t_0) = |\psi(t))(\psi(t_0)|. \tag{2.15}$$

Therefore it is natural to expect the equation

$$\hat{T}_{\mathrm{EH}}(t)^{+}\hat{S}(t, t_0)\hat{T}_{\mathrm{EH}}(t_0) = \hat{S}(t, t_0). \tag{2.16}$$

In fact, it can be verified that the left side of this last equation satisfies the same equation as the right side (with respect to both t and t_0), and for $t = t_0$ both sides are equal to unity; this proves that they are equal for all t and t_0.

Up to this point we have not taken the interaction of the electron with the phonons into account. It may be supposed, however, that the relation (2.11) holds also in the presence of this interaction, since in an equilibrium system of long-wavelength phonons there are no distinguished points in space, just as for the external fields. To convince oneself that this surmise is correct, one must show that the property (2.11) is not lost through the scattering by the phonons, i.e., that it follows from (2.11) that

$$\hat{T}_{\mathrm{H}}^{+} I(\hat{\rho})\hat{T}_{\mathrm{H}} = I(\hat{\rho}). \tag{2.17}$$

It is simpler to construct the proof for infinitesimal translations; that is, to show that

$$[\hat{\mathbf{g}}, I(\hat{\rho})] = 0, \quad \hat{\mathbf{g}} = \hat{\mathbf{p}} + \frac{e}{c}\bar{\mathbf{A}}, \tag{2.18}$$

if

$$[\hat{\mathbf{g}}, \hat{\rho}] = 0. \tag{2.19}$$

To prove (2.17) it is convenient first to calculate the following commutators:

$$[\chi, \hat{\mathbf{g}}] = +\mathbf{q}\hat{\chi}, \qquad [\chi^{+}, \hat{g}] = -\mathbf{q}\chi^{+}; \tag{2.20}$$

$$[\hat{S}(t, t_0), \hat{\mathbf{g}}] = -e\bar{\mathbf{E}}(t - t_0)\hat{S}(t, t_0),$$
$$[\hat{S}(t, t_0)^{+}, \hat{\mathbf{g}}] = +e\bar{\mathbf{E}}(t - t_0)\hat{S}(t, t_0)^{+}. \tag{2.21}$$

The commutators (2.20) are found by an elementary calculation, and (2.21) follows from (2.16) for an infinitesimal translation. Using the Jacobi identity and Eqs. (2.20), (2.21), and (2.19), one can easily check by direct calculation that

$$[[\chi, \hat{S}\chi^{+}\hat{\rho}\hat{S}^{+}], \hat{\mathbf{g}}] = 0,$$
$$[[\chi^{+}, \hat{S}\chi\hat{\rho}\hat{S}^{+}], \hat{\mathbf{g}}] = 0, \tag{2.22}$$

which proves (2.18).

Thus it has been proved that even in the presence of the electron-phonon interaction the density matrix $\hat{\rho}$ of the electron, found from the solution of Eq. (1.16), satisfies the relation (2.11).

3. THE EQUATION FOR THE WIGNER DENSITY

In the coordinate representation the restriction (2.11), imposed on the density matrix by the translational symmetry, takes the following form:

$$(\mathbf{x} + \mathbf{a}|\hat{\rho}(t)|\mathbf{x}' + \mathbf{a}) = \exp\left\{+\frac{ie}{\hbar c}\bar{\mathbf{A}}(\mathbf{a})(\mathbf{x} - \mathbf{x}')\right\}(\mathbf{x}|\hat{\rho}(t)|\mathbf{x}'). \tag{3.1}$$

This equation shows that matrix elements obtained by parallel transport of $\mathbf{x}$ and $\mathbf{x}'$ contain no new information about the state of the system. This relation takes a more transparent form in the Wigner representation

$$w(\mathbf{r}, \mathbf{p}, t) = \frac{1}{(2\pi\hbar)^3}\int (ds)\exp\left\{+\frac{i}{\hbar}\mathbf{ps}\right\}\left(\mathbf{r} - \frac{1}{2}\mathbf{s}|\hat{\rho}(t)|\mathbf{r} + \frac{1}{2}\mathbf{s}\right). \tag{3.2}$$

