SOVIET PHYSICS JETP VOLUME 30, NUMBER 1 JANUARY, 1970

LIMITING CURRENTS IN RELATIVISTIC ELECTRON BEAMS

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Submitted February 3, 1969

Zh. Eksp. Teor. Fiz. 57, 315–320 (July, 1969)

We consider the limiting current of a relativistic electron beam in a drift space. It is shown that for an uncompensated beam the limiting current determined by the space charge increases linearly with electron energy in the relativistic limit. In a compensated beam the limiting current is determined from the condition for the onset of an electrostatic instability. In this case, as the electron energy increases in the relativistic region the limiting currents increase with energy as ϵ^3; this result indicates the possibility of obtaining high currents in compensated relativistic electron beams.

1. In recent years there has been considerable interest in obtaining high-current electron beams in connection with the development of high-power electronic devices, the construction of intense sources of x-ray and microwave radiation, and in connection with the development of accelerator technology. In the case of nonrelativistic beams the question of limiting currents has been discussed at length in the literature (cf.[1,2] and the bibliography given there). However, this question has not been fully resolved at the present time. As far as limiting currents in relativistic electron beams are concerned, we find that this problem has been investigated in[3] for the case of an infinite stabilized beam in an infinite magnetic field. It will be shown below that in the relativistic energy region it is possible to obtain significantly higher currents in bounded compensated electron beams than in uncompensated beams.

Below we investigate the limiting current in an electron beam of radius r_0 which passes along the axis of a waveguide of radius R whose longitudinal dimensions are much larger than its transverse dimensions ($L \gg R$). In order to inhibit the expansion of the beam in the radial direction the system is placed in a strong longitudinal magnetic field, this field satisfying the condition

$$\frac{B_0^2}{8\pi} \gg \frac{nmc^2}{\sqrt{1-u^2/c^2}} \tag{1}$$

where n is the electron density of the beam in the laboratory coordinate system while u is the electron velocity. It is further assumed that the self-magnetic field of the beam is small compared with B_0. This assumption guarantees the stability of the compensated beam with respect to compression (pinch effect). Here we note that the condition given in (1) also allows us to limit our analysis to the case of electrostatic perturbations in the investigation of beam stability.

In the case of an uncompensated nonrelativistic electron beam the limiting current that can be passed through the system considered here is given by the familiar relation[4]

$$I_0 = \begin{cases} 0.88\dfrac{mu^3}{4e} & R \approx r_0 \\ 0.38\dfrac{mu^3}{4e}\cdot\dfrac{1}{\ln R/r_0} & R \gg r_0 \end{cases}, \tag{2}$$

where e is the electron charge and m is the rest mass. It is not difficult to generalize this expression to the ultrarelativistic case, in which the electron energy $\epsilon \gg mc^2$. We find

$$I_0 = \frac{\varepsilon c}{e}\frac{1}{1+2\ln(R/r_0)}. \tag{3}$$

When $\epsilon \sim mc^2$ the expression for the limiting current becomes extremely complicated. It is evident from Eqs. (2) and (3) that in the region of nonrelativistic electron energies $I_0 \sim \epsilon^{3/2}$ while in the region of relativistic energies the dependence of the limiting current on ϵ is weaker: specifically, we find $I_0 \sim \epsilon$. This feature imposes a limitation on the limiting current in "vacuum" systems for relativistic beam energies.

It should be noted that Eqs. (2) and (3) as well as all of the formulas obtained below for limiting currents refer to the stationary case and hence do not take account of the work that goes into the formation of the self-magnetic field of the current. This work, particularly in the region of high currents and relativistic energies, can exceed significantly the kinetic energy of the electrons in the beam; however, it is consumed only in the transient process of establishing the current in the system. In the stationary regime the energy of the magnetic field associated with the current is constant and the work associated with it is not considered.

