CALCULATION OF THE AVERAGE NUMBER OF STATES IN A MODEL PROBLEM

M. M. BENDERSKIĬ and L. A. PASTUR

Physico-technical Low Temperature Institute, Ukrainian Academy of Sciences

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We evaluate the average number of states per unit length N(E) of a one-dimensional Schrödinger equation for two random potentials which are a sequence of potential barriers and wells with lengths which are independent random variables. The main part of the paper is devoted to the situation where the length distribution is exponential. Using arguments normally applied in the theory of random Markov processes we are in that case able to find the density of the zeroes of the solution which is the same as N(E) in the one-dimensional case. In the last part of the paper we consider the simpler case of a slowly varying potential when N(E) can be evaluated directly.

INTRODUCTION

In this paper we give a method for and the results of a calculation of the average number of states in a one-dimensional Schrödinger equation with a random potential of the form $V(x) = V_0 r(x)$, where $r(x)$ is a random process taking on alternately values 0 and 1 at intervals the lengths of which are independent random quantities. We shall mainly consider the case when these random quantities have a distribution density $n_0 e^{-n_0 x}$ and $n_1 e^{-n_1 x}$, respectively. Such a potential could be taken to be a model to describe a two-component one-dimensional alloy.

At the end of the paper we consider the case which is in some well-defined sense the opposite of the first case. Here the distribution of the lengths of the intervals where the potential is constant must be such that the average lengths of the intervals would be infinite. This requirement, which to be sure is a stringent one, by itself already enables us to find the average number of states.

During the whole of the paper we study the calculation of the average number of states without hardly considering the problem about its self-averaging, i.e., about how large the deviation from the average is. One can show that for the potentials considered by us the number of states is a self-averaging quantity. We shall not give the appropriate proofs, primarily because self-averaging occurs in an appreciably more general situation and can be proved without using one or other special model of random potential. We propose to discuss that problem elsewhere.

1. AVERAGE NUMBER OF STATES

We derive in this section formulae for the average number of states which will be the basis of our later calculations. The considerations used here are relatively well known and have often been applied in similar problems (see, for instance,[1]).

We consider the Schrödinger equation in the interval (0, L)

$$-\frac{h^2}{2m}\psi'' + V\psi = E\psi \tag{1}$$

and denote the energy levels in this problem by $E_k(L)$. Let

$$N_L(E) = \frac{1}{L}\sum_{E_k(L)\leqslant E} 1.$$

Our problem is the calculation of the quantity

$$N(E) = \lim_{L\to\infty}\langle N_L(E)\rangle,$$

where $\langle\ldots\rangle$ indicates an average over all possible potentials. The important fact that enables us to evaluate N(E) in one-dimensional problems is the following one:

$$N(E) = \lim_{L\to\infty}\langle M_E(L)\rangle,$$

where $M_E(L) = L^{-1}$ times the number of zeroes in the interval (0, L) of the solution y(x) which together with its first derivative in the zero takes on a well-defined (but not necessarily determined) value. It is just the function

$$\lim_{L\to\infty}\langle M_E(L)\rangle$$

which will be determined in the following.

We introduce new functions $\rho(x)$ and $\theta(x)$ through the equations $y = \rho\cos\theta$, $y' = k\rho\sin\theta$ ($k^2 = 2mE/h^2$). Since $\rho^2 = y^2 + k^2 y'^2$, the zeroes of y(x) only make $\cos\theta$ to vanish. The problem is thus reduced to find the average, per unit length, of the number of zeroes of the equation $\theta(x) = \frac{1}{2}\pi + n\pi$; $n = 0, \pm 1, \ldots$ The function $\theta(x)$, in turn, satisfies a differential equation which one gets from (1):

$$\theta' = -k(1 - \gamma r(x)\cos^2\theta),$$

where we can, since r(x) is equal to zero or unity, assume that $V_0 > 0$ and thus also $\gamma > 0$ without loss of generality.

