THE QUANTUM THEORY OF THE LASER

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The quantum theory of single-mode laser radiation is considered. In the coherent state representation the problem reduces, in the quasiclassical limit, to an investigation of a closed set of equations of the hydrodynamic type. The small parameter of the problem in this case is the quasiclassical parameter $1/\sqrt{n}$ where n is the number of photons. The equations derived are used for determining the photon distribution function and the damping decrement of the mean field due to quantum phase fluctuations. Two limiting cases are analyzed in detail: the case of ion density of excited atoms when collective processes in the spontaneous radiation do not play any role, and the case of high density when effects of the self-consistent field must be taken into account in the spontaneous radiation from the atoms. It is demonstrated that for low radiation energies collective effects result in suppression of the radiation fluctuations (radiation "capture"). At high energies, near the instability region of stationary generation, the fluctuations become much stronger and on the boundary of the region attain a relative magnitude of the order of unity.

1. INTRODUCTION

An appreciable number of recently published papers is devoted to the study of quantum fluctuations of radiation in the quasiclassical region, i.e., when the number of photons in one mode is large. The most interesting from this point of view is the study of the fluctuations of laser emission, since strong coupling between the electromagnetic field and the radiating medium exists in a laser.

It was established in[1-8] that, depending on the laser emission power, there are three characteristic regions with different statistical radiation properties: below the threshold of classical generation the quantum "noise" can be regarded as thermal with a certain effective temperature; at the threshold, the distribution function of the photons is Gaussian, and the relative fluctuations in this region are of the order of unity; above the threshold, the photon distribution approaches a Poisson distribution. Qualitatively these conclusions agree with experiment[9-12]. We note that references[1-8] differ from one another in method, and in some cases also in the results. In earlier investigations[1-3], a semiphenomenological approach was used, connected with introducing noise sources into the classical equations of motion. A strictly quantum approach was developed in the later papers.

We emphasize that in the papers mentioned above they investigated a case typical of a helium-neon laser, when the lifetime of the photon in the resonator $1/\nu$ is large compared with the lifetime τ of the excited atom:

$$\nu\tau \ll 1. \tag{1}$$

In this case the length $c\tau$ of the electromagnetic train in the spontaneous emission of the atoms is shorter than the photon mean free path c/ν and collective effects in the spontaneous emission, which are not linear in the atom density, do not play any role. When condition (1) is satisfied, the equation for the photon density matrix reduces in the quasiclassical region to the Fokker-Planck equation.

In this paper we consider for the quantum description of laser emission a method connected with the use of coherent states. Such an approach is particularly lucid and convenient in the quasiclassical region, where the quantum fluctuations become relatively small. Using the procedure of decoupling the Bogolyubov chain of equations, a closed system of equations is obtained for several of the first distribution functions. The small parameter of the problem is in this case $1/\sqrt{\langle n\rangle}$, where $\langle n\rangle$ is the average number of photons in the system. In the representation of the coherent states, the obtained equations turned out to be equivalent to a system of hydrodynamic equations, which describe two-dimensional flow of a compressible liquid in a strong external field. In Secs. 4.1–4.3, these equations are considered under the conditions of the inequality (1). The photon distribution function is obtained, and the damping decrement of the average field due to the quantum fluctuations of the phase is determined. Above the generation threshold, the dispersion of the distribution function of the photons differs from that obtained by others.

Further, (Secs. 5.1–5.4) we consider the case $\nu\tau \gtrsim 1$, when collective effects due to the "overlap" of the radiation processes and the absorption of photons by different atoms become significant.