2. Since the limiting current that can pass through a vacuum system is due to the space charge of the beam, one expects that this limitation on the current will not appear in systems in which the beam charge is compensated. This possibility was first indicated by Pierce.[5] However, Pierce directed attention to the fact that compensated electron beams are subject to instabilities that can limit the current and determined the instability limit for a nonrelativistic electron beam.

The stability of a compensated relativistic electron beam in a waveguide is investigated below and the limiting current that can pass through such a system is determined from the instability limit. Assuming that the electron beam is uniform along the axis of the waveguide, we can write the potential for the perturbed electric field in the form $\Phi = \Phi(r)e^{i(-\omega t + l\varphi + k_z z)}$ where l and k_z are the azimuthal and longitudinal wave numbers. In this case the equation for the electrostatic oscillations of the field is given by

$$\left(1-\frac{\omega_{Li}^2}{\omega^2}\right)\Delta\Phi + \frac{\partial\Phi}{\partial r}\frac{\partial}{\partial r}\left(1-\frac{\omega_{Li}^2}{\omega^2}\right)$$

$$+\frac{\omega_{Le}^2(1-\beta^2)^{3/2}}{(\omega-k_zu)^2}k_z^2\Phi-\frac{l}{r}\Phi\frac{\partial}{\partial r}\left[\frac{\omega_{Le}^2}{\Omega_e(\omega-k_zu)}\right]=0, \quad (4)$$

where $\beta = u/c$, ω_{Le} and ω_{Li} are the plasma frequencies of the electrons and ions while Ω_e and Ω_i are the Larmor frequencies in the laboratory coordinate system. In writing this equation we have neglected the effect of the external magnetic field on the ion motion ($\omega > \Omega_i$) and have also taken account of the condition in (1).

Equation (4) must be supplemented by boundary conditions. At $r = r_0$ these conditions are obtained by integrating the equation itself over a narrow layer close to the surface of the beam. The conditions assume the form

$$\{\Phi\}_{r=r_0}=0,$$
$$\left\{\left(1-\frac{\omega_{Li}^2}{\omega^2}\right)\frac{\partial\Phi}{\partial r}-\frac{l}{r}\Phi\frac{\omega_{Le}^2}{\Omega_e(\omega-k_zu)}\right\}_{r=r_0}=0. \quad (5)$$

At the surface of the waveguide, where $r = R$,

$$\Phi|_{r=R}=0. \quad (6)$$

Solving Eq. (4) with the boundary conditions (5) and (6) under the assumption that the beam is uniform for $r < r_0$, we obtain the following relation

$$\left(1-\frac{\omega_{Li}^2}{\omega^2}\right)\frac{1}{J_l(i\alpha k_zr_0)}\frac{dJ_l(i\alpha k_zr_0)}{dr_0}=\frac{l}{r_0}\frac{\omega_{Le}^2}{\Omega_e(\omega-k_zu)}-f_l, \quad (7)$$

where

$$\alpha^2=1-\frac{\omega_{Le}^2(1-\beta^2)^{3/2}}{(\omega-k_zu)^2}\left(1-\frac{\omega_{Li}^2}{\omega^2}\right)^{-1},$$
$$f_l=\frac{I_l(k_zR)dK_l(k_zr_0)/dr_0-K_l(k_zR)dI_l(k_zr_0)/dr_0}{I_l(k_zr_0)K_l(k_zR)-I_l(k_zR)K_l(k_zr_0)}$$

and J_l, I_l and K_l are Bessel functions.

The relation in (7) for an unbounded waveguide represents the dispersion equation that determines $\omega = \omega(k_z)$ in which the quantity k_z is assumed to be a real quantity. However, if the waveguide is bounded then (7) is to be regarded as a characteristic equation for determining k_z which, generally speaking, can be complex. In this case $\Phi = \sum_n C_n \exp\{ik_{zn}z\}$ where the k_{zn} are the roots of the characteristic equation (7). In order to obtain the dispersion equation this solution must be substituted in the boundary conditions (at the ends of the waveguide); the number of boundary conditions must correspond to the number of roots k_{zn}. These boundary conditions will be written below.