Let $\tilde{p}(x, \theta)$ be the density of the probability distribution of the random quantity $\theta(x)$. In that case

$$\langle M_E(L)\rangle = \frac{k}{L}\int_0^L \sum_n \left\langle \delta\left(\theta(x) - \frac{\pi}{2} + n\pi\right)\right\rangle dx = $$
$$= \frac{k}{L}\int_0^L \sum_n \tilde{p}\left(x, \frac{\pi}{2} + n\pi\right) dx.$$

It is, however, more convenient to consider the phase $\varphi(x)$ obtained by reducing θ to the interval $(0, \pi)$ rather than $\theta(x)$. The sum in the last integral will then be just the probability distribution $p(x, \varphi)$ of the random quantity $\varphi(x)$ in the point $\varphi = \frac{1}{2}\pi$ and, hence,

$$\langle M_E(L)\rangle = \frac{k}{L}\int_0^L p\left(x, \frac{\pi}{2}\right)dx.$$

If $\varphi(x)$ stabilizes at large x, i.e., if $p(x, \varphi)$ as $x \to \infty$ tends to a stationary distribution $p(\varphi)$, we have

$$N(E) = kp(\pi/2). \quad (2)$$

The initial problem is thus in this situation reduced to finding the stationary distribution $p(\varphi)$ for the phase φ in the point $\frac{1}{2}\pi$.

2. CALCULATION OF $p(\varphi)$

It is convenient to change in the following to the dimensionless variable $t = kx$. The equation for θ then becomes

$$\theta' = -1 + \gamma r(t)\cos^2\theta$$

and now $r(t)$ is a process taking the values 0 or 1 in intervals the lengths of which are independent random quantities with distribution densities $\nu_0 e^{-\nu_0 t}$ and $\nu_1 e^{-\nu_1 t}$ ($\nu_r = n_r/k$, $r = 0, 1$). It is important that $r(t)$ is a Markov process, i.e., that its probability properties for all $t > \tau$ are uniquely determined by its value in the point τ and are independent of the values of the process in points preceding τ. Indeed, let $h(t_1, t_2)$ be the conditional probability that $r(t)$ which was already constant along an interval of length t_1 will still retain the same value along an interval of length not less than t_2. From our assumptions about $r(t)$ it is clear that

$$h(0, t_1 + t_2) = e^{-\nu(t_1+t_2)}, \qquad h(0, t_1) = e^{-\nu t_1}.$$

According to the theorem about multiplying probabilities

$$h(0, t_1 + t_2) = h(0, t_1)h(t_1, t_2),$$

whence

$$h(t_1, t_2) = e^{-\nu t_2} = h(0, t_2).$$

From this it also follows that in an interval of small length Δt the process $r(t)$ remains equal to r with a probability $1 - \nu_r\Delta t + o(\Delta t)$, and takes on the value $1 - r$ with a probability $\nu_r\Delta t + o(\Delta t)$.

As $\theta(t)$ satisfies a first-order differential equation in which $r(t)$ is a coefficient the values of $\theta(t)$ for $t > \tau$ are determined by the value $\theta(\tau)$ and the behavior of $r(s)$ for $\tau \le s \le t$. The pair (θ, r) thus forms a two-dimensional Markov process. Let $\tilde{p}(t, \theta, r)$ be the probability that $\theta(t) = \theta$, $r(t) = r$. We then obtain $\tilde{p}(t, \theta)$ by summing $\tilde{p}(t, \theta, r)$ over r. We can introduce for the function $\tilde{p}(t, \theta, r)$ an equation which plays the role of the Fokker-Planck equation in this case.[1,2] To do that we calculate how the function $\tilde{p}(t, \theta, r)$ changes in the interval $(t, t + \Delta t)$. The state (θ, r) in the point $t + \Delta t$ can be obtained from the state $(\theta - (-1 + \gamma r\cos^2\theta)\Delta t, r)$ at the time t, if in the interval Δt the process $r(t)$ has not changed its value, or from the state $(\theta, 1 - r)$ if there is one jump in the interval Δt. Evaluating the corresponding probability, taking into account that the probability of more than one change in $r(t)$ along Δt is $o(\Delta t)$, we are led to the following equation:

$$\frac{\partial\tilde{p}(t,\theta,r)}{\partial t} = \frac{\partial}{\partial\theta}[(1-\gamma r\cos^2\theta)\tilde{p}(t,\theta,r)] - \nu_r\tilde{p}(t,\theta,r) + \nu_{1-r}\tilde{p}(t,\theta,1-r). \quad (3)$$

As we showed in Sec. 1 to find N(E) we must know the density of the probability distribution of the reduced phase

$$p(x, \varphi) = \sum_n \tilde{p}(x, \varphi + \pi n).$$

This function is π-periodic in φ and can be found as the π-periodic solution of Eq. (3) satisfying well-defined boundary conditions, i.e.,

$$p(0, \varphi, r) = p_0(f, r),\ p_0(\varphi + \pi, r) = p_0(\varphi, r).$$

Using essentially the same considerations as Frisch and Lloyd[1] we can show that the process (φ, r) is ergodic. Hence it follows that the solution of Eq. (3) is stabilized, i.e., that as $t \to \infty$, $p(t, \varphi, r)$ tends to a limiting function $p(\varphi, r)$ which is independent of the form of $p_0(\varphi, r)$ and which is a unique stationary solution of Eq. (3).