Substituting (3.1) in this, we find

$$w(\mathbf{r}, \mathbf{p}, t) = \frac{1}{(2\pi\hbar)^3}\int (ds)\exp\left\{+\frac{i}{\hbar}\left(\mathbf{p} - \frac{e}{c}\bar{\mathbf{A}}(\mathbf{r})\right)\mathbf{s}\right\}\left(-\frac{1}{2}\mathbf{s}|\hat{\rho}(t)|+\frac{1}{2}\mathbf{s}\right). \tag{3.3}$$

We now have the following important result: w does not depend on $\mathbf{r}$ and $\mathbf{p}$ separately, but depends on them only through the kinetic momentum in the magnetic field, $\bar{\mathbf{k}} = \mathbf{p} - e\bar{\mathbf{A}}/c$. This means that we can write

$$w(\mathbf{r}, \mathbf{p}, t) = \bar{f}(\bar{\mathbf{k}}, t), \quad \bar{\mathbf{k}} = \mathbf{p} - \frac{e}{c}\bar{\mathbf{A}}(\mathbf{r}). \tag{3.3}$$

A natural consequence is that we can go over to the actual kinetic momentum. We write

$$w(\mathbf{r}\ \mathbf{p}, t) = f(\mathbf{k}, t), \quad \mathbf{k} = \mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r}, t) = \bar{\mathbf{k}} - \frac{e}{c}\tilde{\mathbf{A}}(t). \tag{3.4}$$

The relation (3.4) is the central result of the present paper. It enables us to write the equation for the density matrix in precisely the same variables that appear in the classical kinetic equation for the homogeneous system.

Before going on to the derivation of the equation for f, let us write the relation (2.16) in the coordinate representation:

$$\left(\mathbf{r} - \frac{1}{2}\mathbf{s}|\hat{S}(t, t_0)|\mathbf{r} + \frac{1}{2}\mathbf{s}\right) = \exp\left\{-\frac{ie}{\hbar c}\bar{\mathbf{A}}(\mathbf{r})\mathbf{s}\right\} \times\exp\left\{-\frac{ie}{\hbar}\varphi(\mathbf{r})(t - t_0)\right\}S(\mathbf{s}|t, t_0), \tag{3.5}$$

where we have used the notation

$$S(\mathbf{s}|t, t_0) = \left(-\frac{1}{2}\mathbf{s}|\hat{S}(t, t_0)|+\frac{1}{2}\mathbf{s}\right) = S(-\mathbf{s}|t_0, t)^{*}. \tag{3.6}$$

By calculating the matrix element $(-\frac{1}{2}\mathbf{s}|\ldots|+\frac{1}{2}\mathbf{s})$ in Eq. (1.16) and subjecting it to a Fourier transformation with $\exp\{+(\mathrm{i}/\hbar)\bar{\mathbf{k}}\cdot\mathbf{s}\}$, we can obtain the equation for $\bar{f}$. This is a cumbersome but straight forward calculation; one needs only to use (3.1) and (3.5) at all stages of the calculation. After this we can go over from the variables $\bar{\mathbf{k}}$, t to the variables $\mathbf{k}$, t, keeping in mind that

$$\frac{\partial}{\partial t}\bar{f}(\bar{\mathbf{k}}, t) = \frac{\partial}{\partial t}f(\mathbf{k}, t) + \frac{\partial}{\partial\mathbf{k}}f(\mathbf{k}, t)e\tilde{\mathbf{E}}(t). \tag{3.7}$$

We present only the final result:

$$\left\{\frac{\partial}{\partial t} + \left[e\mathbf{E}(t) + \frac{e}{c}[\mathbf{v}(\mathbf{k})\mathbf{H}]\right]\frac{\partial}{\partial\mathbf{k}}\right\}f(\mathbf{k}, t) = I(f|\mathbf{k}, t), \tag{3.8}$$
$$\mathbf{v}(\mathbf{k}) = \mathbf{k}/m.$$