2. EQUATIONS OF MOTION IN THE COHERENT-STATE REPRESENTATION

The simplest model of the single-mode laser is a quantum oscillator of frequency ω_0, interacting with a system of N two-level resonant atoms with transition frequency ω_{ab}. Such a model corresponds to the well known spin Hamiltonian (in the interaction representation; $\hbar = 1$)

$$H = {}^1\!/_2\omega\sigma_3 + g(a^+\sigma_- + a\sigma_+), \quad \omega = \omega_{ab} - \omega_0, \quad \sigma = \sum_i \sigma^i, \quad g = d_{ab}\sqrt{\omega_0/2V}. \tag{2}$$

Here a^+ and a are Bose creation and annihilation operators, σ^i are spin matrices, σ^i_+ and σ^i_- are the matrices of upward and downward spin flip:

$$\sigma_+{}^i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_-{}^i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3{}^i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

d_{ab} is the dipole moment of the transition ab, V is the volume of the system, and the summation is over all the spins from 1 to N. For simplicity, the atoms are assumed to be immobile and the coupling constant g for all the atoms is assumed to be the same. The processes of relaxation and excitation of the atoms will be taken into account in the next section.

It will be convenient to write the initial equations from the very outset in the coherent-state representation. In this representation, the operator a is diagonal[13]:

$$a|z\rangle = z|z\rangle, \quad z = x + iy. \tag{4}$$

In the coherent-state representation, the eigenvectors are not orthgonal. For the two vectors $|z\rangle$ and $|z'\rangle$ we have

$$|\langle z'|z\rangle|^2 = e^{-|z-z'|^2}. \tag{5}$$

In order to obtain the equations of motion in the z-representation, we write the density matrix of the entire system (spins plus radiation) z, following Sudarshan[14] and Glauber[13], in the so-called P-representation:

$$r = \int d^2z|z\rangle\langle z|r(z,t). \tag{6}$$

We now substitute (6) in the equation for the density matrix

$$i\frac{\partial r}{\partial t} = [H, r], \quad \mathrm{Sp}\, r = 1, \tag{7}$$

where Sp denotes the trace over all the spin variables and the oscillator variables. Multiplying (7) from the left and from the right by the vectors $\langle z'|$ and $|z'\rangle$, and taking (5) into account, we obtain after integrating by parts an equation for the density matrix in the z-representation:

$$i\frac{\partial r}{\partial t} = \mathscr{H} r - r\mathscr{H}^+, \tag{8}$$

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1, \quad \mathscr{H}_0 = \sum_i \mathscr{H}_{0i} = {}^1/{}_2\omega\sigma_3 + g(\bar{z}\sigma_- + z\sigma_+),$$
$$\mathscr{H}_1 = \sum_i \mathscr{H}_{1i} = -g\sigma_-\nabla, \quad \nabla = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{9}$$

The superior bar denotes complex conjugation. The quantity $\mathscr{H}_0 = \mathscr{H}_0^+$ is obviously the Hamiltonian of the interaction between the spins and the classical field z.

The non-Hermitian Hamiltonian $\mathscr{H}_1$ describes the processes of emission and absorption of photons in the system. If we discard $\mathscr{H}_1$, then the Hamiltonian $\mathscr{H}_0$ makes it obviously possible to represent the density matrix of the system r in the form of a product of the density matrices of the spins and of the oscillator, the oscillator density matrix being constant in time. Such an approximation is justified only when the number of photons is large and $|z|^2 \gg N$. Therefore $\mathscr{H}_1$ cannot be regarded as a perturbation relative to $\mathscr{H}_0$ when $|z|^2 \lesssim N$, when the density matrix of the oscillator changes appreciably in the process of interaction with the spins. In this case, in the classical limit, the self-consistent behavior of the spins and of the radiation were considered earlier in[15,16].

Thus, $\mathscr{H}_1$ includes the classical effect of the self-consistent field of order $N/|z|^2$, and also the quantum effects connected with the photon diffusion and having an order $1/|z|^2$. We introduce the oscillator distribution function

$$\rho(z) = \langle r(z, t)\rangle_{12\ldots N}, \tag{10}$$

where $\langle\ldots\rangle_{12\ldots N}$ denotes the trace over the variables of spin 12...N. Then the density matrix of the oscillator takes the form

$$R = \int d^2z|z\rangle\langle z|\rho(z,t), \quad \int d^2z\rho(z,t) = 1. \tag{11}$$

Strictly speaking, $\rho(z, t)$ need not necessarily be a positive definite quantity for all z. However, in the quasiclassical limit ($|z|^2 \gg 1$), in which we are interested, $\rho(z, t)$ is essentially the ordinary distribution function.