We note also that the self-averaging property of N(E) is also a consequence of the ergodicity of the process (φ, r).

Before solving Eq. (3) we integrate it over φ. The result is

$$\frac{dp(t,r)}{dt} = -\nu_r p(t,r) + \nu_{1-r}p(t,1-r),$$

where $p(t, r) = \int_0^\pi p(t, \varphi, r)d\varphi$ is the probability that $r(t) = r$. This is a Fokker-Planck equation for the process $r(t)$ which we could, of course, also have derived directly using the above-described properties of the process. The solution of this equation for the given boundary conditions can be found easily and is such that

$$\lim_{t\to\infty} p(t,r) = \frac{\nu_{1-r}}{\nu_0 + \nu_1}.$$

The process $r(t)$ is thus stabilized.

Let us now turn to Eq. (3). As we mentioned already we are interested in a solution of this equation which is stationary, i.e., which is independent of the time. It must satisfy (3) with the left-hand side equal to zero:

$$\frac{\partial}{\partial\varphi}[(1-\gamma r\cos^2\varphi)p(\varphi,r)] - \nu_r p(\varphi,r) + \nu_{1-r}p(\varphi,1-r) = 0. \quad (4)$$

Moreover, $p(\varphi, r)$ must be a π-periodic function of φ and normalized to unity, i.e.,

$$\sum_{r=0,1}\int_0^\pi p(\varphi,r)d\varphi = 1.$$

Summing Eq. (4) over r we get the following relation ($p_r(\varphi) \equiv p(\varphi, r)$):

$$p_0(\varphi) + (1 - \gamma\cos^2\varphi)p_1(\varphi) = \text{const}, \quad (5)$$

where the constant on the right is at once equal to $k^{-1}N(E)$ by virtue of (2).

Eliminating now by means of (5) the function $p_1(\varphi)$ from Eq. (4) we get for $p_0(\varphi)$ the equation

$$(1 - \gamma\cos^2\varphi)p_0' - [\nu_0(1 - \gamma\cos^2\varphi) + \nu_1]p_0 = -\nu_1 k^{-1}N(E). \quad (6)$$

Using (3) and (4) one can verify that the initial set of conditions on the functions $p_0(\varphi)$ and $p_1(\varphi)$ are equivalent to the following conditions: $p_0(\varphi)$ must be a π-periodic function of φ and must satisfy a normalization condition of the following form

$$\int_0^\pi p_0\,d\varphi = \frac{\nu_1}{\nu_0 + \nu_1}.$$

From Eq. (6) and the conditions formulated here we can find N(E).

When solving (6) it is necessary to distinguish two cases depending on whether or not the coefficient of the derivative vanishes. This depends on whether the parameter γ is larger or less than unity and this in turn is connected (as $\gamma = V_0/E$) with the region of the spectrum ($E > V_0$ or $E < V_0$) we are interested in. We give the results for those two cases separately.

1) $E > V_0$ (region above the barrier):

$$N(E) = \frac{k\nu_1}{\nu_0 + \nu_1} F^{-1},$$

$$F = \pi - \frac{4\nu_0}{\operatorname{sh} f(\pi/2)} \int_0^{\pi} \operatorname{ch}\left[f\left(\frac{\pi}{2}\right) - f(\varphi)\right] d\varphi \int_0^{\varphi} \operatorname{ch} f(t)\, dt, \tag{7}$$

$$f(t) = \nu_0 t + \frac{\nu_1}{\sqrt{1-\gamma}} \operatorname{arctg} \frac{\operatorname{tg} t}{\sqrt{1-\gamma}}.$$