This equation is the quantum analog of the kinetic equation. The terms not associated with collisions are formally the same as the corresponding terms of the Boltzmann kinetic equation, in full correspondence with the fact that in the transition to classical theory

the Wigner density goes over into the classical distribution function. The collision term is decidedly different from the classical form:

$$I(f|\mathbf{k},t)=\int_{-\infty}^{t}dt_0\int(d\mathbf{k}_0)f(\mathbf{k}_0,t_0)I(\mathbf{k},\mathbf{k}_0;t,t_0). \tag{3.9}$$

Corresponding to the separation of I(f) into four terms—leaving and entering, and owing to absorption and to emission of a phonon, Eqs. (1.19) and (1.21)—the kernel I also separates into four terms, for which we keep the same symbols. When we also sum the collision term over types of phonons, we have

$$\begin{aligned}
A^{(+)}(\mathbf{k},\mathbf{k}_0;t,t_0)&=\int(d\mathbf{q})W_{\mathbf{q}}^{(+)}\,\Delta_{\mathbf{q}}\left(\mathbf{k}-\frac{\mathbf{q}}{2},\mathbf{k}_0-\frac{\mathbf{q}}{2};t,t_0\right),\\
A^{(-)}(\mathbf{k},\mathbf{k}_0;t,t_0)&=\int(d\mathbf{q})W_{\mathbf{q}}^{(-)}\,\Delta_{\mathbf{q}}\left(\mathbf{k}+\frac{\mathbf{q}}{2},\mathbf{k}_0+\frac{\mathbf{q}}{2};t,t_0\right),\\
B^{(+)}(\mathbf{k},\mathbf{k}_0;t,t_0)&=\int(d\mathbf{q})W_{\mathbf{q}}^{(+)}\,\Delta_{\mathbf{q}}\left(\mathbf{k}+\frac{\mathbf{q}}{2},\mathbf{k}_0-\frac{\mathbf{q}}{2};t,t_0\right),\\
B^{(-)}(\mathbf{k},\mathbf{k}_0;t,t_0)&=\int(d\mathbf{q})W_{\mathbf{q}}^{(-)}\,\Delta_{\mathbf{q}}\left(\mathbf{k}-\frac{\mathbf{q}}{2},\mathbf{k}_0+\frac{\mathbf{q}}{2};t,t_0\right),
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
\Delta_{\mathbf{q}}(\mathbf{k},\mathbf{k}_0;t,t_0)=\frac{1}{2\pi\hbar}\int(d\mathbf{x})\int(d\mathbf{x}_0)\,\delta(\mathbf{p}-\mathbf{p}_0-e\bar{\mathbf{E}}(t-t_0))\\
\times\exp\left\{-\frac{i}{\hbar}\left[\hbar\Omega_{\mathbf{q}}(t-t_0)-\frac{\mathbf{q}}{2}(\mathbf{x}+\mathbf{x}_0)\right]\right\}\\
\times\exp\left\{-\frac{i}{\hbar}\frac{1}{2}(\mathbf{p}+\mathbf{p}_0)(\mathbf{x}+\mathbf{x}_0)\right\}S(-\mathbf{x}|t,t_0)S(-\mathbf{x}_0|t_0,t)+\text{c.c.}\\
\mathbf{p}=\mathbf{k}+\frac{e}{c}\mathbf{A}(\mathbf{x},t),\quad \mathbf{p}_0=\mathbf{k}_0+\frac{e}{c}\mathbf{A}(\mathbf{x}_0,t_0)
\end{aligned} \tag{3.11}$$

and

$$W_{\mathbf{q}}^{(\pm)}=\frac{2\pi}{\hbar}|c_{\mathbf{q}}|^2\left(N_{\mathbf{q}}+\frac{1}{2}\pm\frac{1}{2}\right) \tag{3.12}$$

are the classical probabilities for emission and absorption of a phonon $\mathbf{q}$.