3. Proceeding to the analysis of (7) we note first of all that when $r_0 = R$, that is to say, when the beam fills the waveguide completely, $f_l \to \infty$ and Eq. (7) assumes the form

$$\frac{\omega_{Le}^2(1-\beta^2)^{3/2}}{(\omega-k_zu)^2}\left(1-\frac{\omega_{Li}^2}{\omega^2}\right)^{-1}=1+\frac{\mu_{nl}^2}{k_z^2r_0^2}, \quad (8)$$

where the μ_{nl} are the roots of the Bessel function $J_l(\mu_{nl})$ (aside from $\mu_{nl} = 0$). For an unbounded waveguide Eq. (8) generalizes the well-known equation of Buneman[6] to the case of a relativistic electron beam. For a bounded waveguide Eq. (8) defines four roots k_{zn}; hence, to determine the dispersion equation it is necessary to satisfy four boundary conditions at the ends of the waveguide, i.e., at $z = 0$ and $z = L$. If the ends of the waveguide are metal then obviously

$$\Phi|_{z=0,\,L}=0. \quad (9)$$

As the other two boundary conditions we make use of the conditions given by Pierce:[5]

$$v_{z1}=\frac{e}{m}(1-\beta^2)^{3/2}\sum_n\frac{k_{zn}}{\omega-k_{zn}u}C_ne^{ik_{zn}z}|_{z=0}=0,$$
$$n_1=\frac{e}{m}(1-\beta^2)^{3/2}\sum_n\frac{k_{zn}^2}{(\omega-k_{zn}u)^2}C_ne^{ik_{zn}z}|_{z=0}=0, \quad (10)$$

where v_{z1} and n_1 are the perturbations in the velocity and density of beam electrons due to the waves.

In what follows we will be interested in long-wave perturbations for which $k_z \sim 1/L$. It is precisely these perturbations that determine the limiting current in the system. For these perturbations Eq. (8) reduces to a quadratic equation for which

$$k_{z1,2}=\frac{\pm\mu_{nl}\omega}{r_0\omega_{Le}(1-\beta^2)^{3/4}(1-\omega_{Li}^2/\omega^2)^{-1/2}\pm u\mu_{nl}}. \quad (11)$$

The boundary conditions (9) then reduce to the following dispersion equation for the waves:

$$\frac{\omega}{k_su}\frac{r_0\omega_{Le}(1-\beta^2)^{3/4}}{u\mu_{nl}(1-\omega_{Li}^2/\omega^2)^{1/2}}=\frac{r_0^2\omega_{Le}^2(1-\beta^2)^{3/2}}{u^2\mu_{nl}^2(1-\omega_{Li}^2/\omega^2)}-1, \quad (12)$$

where

$$k_s=\pi s/L,\quad s=1,2,3,\ldots$$

It should be noted that in solving Eq. (8) for the long-wave perturbations we have retained only two roots $k_{z1,2}$. The other two roots are negligibly small when $r_0^2/L^2 \ll 1$. In this case the conditions in (10) are satisfied automatically if $\omega \ll k_{zn}u \sim k_su$. It is easy to show from the dispersion equation (12) that the latter inequality is satisfied if

$$\delta_s=\left(2\frac{m}{M}(1-\beta^2)^{-3/2}\frac{\mu_{nl}^2}{k_s^2r_0^2}\right)^{1/3}\ll 1. \quad (13)$$

In this case we have oscillations with frequency $\omega = k_su\delta_s$. The minimum current for which the instability arises in the system (this will be called the critical current) corresponds to the excitation of the fundamental mode $\mu_{00} = 2.4$, $k_1 = \pi/L$ and is given by the expression

$$I_{cr}=\frac{mu^3}{4e(1-\beta^2)^{3/2}}\frac{(2{,}4)^2}{1+{}^3/_2\delta_1}. \quad (14)$$