2) $E < V_0$ (region of the fluctuation spectrum):

$$N(E) = \frac{k\nu_1}{\nu_0 + \nu_1} F^{-1},$$

$$F = \pi - \nu_0 \int_{-\pi+\alpha}^{-\alpha} d\varphi \int_{\varphi}^{-\alpha} dt\, e^{f(\varphi)-f(t)} + \nu_0 \int_{-\alpha}^{\alpha} d\varphi \int_{-\alpha}^{\varphi} dt\, e^{f(\varphi)-f(t)},$$

$$f(t) = \nu_0 t + \frac{\nu_1}{2\sqrt{\gamma - 1}} \ln\left|\frac{\sin(\varphi - \alpha)}{\sin(\varphi + \alpha)}\right|, \tag{8}$$

where α is the root of the equation $1 - \gamma\cos^2\varphi = 0$ which lies in the first quadrant; $\nu_r = n_r/k$; $k^2 = 2mE/h^2$.

Equations (7) and (8) give the solution of the original problem which enables us to evaluate the average number of states for any E.

3. DISCUSSION OF THE OBTAINED EQUATIONS

In the present section we shall indicate the very simple asymptotic formulae for N(E) on characteristic ranges of the spectrum and in first instance at its ends, and we also discuss the form of N(E) for various limiting cases for the change in the parameters n_0 and n_1.

1. The right-hand end of the spectrum ($E \to \infty$). We have

$$N(E) \approx \frac{k}{\pi}\left\{1 + \frac{n_1}{2(n_0+n_1)}\frac{V_0}{k^2} + o\left(\frac{1}{k^4}\right)\right\}.$$

2. The left-hand end of the spectrum ($E \to 0$). To elucidate the asymptotic behavior of N(E) in this case it is necessary to use Eq. (8), letting in it γ, ν_0, and ν_1 tend to infinity, but in such a way that $\nu_r/\sqrt{\gamma} = n_r/\sqrt{V_0}$ remains fixed.

It is at once clear that $N(E) \to 0$ as $E \to 0$ and F must then be large. The third term in (8) is thus the only important one since in the second term the integrand does not exceed unity. In turn, the value of this integral is mainly determined by the value of the integrand near the point $\varphi = \alpha$, $t = -\alpha$. Splitting that contribution off, we get

$$N(E) \approx \frac{n_0 n_1}{n_0 + n_1} H^{-2}\left(\frac{n_0}{\sqrt{V_0}}, \frac{n_1}{\sqrt{V_0}}\right) e^{-\pi n_0/k}, \tag{9}$$

where

$$H(x, y) = \int_0^{\infty} e^{-x-t}\left(\frac{t}{t+2x}\right)^{y/2} dt.$$

We note that the exponential character of the asymptotic behavior of N(E) (including the form of the index of the exponential) was shown by I. M. Lifshitz[3] from qualitative considerations connected with the evaluation of the probability of the appearance of low-lying levels.

3. $E \sim V_0$. Although the formulae giving N(E) to the left and to the right of $E = V_0$ are different, one can show that

$$N(V_0 - 0) = N(V_0 + 0) = \frac{n_1}{n_0 + n_1}\sqrt{\frac{2mV_0}{h^2}} F^{-1},$$

$$F = \pi - 4\nu_0 \int_0^{\pi/2} \operatorname{ch}[\nu_0\varphi + \nu_1 \operatorname{tg}\varphi]\, d\varphi \int_0^{\varphi} e^{-\nu_0 t - \nu_1 \operatorname{tg} t}\, dt.$$

Of definite interest is a study of different limiting cases when the parameters n_0 and n_1 are very large or small.

As n_0^{-1} and n_1^{-1} are the average lengths of the wells and the barriers, the cases which we shall consider correspond to situations where either the wells or the barriers or both together become either very wide or very narrow.

We note at once that the case when only one of the parameters tends to zero or infinity is least interesting since all such limiting transitions[1)] give for N(E) expressions of the form $\pi^{-1}[2mE/h^2]_+^{1/2}$ or $\pi^{-1}[2m(E - V_0)/h^2]_+^{1/2}$ (here we have written $[x]_+^{1/2} = \sqrt{x}$ when $x \geq 0$, $[x]_+^{1/2} = 0$ when $x < 0$) corresponding to the situations when the potential is almost everywhere equal to either zero or V_0.