We define further the single-, two- and three-particle spin density matrices:

$$\rho r_i(z, t) = \langle r\rangle_{1\ldots i-1,\, i+1\ldots N},$$
$$\rho r_{ij}(z, t) = \langle r\rangle_{1\ldots i-1,\, i+1\ldots j-1,\, j+1\ldots N}, \tag{12}$$
$$\rho r_{ijk}(z, t) = \langle r\rangle_{1\ldots i-1,\, i+1\ldots j-1,\, j+1\ldots k-1,\, k+1\ldots N}.$$

The density matrices defined in this manner satisfy the normalization condition

$$\langle r_{ijk}\rangle_k = r_{ij}, \quad \langle r_{ij}\rangle_j = r_i, \quad \langle r_i\rangle_i = 1. \tag{13}$$

The function $\rho(z)r_i(z)$ is the joint distribution function of the oscillator and of the i-th spin in the z-plane. Analogously, ρr_{ij} is the joint distribution function of the oscillator and two spins, etc. We emphasize that Eqs. (12) do not mean factorization of the joint distribution functions into spin and oscillator functions, since r_i, r_{ij}, and r_{ijk} depend on ρ.

We obtain from (8) a system of coupled equations for the distribution functions (10) and (12). Putting $r_{ij} = r_i r_j + \delta r_{ij}$, we get

$$i\frac{\partial\rho}{\partial t} + \nabla(gP_-\rho) + \overline{\nabla}(gP_+\rho) = 0, \tag{14}$$

$$i\frac{\partial r_i}{\partial t} + g(P_-\nabla - P_+\overline{\nabla})r_i = [\mathscr{H}_{0i}, r_i] + S_i + \delta S_i, \tag{15}$$

$$P_\pm = \sum_i p_\pm{}^i = \sum_i \langle\sigma_\pm{}^i r_i\rangle_i,$$

$$S_i = -\frac{g}{\rho}[\nabla(\tilde{\sigma}_-{}^i r_i\rho) - \text{э.c.}], \quad \tilde{\sigma}_\pm{}^i = \sigma_\pm{}^i - p_\pm{}^i, \tag{16}$$

$$\delta S_i = -\frac{g}{\rho}\sum_{j\neq i}(\nabla\langle\sigma_-{}^j\delta r_{ij}\rho\rangle_j - \text{h.c.}). \tag{17}$$

Equation (14) is the continuity equation for two-dimensional flow (in the z plane) of an incompressible liquid, expressed in complex form. Here $gP_\pm$ plays the role of the flow velocity. Equation (15) without the quantum corrections S_i and δS_i is none other than the self-consistent equations of motion of the spins and of the classical field, written in Euler variables. Indeed, in Lagrange variables we obtain in this case in lieu of (15) a system of equations for $r_i(t)$ and the classical field z(t):

$$i\frac{dr_i}{dt} = [\mathscr{H}_{0i}(z(t)), r_i], \quad i\frac{dz}{dt} = gP_-(t). \tag{18}$$

Allowance for the terms S_i and δS_i in Eq. (15) leads to the correlation of the distribution functions of the oscillator and of the spin and to the appearance of the diffusion of photons in z-plane.

In the classical limit S_i is a small quantity, and its order relative to $[\mathscr{H}_{0i}, r_i]$ is determined by the quantity $\rho^{-1}\partial\rho/\partial|z|^2$, since the remaining functions in S_i are smoother than $\rho(z)$. For a distribution of the Poisson type with a dispersion of the order of $\sqrt{\langle n\rangle}$, we have $\rho^{-1}\partial\rho/\partial|z|^2 \sim 1/\sqrt{\langle n\rangle}$.