For the case of a nonrelativistic beam with infinitely heavy ions ($\delta_1 \to 0$) Eq. (14) coincides with the familiar expression for the limiting current given by Pierce.[5] In this limit the critical current (14) is approximately 6 times larger than the critical current for an uncompensated beam (2) for $r_0 \sim R$. The situation is different in the region of relativistic beam energies. From a comparison of Eqs. (3) and (14) we see that when $r_0 \sim R$, for ultrarelativistic energies $(I_{cr}/I_0) \approx (\epsilon/mc^2)^2$, that is to say, the critical current in the uncompensated beam can be much greater than the current transmitted through a vacuum system.

4. The analysis given above not only applies in the case in which the beam fills the waveguide completely, but also applies when there is a gap between the beam and the metal wall, provided the following condition is satisfied:

$$r_0f_l/\mu_{nl}\gg 1. \quad (15)$$

If we assume that for the longwave perturbations ($k_z R \ll 1$)

$$f_l = \begin{cases} \dfrac{1}{r_0 \ln (R/r_0)} & l = 0 \\ \dfrac{l}{r_0} \dfrac{1+(r_0/R)^{2l}}{1-(r_0/R)^{2l}} & l \neq 0 \end{cases}, \quad (16)$$

then, in order for (14) to be valid, in accordance with (15) we require that $2.4 \ln (R/r_0) \ll 1$.

If the gap is large, so that the inverse inequality to (15) is satisfied, then the critical current is determined by excitation of waves that have long wavelengths in the radial direction, for which we have the condition $\alpha k_z r_0 \ll 1$.[1)] As a result, from Eq. (4) we find

$$1 - \frac{\omega_{Li}^2}{\omega^2} - \frac{\omega_{Le}^2(1-\beta^2)^{3/2}}{(\omega - k_z u)^2} \frac{k_z^2 r_0^2}{k_z^2 r_0^2 + 2l(l+1)} - \frac{2l(l+1)}{k_z^2 r_0^2 + 2l(l+1)} \left[\frac{\omega_{Le}^2}{\Omega_e(\omega - k_z u)} - \frac{r_0}{l} f_l \right] = 0 \quad (17)$$

For axially symmetric perturbations ($l = 0$) in the region $\omega < k_z u$ Eq. (17) yields two roots:

$$k_{z1,2} \cong \pm \frac{\omega}{\omega_{Li}} \left[\frac{2}{r_0^2 \ln (R/r_0)} - \frac{\omega_{Le}^2(1-\beta^2)^{3/2}}{u^2} \right]^{1/2}. \quad (18)$$

The boundary conditions (9) lead to the following dispersion equation (we note that the condition in (10) is satisfied automatically in this case):

$$\frac{\omega^2}{\omega_{Li}^2} \left[\frac{2}{r_0^2 \ln (R/r_0)} - \frac{\omega_{Le}^2(1-\beta^2)^{3/2}}{u^2} \right] = \left(\frac{\pi s}{L} \right)^2 \quad (19)$$

Whence we find the critical current for excitation of axially symmetric modes for $R > r_0$:

$$I_{cr} = \frac{mu^3}{4e(1-\beta^2)^{3/2}} \frac{2}{\ln (R/r_0)} \quad (20)$$

Taking account of terms $\sim \omega / k_z u$ in this expression leads to a factor that is approximately equal to unity: $[1 + (m/M)^{1/3}(1-\beta^2)^{-1/2}]^{-1}$.