A. The quantities n_0 and n_1 tend to zero simultaneously, i.e., $n_0 \to 0$, $n_1 \to 0$ while the ratio n_0/n_1 is fixed. Then

$$\pi N(E) \approx \frac{n_1}{n_0 + n_1}\left[\frac{2m}{h^2}E\right]_+^{1/2} + \frac{n_0}{n_0 + n_1}\left[\frac{2m}{h^2}(E - V_0)\right]_+^{1/2}.$$

This form of N(E) can easily be explained in this limiting case.

Indeed, as $n_0, n_1 \to 0$ simultaneously the potential remains practically constant along very long stretches and therefore simply gives a random shift in E (we recall that $n_1/(n_0 + n_1)$ and $n_0/(n_0 + n_1)$ are directly the probabilities that V(x) is equal to 0 or V_0, respectively).

B. The quantities n_0 and n_1 tend simultaneously to infinity (i.e., $n_0 \to \infty$, $n_1 \to \infty$, while n_0/n_1 is fixed). Then

$$N(E) \approx \frac{1}{\pi}\left[\frac{2m}{h^2}\left(E - \frac{n_0}{n_0 + n_1}V_0\right)\right]_+^{1/2},$$

i.e., in this case we get simply a shift in the quantity $n_0V_0/(n_0 + n_1)$ which is equal to the average value of the potential. This is understandable since as $n_0, n_1 \to \infty$ practically all wells and barriers are very narrow so that the potential becomes a very fast oscillating one. However, the wave function is a rather smooth functional of the potential and therefore will change considerably more slowly. Then, if we apply to Eq. (1) the operation

$$\frac{1}{\Delta}\int_x^{x+\Delta} \ldots dx \qquad (n_0^{-1}, n_1^{-1} \ll \Delta \ll k^{-1}),$$

it will not act upon the wave function, but the potential is changed to its average value.

C. The quantity $n_0 = n_1 = n \to \infty$, $V_0 \to \infty$, $V_0^2/8n = d$ finite. In such a limiting transition the potential must change to a white noise process, the values of which at

[1)]Here and henceforth we shall talk only about the main terms of the appropriate asymptotic formula, i.e., merely about the limits of N(E).

different points are independent random quantities and $\langle V(x)\rangle = 0$, $\langle V(x)V(x')\rangle = d\delta(x - x')$. N(E) has been evaluated before for such a potential.[1,4]

In our case the correlation function of the potential is ($n_0 = n_1$)

$$\langle [V(x) - \langle V(x)\rangle][V(x') - \langle V(x')\rangle]\rangle = {}^1/_4 V_0^2 e^{-2n|x-x'|}$$

and in order that it tends to a δ-function we must change the parameters V_0 and n just in the way we have indicated. Shifting, moreover, the energy origin to the point $V_0/2$ we get

$$N^{-1}(E) = d^{-1/3}\int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy e^{-f(x)+f(y)},$$

$$f(x) = -\frac{x^3}{3} - \frac{2mE}{h^2 d^{2/3}}x,$$

which is the same as the results obtained in[1,4] (see, for instance, Eq. (1.58) in[4]).

D. The quantity $V_0 \to \infty$, $n_1 \to \infty$, while $n_1/V_0 = \sigma_1$ and n_0 are fixed. Such a limiting transition corresponds to a "contraction" of the barriers and their change to δ-functions. The resulting potential will be equal to $\sum s_j \delta(x - x_j)$ where the distances between the δ-functions $(x_{j+1} - x_j)$ are independent random quantities with a density of the probability distribution $n_0 e^{-n_0 x}$ ($x \geq 0$) while the coefficients s_j are also independent non-negative random quantities (barrier areas) distributed as $\sigma_1 e^{-\sigma_1 s}$ ($s \geq 0$). We have then

$$kN^{-1}(E) = \pi + \int_{-\pi/2}^{\pi/2} d\varphi \int_{-\pi/2} dt e^{f(\varphi)-f(t)}, \quad f(\varphi) = \nu_0\varphi - \sigma_1 \operatorname{tg}\varphi. \quad (10)$$

As $E \to 0$ we get from this

$$N(E) \approx \frac{n_0}{H\cdot 2(n_0\sigma_1)} e^{-\pi n_0/k},$$

$$H_1(x) = \int_0^{\infty} e^{-t-x/t}dt = \sqrt{4x}K_1(\sqrt{4x}).$$

It is interesting that this asymptotic behavior could also have been obtained straight from (9).

We note that a potential of the form $\sum s_j \delta(x - x_j)$ with the same x_j but with an arbitrary distribution of the s_j was considered in[1].