The presence of the fields $\mathbf{E}$ and $\mathbf{H}$ in $\Delta_{\mathbf{q}}$ reflects the influence of the fields on the act of scattering, or, in other words, the fact that the scattering occurs not between states described by the plane waves of a free electron, but between those of an electron in the fields $\mathbf{E}$ and $\mathbf{H}$. If we formally set $\mathbf{E}=0$ and $\mathbf{H}=0$ in (3.11), we get

$$\begin{aligned}
\Delta_{\mathbf{q}}(\mathbf{k},\mathbf{k}_0;t,t_0)=\delta(\mathbf{k}-\mathbf{k}_0)\frac{1}{2\pi\hbar}\\
\times\exp\left\{-\frac{i}{\hbar}\left[\varepsilon\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)-\varepsilon\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)+\hbar\Omega_{\mathbf{q}}\right](t-t_0)\right\}+\text{c.c.}
\end{aligned} \tag{3.13}$$

If, again proceeding formally, we let $\hbar\to 0$ in (3.13), then, keeping in mind the convergence factor, we can use the relation

$$\exp\left\{+\frac{i}{\hbar}Et\right\}=\hbar\delta(t)\left[\pi\delta(E)+i\frac{\mathscr{P}}{E}\right]. \tag{3.14}$$

It then follows from (3.13) that

$$\begin{aligned}
\Delta_{\mathbf{q}}(\mathbf{k},\mathbf{k}_0;t,t_0)\\
=\delta(\mathbf{k}-\mathbf{k}_0)\delta(t-t_0)\delta\left[\varepsilon\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)-\varepsilon\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)+\hbar\Omega_{\mathbf{q}}\right].
\end{aligned} \tag{3.15}$$

It is not hard to see that when (3.15) and (3.10) are substituted into it the expression (3.9) takes the form of the classical collision term.

4. THE EQUILIBRIUM DISTRIBUTION

We shall now show that the equilibrium distribution

$$\hat{\rho}_T=Z^{-1}\exp\{-\beta\hat{H}\},\quad Z=\operatorname{sp}\exp\{-\beta\hat{H}\}, \tag{4.1}$$

is, up to terms of order $|c|^2$, a stationary solution of Eq. (1.16) with the stationary Hamiltonian $\hat{H}$. Recalling that I contains the small factor $|c|^2$, we have the stationary solution in the form

$$\hat{\rho}=\hat{\rho}^0+\hat{\rho}', \tag{4.2}$$

where $\hat{\rho}'$ is a small quantity of the order $|c|^2$. Then (1.16) is converted into the system of equations

$$\frac{i}{\hbar}[\hat{H},\hat{\rho}^0]=0, \tag{4.3}$$

$$\frac{i}{\hbar}]\hat{H},\hat{\rho}']=I(\hat{\rho}^0). \tag{4.4}$$

It follows from the zeroth approximation (4.3) (we assume at first that $\hat{H}$ has no degeneracy) that

$$\hat{\rho}^0=\rho^0(\hat{H}), \tag{4.5}$$

where ρ^0 is an arbitrary real function of its argument. The form of this function is determined from the solubility conditions of the next approximation (4.4). We note that the operator in the left members of (4.3) and (4.4) satisfies the identity

$$\operatorname{sp}\{\Phi(\hat{H})[\hat{H},\hat{\rho}]\}=0\quad\text{for arbitrary} \tag{4.6}$$

where Φ is an arbitrary function. Therefore the solubility condition of Eqs. (4.4) is

$$\operatorname{sp}\{\Phi(\hat{H})I(\hat{\rho}^0)\}=0\quad\text{for arbitrary} \tag{4.7}$$

This condition is equivalent to

$$\operatorname{sp}\{\delta(\hat{H}-E)I(\hat{\rho}^0)\}=0\quad\text{for arbitrary} \tag{4.8}$$

The trace is easily calculated in the system of eigenfunctions of $\hat{H}$. Here it must be remembered that in the case in question

$$\hat{S}(t,t_0)=\exp\left\{-\frac{i}{\hbar}\hat{H}(t-t_0)\right\}. \tag{4.9}$$

The resulting integral over t_0 is to be understood in the sense

$$\int_{-\infty}^{t}dt_0\exp\{i\omega(t-t_0)\}=\pi\delta(\omega)+i\frac{\mathscr{P}}{\omega}; \tag{4.10}$$

this follows from the presence of the convergence factor $\exp\{\lambda t_0\}$. After the calculation we find the individual terms of the left member of (4.8):