Equation (15) without the term δS_i is equivalent to the equation of motion of a low-temperature ideal liquid situated in a strong external field. To estimate the last term in (15), we write the equation for δr_{ij}. Putting

$$r_{ijk} = r_i r_j r_k + r_i \delta r_{jk} + r_j \delta r_{ik} + r_k \delta r_{ij} + \delta r_{ijk},$$
$$\langle \delta r_{ijk}\rangle_k = \langle \delta r_{ijk}\rangle_j = \langle \delta r_{ijk}\rangle_i = 0,$$

we get from (8)

$$i\frac{\partial \delta r_{ij}}{\partial t} + g(P_-\nabla - P_+\overline{\nabla})\delta r_{ij} = [\mathscr{H}_{0i} + \mathscr{H}_{0j},\ \delta r_{ij}] \qquad (19)$$

$$-g\sum_{k\neq i\neq j}(\langle\sigma_-^k\delta r_{jk}\rangle_k\nabla r_i + \langle\sigma_-^k\delta r_{ik}\rangle_k\nabla r_j - \text{h.c.}) - g(\tilde{\sigma}_-^i r_i\nabla r_i + \tilde{\sigma}_-^j r_j\nabla r_i - \text{h.c.}). \qquad (20)$$

Since δr_{ij} enters in Eq. (15) only in the form of a quantum correction, which can be comparable with S_i, it suffices to write the equation for δr_{ij} itself only with accuracy of the order of $1/\sqrt{\langle n\rangle}$. We have therefore discarded in (20) the small terms connected with $\mathscr{H}_{1i}$, $\mathscr{H}_{1j}$, and δ_{ijk}.

We note also that from the definition of the irreducible part δr_{ij} we get the condition $\langle \delta r_{ij}\rangle_i = \langle \delta r_{ij}\rangle_j = 0$, and that Eq. (20) agrees with this condition. With the aid of (20) it is easy to establish that the order of magnitude of $\delta S_i/S_i$ is $N/|z|^2$, and thus the region of the collective processes in the radiation diffusion coincides with the region $|z|^2 \lesssim N$ of the self-consistent-field approximation. As applied to a laser, the quantity $N/|z|^2$ is the ratio of the lifetimes of the excited atom and the photon in the resonator, so that the region of the collective effects, as already noted, is $\nu\tau \gtrsim 1$.

3. PROCESSES OF RELAXATION AND EXCITATION OF THE ATOMS

In order to take into account the relaxation processes connected with the finite lifetimes of the atoms and the photon in the resonator, and also processes of excitation of the atoms by the pump, we proceed in the following manner. Since we are interested only in equal-time distribution functions, the corresponding operators of relaxation and excitation can be introduced directly in the initial equation for the density matrix of the entire system. Of course, these operators should not violate the normalization and the hermiticity of the density matrix r.

Equation (7) then takes the form

$$\frac{\partial r}{\partial t} = \hat{\nu} r + \sum_i \hat{\tau}_i^{-1} r + \frac{1}{i}[H, r], \qquad (21)$$

$$\hat{\nu} r = \nu(-a^+ a r + a r a^+ + \text{h.c.}), \qquad (22)$$

$$\hat{\tau}_i^{-1} r = (r_i^{(0)}\langle r\rangle_i - r)/\tau. \qquad (23)$$

Here $r_i^{(0)}$ is the normalized ($\langle r_i^{(0)}\rangle_i = 1$) density matrix of the i-th spin in the absence of radiation. It is natural to assume that without radiation the atoms are unpolarized. Therefore we have $r_i^{(0)} = (1 - \sigma_3^i)/2$ for an excited atom and $r_i^{(0)} = (1 - \sigma_3^i)/2$ for an unexcited atom. For simplicity we also assume that the times of longitudinal and transverse relaxations of the spin are equal. The number of excited spins will be denoted by N_a, and the number of unexcited spins by N_b. Thus, $N = N_a + N_b$ is the total number of the spins, and $\Delta N = N_a - N_b > 0$ is the excess population. We shall also assume that ΔN is not very small, $\Delta N \gg \sqrt{N}$, for otherwise the excess population will have a fluctuation origin. The relaxation operators (22) and (23) obviously satisfy the necessary conditions indicated above, since