In[1] they found for the function T(z), which was such that $\lim_{z\to\pm\infty} z^2 T(z) = n_0 N(E)$ (T(z) is the probability density for the quantity $z = k \tan \varphi$ in our notation) the equation

$$\frac{d}{dz}[(z^2 + k^2)T] + n_0 \int_{-\infty}^{\infty} [T(z-s) - T(z)]p(s)ds = 0,$$

where p(s) is the density of the probability distribution of the random quantity s. It is, however, not clear how one can solve this equation in the case of an arbitrary function p(s). Our case, when

$$p(s) = \begin{cases} \sigma_1 e^{-\sigma_1 s}, & s \geq 0 \\ 0 & s < 0 \end{cases},$$

is, apparently, one of the few when this equation can be solved. The result obtained is, of course, the same as (10).

One can similarly consider the case $V_0 \to \infty$, $n_0 \to \infty$, while $n_0/V_0 = \sigma_0$ and n_1 are fixed; this corresponds to a "contraction" of the wells to δ-functions. It is here, in contrast to the previous case necessary to change the energy origin right from the start to the point V_0. The resulting potential has the same form $\sum s_j \delta(x - x_j)$, but the s_j will now be negative and have a distribution density

$$p(s) = \begin{cases} 0, & s > 0 \\ \sigma_0 e^{-\sigma_0 s}, & s \leq 0 \end{cases}.$$

We shall not write out the formulae obtained here as they are rather complicated. We only give the asymptotic behavior as $E \to -\infty$:

$$N(E) \approx n_1 e^{-2\sigma_0\varkappa}, \quad \varkappa^2 = -2mE/h^2.$$

This case like the preceding one can also be studied using the methods of[1].

In concluding this section we note that the method which we have applied above to find the density of the probability distribution of the phase and hence also the average number of states consisting of a consideration of the phase and the potential as a vector Markov process can, in principle, also be applied to all one-dimensional problems in which either the potential itself is a Markov process or is one component of some vector Markov process. However, the equations obtained in all cases known to the authors for a stationary probability density (analogous to Eq. (4)) does not allow us either to solve it or even to study it so fully that we can derive complete information about N(E).

4. EVALUATION OF N(E) FOR A SLOWLY CHANGING POTENTIAL

As we have already mentioned in the Introduction we shall in this section calculate the average number of states for a potential which takes on alternately the values, say, 0 and V_0 on sections the lengths of which are independent, equally distributed random quantities for which the average values are infinite, i.e., if $f(l)$ is the density of the probability distribution of the lengths of the intervals along which the potential is constant (wells and barriers), we have $\int_0^\infty l f(l) dl = \infty$ ($\int_0^\infty f(l) dl = 1$, is finite).

As an example of such a function we can take the Cauchy distribution

$$f(l) = \begin{cases} \dfrac{1}{2\pi}\dfrac{a}{a^2 + l^2}, & l \geq 0 \\ 0, & l < 0 \end{cases}.$$

We denote the intervals along which the potential is constant, starting from zero by l_1, l_2, ... and let n(L) be the number of intervals lying on the stretch (0, L) on which we consider Eq. (1), i.e.,

$$L = \sum_1^{n(L)} l_k + \Delta_{n(L)}, \quad 0 \leq \Delta_{n(L)} \leq l_{n(L)+1}.$$

Let us now consider instead of one boundary value problem on the interval (0, L) n(L) + 1 problems on the sections $l_1, \ldots, l_{n(L)}, \Delta_{n(L)}$ with the conditions of vanishing at the ends and n(L) + 1 problems along the same sections with the condition that the derivatives vanish. The corresponding number of states per unit length we denote by $N_1(l_k, E)$ and $N_2(l_k, E)$. According to the variational principle (see, for instance,[5]) we then get the following two-sided estimate:

$$\Delta_{n(L)}N_1(\Delta_{n(L)},E)+\sum_1^{n(L)} l_kN_1(l_k,E)\leqslant LN_L(E)\leqslant$$

$$\leqslant\sum_1^{n(L)} l_kN_2(l_k,E)+\Delta_{n(L)}N_2(\Delta_{n(L)},E).$$

As the potential is constant on each of the l_k, we can easily find $N_{1,2}(l_k, E)$ and the result is

$$\left|N_L(E)-\frac{1}{\pi}\left[\frac{2m}{h^2}E\right]_+^{1/2}\frac{1}{L}\sum_{\text{even}} l_k-\frac{1}{\pi}\left[\frac{2m}{h^2}(E-V_0)\right]_+^{1/2}\frac{1}{L}\sum_{\text{odd}} l_k\right|\leqslant\frac{n(L)+1}{L},$$

where $\sum_{\text{even}} l_k$ ($\sum_{\text{odd}} l_k$) indicates a sum of the lengths of the intervals with even (odd) numbers with the addition of $\Delta_{n(L)}$ in the case of odd (even) n(L).