For axially asymmetric perturbations Eq. (17) leads to a quadratic equation (for modes with $k_z r_0 < 1$) in k_z, the roots being given by

$$k_{z\,1,2} = a \pm b, \quad (21)$$
$$a = \frac{2u\omega(2 - \omega_{Li}^2/\omega^2) - u\omega_{Le}^2/\Omega_e}{2[u^2(2-\omega_{Li}^2/\omega^2) - r_0^2 \omega_{Le}^2 (1-\beta^2)^{3/2}/2l(l+1)]},$$
$$b = \frac{1}{2} \left\{ \frac{u^2 \omega_{Le}^4}{\Omega_e^2} + 4 \frac{r_0^2 \omega_{Le}^2 (1-\beta^2)^{3/2}}{2l(l+1)} \left[\omega^2 \left(2 - \frac{\omega_{Li}^2}{\omega^2} \right) - \frac{\omega \omega_{Le}^2}{\Omega_e} \right] \right\}^{1/2} \times \left[u^2 \left(2 - \frac{\omega_{Li}^2}{\omega^2} \right) - \frac{r_0^2 \omega_{Le}^2 (1-\beta^2)^{3/2}}{2l(l+1)} \right]^{-1}.$$

Under these conditions the boundary conditions (9) lead to the dispersion equation $b = k_s$ which, in the limit $\omega < k_s u$ [(for these waves the condition in (10) is again satisfied automatically], is of the form

$$2 - \frac{\omega_{Li}^2}{\omega^2} = \frac{r_0^2 \omega_{Le}^2 (1-\beta^2)^{3/2}}{2l(l+1)u^2} \left[1 + \frac{ul(l+1)}{k_s \Omega_e r_0^2 (1-\beta^2)^{3/2}} \right]. \quad (22)$$

Thus, we find the current required for excitation of modes for which $l \neq 0$:

$$I = \frac{mu^3 l(l+1)}{e(1-\beta^2)^{3/2}} \left(1 + \frac{ul(l+1)L}{\pi r_0^2 \Omega_e (1-\beta^2)^{3/2}} \right)^{-1}. \quad (23)$$

If the following inequality is satisfied (for $l \neq 0$)

$$1 + \frac{ul(l+1)L}{\pi r_0^2 \Omega_e (1-\beta^2)^{3/2}} < 2l(l+1) \ln \frac{R}{r_0}, \quad (24)$$

then the critical current corresponds to the excitation of a mode with $l = 0$ and is determined by Eq. (20). In this case, in the region of nonrelativistic energies this current is six times larger than the current (2) that can be transmitted through a vacuum system. For relativistic energies, however, the ratio of the current in (20) to the vacuum current (3) is $I_{cr}/I_0 \sim (\epsilon/mc^2)^2$ and increases with electron energy. The situation is changed when the inequality inverse to (24) is satisfied in which case the critical current is given by (23) and corresponds to excitation of modes for which $l \neq 0$. As is evident from (23), the critical current in the system exhibits a saturation with increasing electron energy:

$$I_{\text{кр}} \to \frac{mc^2}{e} \frac{\pi r_0^2 \Omega_e}{L}. \quad (25)$$

Thus, the maximum possible currents can be achieved in systems in which (24) is satisfied, in which case (20) holds. Under these conditions the critical currents in the relativistic energy range are a factor $(\epsilon/mc^2)^2$ times greater than the limiting current that can be transmitted through a vacuum system. For example, with $\epsilon \approx 5$ MeV and $R/r_0 = 10$, according to (3) the vacuum current is $I_0 \sim 3 \times 10^4$ A whereas the critical current given by Eq. (20) is of the order of 3×10^6 A. It should be noted, however, that for these electron energies the inequality in (24) can only be satisfied in very high magnetic fields. Thus, if $L \sim 10^2$ cm and $r_0 \sim 1$ cm the required fields are given by $B_0 \gtrsim 10^7$ Oe and can only be achieved in pulsed systems.

In conclusion, the authors wish to thank V. P. Silin for discussion of the results and for valuable comments.

[1)] Actually for shortwave oscillations, in which $\alpha k_z r_0 \geqslant 1$, to a high degree of accuracy the roots of Eq. (4) coincide with the roots of the equation $I'_l(i\alpha k_z r_0) = 0$. In this case, the analysis of the dispersion equation is similar to the one above and yields a limiting current larger than that in (14).

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Translated by H. Lashinsky
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