$$\mathrm{Sp}_a(\hat{\nu} r) = 0,\ \langle \hat{\tau}_i^{-1} r\rangle_i = 0.$$

In the z-representation, the oscillator damping operator is given by

$$\hat{\nu} r = \nu(\nabla(zr) + \overline{\nabla}(\bar{z}r)). \qquad (24)$$

The oscillator damping operator leads to the following substitution in the left sides of Eqs. (14), (15), and (20):

$$gP_- \to v_- = gP_- - i\nu z, \quad gP_+ \to v_+ = gP_+ + i\nu\bar{z}. \qquad (25)$$

We note that the coupling constant g, which has the dimension of frequency, is a very small quantity. Thus, for a gas laser, the typical order of magnitude is $g\tau \sim 10^{-3}$. The characteristic order of magnitude of $|z|$ (above the generation threshold) is $(g\tau)^{-1}$.

It will be convenient further to consider separately the cases $\nu\tau \ll 1$ and $\nu\tau \gtrsim 1$.

4.1. THE CASE $\nu\tau \ll 1$

In this case, as already noted above, the collective effects do not play any role and δS_i can be omitted from Eq. (15). The system of Eqs. (14) and (15) then becomes closed. Since the relaxation in the spin system occurs much more rapidly than the change of the photon distribution function (within a time of the order of $1/\nu$), the equation for $r_i(z, t)$ can be considered in the quasistationary approximation, discarding in it the terms $v_-\nabla$ and $v_+\overline{\nabla}$, which are of the order of $\nu\tau$.

After this, the equation for r_i assumes the form

$$\frac{r_i}{\tau} = \frac{r_i^{(0)}}{\tau} + \frac{1}{i}[\mathscr{H}_{0i}, r_i] + \frac{1}{i}S_i, \qquad (26)$$

$$S_i \approx -g(\tilde{\sigma}_-^i r_i \nabla\rho/\rho - \text{э.c.}). \qquad (27)$$

With respect to Eq. (26), it should be noted that although the terms of order $\nu\tau$, which were discarded in it, are generally speaking larger than S_i, nonetheless they are insignificant, since they do not contain the gradient of the distribution function $\rho(z, t)$, and therefore do not influence the photon diffusion coefficient[1)]. For the same reason, only the term with $\nabla\rho/\rho$ was retained in expression (27) for S_i.

The problem consists of finding the quantum correc-

[1)]We note that in the stationary case we have $v_+ = v_- = 0$ when $\omega = 0$, meaning that there is no probability flux.

tions that must be added to the classical radiation current because of the diffusion of the photons in the z-plane.

We note that the approximation of Scully and Lamb[7] is equivalent to the representation of the equation for $r_i(z, t)$ in the form (26), the only difference being that $\sigma_\pm^i$ enters in place of $\tilde{\sigma}_\pm^i$. This difference is connected with the fact that in[7], in determining the joint density matrix of the spin and of the oscillator, it was assumed that the photon density matrix does not change within a time of the order of τ. Yet the determination of the diffusion coefficient requires a higher accuracy. The difference between $\tilde{\sigma}_\pm^i$ and $\sigma_\pm^i$ is significant only at intermediate values of the radiation energy; at low and high radiation energies we have $|p_\pm^i| \ll 1$ and $\tilde{\sigma}_\pm^i \approx \sigma_\pm^i$.