We now need average this inequality over all possible potentials. However,

$$\left\langle\frac{1}{L}\sum_{\text{even}} l_k\right\rangle=\left\langle\frac{1}{L}\int_0^L r(x)\,dx\right\rangle=\frac{1}{L}\int_0^L p(x,1)\,dx,$$

where p(x, r) is the probability that in the point x the quantity r(x) = r, and

$$\left\langle\frac{1}{L}\sum_{\text{odd}} l_k\right\rangle=\left\langle 1-\frac{1}{L}\sum_{\text{even}} l_k\right\rangle=\frac{1}{L}\int_0^L p(x,0)\,dx.$$

We can now show by standard probability theory considerations (for details see[2]) that $\frac{1}{L}\int_0^L p(x,0)dx$, and hence also $\frac{1}{L}\int_0^L p(x,1)dx$ tends to ½ as L → ∞. Therefore, if we in addition to what we have already said also let ⟨n(L)/L⟩ → 0 as L → ∞, we get in the limit

$$N(E)=\frac{1}{2\pi}\left(\left[\frac{2mE}{h^2}\right]_+^{1/2}+\left[\frac{2m}{h^2}(E-V_0)\right]_+^{1/2}\right).$$

By using again probability-theory formulae we can, however, show that

$$\lim_{L\to\infty}\left\langle\frac{n(L)}{L}\right\rangle=\lim_{p\to 0}\frac{p}{1-F(p)},\tag{11}$$

where $F(p)=\int_0^\infty e^{-pl}f(l)dl$ is the Laplace transform of the distribution function for the probabilities for the lengths over which the potential V(x) is constant.

The following chain of inequalities shows that the vanishing of the right-hand side of (11) is equivalent to the average length of the intervals becoming infinite, i.e., to the divergence of the integral $\int_0^\infty lf(l)dl$:

$$(1-e)\int_0^{1/p} lf(l)\,dl\leqslant\int_0^{1/p}\frac{1-e^{-pl}}{pl}lf(l)\,dl\leqslant\frac{1-F(p)}{p}=$$

$$=\int_0^\infty\frac{1-e^{-pl}}{pl}lf(l)\,dl\leqslant\int_0^\infty lf(l)\,dl.$$

We have used here the elementary inequalities

$$1-e^{-x}\leqslant x,\qquad x\geqslant 0;\qquad 1-e^{-x}\geqslant(1-e)x,\qquad 0\leqslant x\leqslant 1.$$

The scheme to evaluate N(E) given here can without great changes be applied also to the more complicated case when on each of the intervals, the average lengths of which as before are infinite, V(x) takes on arbitrary values which are independent both of l_k and of the values taken by V(x) on the other intervals, with a well-defined probability p(V). In that case

$$N(E)=\frac{1}{\pi}\int_{-\infty}^{\infty}[E-V]_+^{1/2}p(V)\,dV.$$

This scheme can also be generalized to the many-dimensional case to some, to be true, rather insignificant degree.

In conclusion we note that the results given in this section are intuitively obvious and correspond to the case of a "slowly changing" potential (the average values of the lengths of the intervals over which the potential is constant are infinite!) when we can consider V(x) to be simply a random shift in E.

[1] H. L. Frisch and S. P. Lloyd, Phys. Rev. 120, 1175 (1960).

[2] W. Feller, Introduction to Probability Theory and Its Applications, Vol. 2, J. Wiley and Sons, New York, 1957.

[3] I. M. Lifshitz, Usp. Fiz. Nauk 83, 611 (1964) [Sov. Phys.-Uspekhi 7, 855 (1965)].

[4] B. I. Halperin, Phys. Rev. 139, A104, 1965.

[5] R. Courant and D. Hilbert, Methods of Mathematical Physics, Interscience, 1953.

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