$$\begin{aligned}
\overline{A^{(+)}(\hat{\rho}^0)}&=(N+1)\frac{2\pi}{\hbar}|c|^2\rho^0(E)M(E-\hbar\Omega,E),\\
\overline{A^{(-)}(\hat{\rho}^0)}&=(N)\frac{2\pi}{\hbar}|c|^2\rho^0(E)M(E,E+\hbar\Omega).\\
\overline{B^{(+)}(\hat{\rho}^0)}&=(N+1)\frac{2\pi}{\hbar}|c|^2\rho^0(E+\hbar\Omega)M(E,E+\hbar\Omega),\\
\overline{B^{(-)}(\hat{\rho}^0)}&=(N)\frac{2\pi}{\hbar}|c|^2\rho^0(E-\hbar\Omega)M(E-\hbar\Omega,E),
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
\overline{(\ldots)}=\operatorname{sp}\{\delta(\hat{H}-E)(\ldots)\},\\
M(E,E')=\operatorname{sp}\{\hat{\chi}\delta(\hat{H}-E)\hat{\chi}^+\delta(\hat{H}-E')\}.
\end{aligned} \tag{4.12}$$

Noting that

$$(N+1)=(N)\exp\{\beta\hbar\Omega\}, \tag{4.13}$$

we see that the equilibrium distribution

$$\rho^0(E)=\rho_T(E)=Z^{-1}\exp\{-\beta E\} \tag{4.14}$$

is a solution of (4.8). We note that I becomes zero for each oscillator separately (i.e., without summation over s). Here $A^{(+)}$ and $B^{(-)}$ cancel, and $A^{(-)}$ and $B^{(+)}$ cancel, in complete analogy with the detailed balancing in the classical kinetic equation.

It is obvious that the solution (4.14) has meaning only when it is normalizable, i.e., when Z is finite. Since

$$Z=\int dED(E)\exp\{-\beta E\}, \tag{4.15}$$

where D(E) is the density of states for the Hamiltonian $\hat{H}$, the actual meaning of the requirement that Z be finite for all β is that the spectrum of $\hat{H}$ must be bounded below.

We now return to the electron in the uniform fields $\overline{E}$ and H. It is obvious that the proof considered has meaning only for $\overline{E}$ = 0, since otherwise the spectrum of $\hat{H}$ is not bounded for $E \to -\infty$. We further note that there is a degeneracy for the electron in the field H, since H commutes with $\hat{g}$. The proof given remains valid, however, since we are limited to only one space of values of $\hat{\rho}$ which is invariant with respect to g, namely that space in which $\hat{\rho}$ is invariant with respect to g.

It is interesting to calculate the equilibrium distribution $f_T(\mathbf{k})$ in the magnetic field. Starting from the well known wave functions $\psi_{lp_xp_z}(\mathbf{x})$ of an electron in a magnetic field, one can easily calculate the equilibrium density matrix in the x representation. It is obvious that here the integration over p_x and p_z is elementary, and the summation over the Landau quantum numbers is performed by means of the formula for summing Hermite polynomials[7]

$$\sum_{l=0}^{\infty}\frac{z^l}{2^l l!}H_l(x)H_l(y)=(1-z^2)^{-1/2}\exp\left\{\frac{2xyz-z^2(x^2+y^2)}{1-z^2}\right\}. \tag{4.16}$$

When we then go over from the x representation to the Wigner representation by means of (3.2), we find

$$f_T(\mathbf{k})=n(2\pi mT)^{-1/2}\exp\left\{-\frac{k_\parallel^2}{2mT}\right\}\cdot$$

$$\times(2\pi mE_T(\omega_c))^{-1}\exp\left\{-\frac{k_\perp^2}{2mE_T(\omega_c)}\right\}. \tag{4.17}$$

Here n is the concentration of electrons, $k_\parallel$ and $k_\perp$ are the components of k relative to the direction of H, and

$$E_T(\omega_C)=\frac{1}{2}\hbar\omega_C\,\mathrm{cth}\left(\frac{1}{2}\frac{\hbar\omega_C}{T}\right),\quad \omega_C=\frac{eH}{mc}, \tag{4.18}$$

is the average energy of the oscillator with frequency ω_C. When (4.17) is compared with the classical Boltzmann distribution it is seen that the quantization in the magnetic field reduces to replacement of the "transverse temperature", i.e., the energy of the classical oscillator, by the energy of the quantum oscillator.

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