Equation (26) can be easily solved by perturbation theory

$$r_i = r_i^{(1)} + r_i^{(2)} + \ldots;$$

$$\frac{r_i^{(1)}}{\tau} = \frac{r_i^{(0)}}{\tau} + \frac{1}{i}[\mathcal{H}_{0i}, r_i^{(1)}], \tag{28}$$

$$\frac{r_i^{(2)}}{\tau} = \frac{1}{i}[\mathcal{H}_{0i}, r_i^{(2)}] + \frac{1}{i}S_i(r_i^{(1)}). \tag{29}$$

From this we get the radiation current from the i-th spin

$$p_+{}^i = p_+^{i(1)} + p_+^{i(2)} + \ldots, \quad p_+^{i(1)} = \langle\sigma_+{}^i r_i^{(1)}\rangle_i, \quad p_+^{i(2)} = \langle\sigma_+{}^i r_i^{(2)}\rangle_i;$$

$$p_+^{i(1)} = \frac{1}{i}\frac{g\tau\bar{z}\lambda q_i}{1-i\varepsilon}, \quad \lambda = \left(1+\frac{4k}{1+\varepsilon^2}\right)^{-1}, \quad k = |g\tau z|^2, \quad \varepsilon = \omega\tau; \tag{30}$$

$$p_+^{i(2)} = a^i\frac{\nabla\rho}{\rho} + \frac{\bar{z}}{z}b^i\frac{\bar{\nabla}\rho}{\rho}; \tag{31}$$

$$a^i = \frac{ig\tau\lambda}{1-i\varepsilon}\left[\left(\frac{1+\lambda q_i}{2} - \frac{k\lambda^2}{1+\varepsilon^2}\right)\left(1+\frac{2k}{1+i\varepsilon}\right) + \frac{2k^2\lambda^2}{(1+i\varepsilon)^3} - \frac{k\lambda^2(1+\lambda q_i)q_i}{1+i\varepsilon}\right],$$

$$b^i = -\frac{ig\tau\varepsilon\lambda}{1-i\varepsilon}\left[\frac{\lambda}{(1-i\varepsilon)^2}\left(1+\frac{2k}{1+i\varepsilon}\right) + \frac{2}{1+i\varepsilon}\left(\frac{1+\lambda q_i}{2} - \frac{k}{1+\varepsilon^2}\right) + \frac{\lambda(1+\lambda q_i)}{1-i\varepsilon}\right].$$

Here $q_i = +1$ for spins that are directed upward in the absence of radiation, and $q_i = -1$ for spins directed downward.

The function $\lambda(|z|^2)$ describes the effect of saturation of the difference of the populations of the upper and lower levels of the atom by the radiation field. Summing the radiation currents from all the spins and substituting the result in the continuity equation

$$i\frac{\partial\rho}{\partial t} + \nabla(v_-\rho) - \bar{\nabla}(v_+\rho) = 0, \tag{14'}$$

we obtain the Fokker-Planck equation for the distribution function $\rho(z, t)$. Since the coefficients a^i and b^i depend only on $|z|^2$, the final equation is best written in cylindrical coordinates $z = \sqrt{\xi}e^{i\varphi}$, $\bar{z} = \sqrt{\xi}e^{-i\varphi}$:

$$\frac{\partial\rho}{\partial t} = 2\nu\left\{\frac{\partial}{\partial\xi}\left(A_1\rho + B_1\frac{\partial\rho}{\partial\xi}\right) + \frac{1}{\xi}\frac{\partial}{\partial\varphi}\left(A_2\rho + \frac{B_2}{\xi}\frac{\partial\rho}{\partial\varphi}\right)\right\}. \tag{32}$$

In Eq. (32) we have discarded the term with the mixed derivative, since it produces only a small quantum correction to the classical generation frequency shift due to the coefficient A_2. We introduce further the generation parameter η:

$$\eta = \frac{g^2\tau\Delta N}{(1+\varepsilon^2)\nu}. \tag{33}$$

In the classical limit, the generation region is determined by the condition $\eta > 1$; the condition $\eta = 1$ determines the generation threshold.

With the aid of (30) and (31) we get

$$A_1(\xi) = \xi\left[1 + \frac{g}{\nu}\sum_i \mathrm{Im}\left(\frac{p_+^{i(1)}}{\bar{z}}\right)\right] = \xi(1-\eta\lambda), \tag{34}$$

$$A_2(\xi) = \frac{g\xi}{2\nu}\sum_i \mathrm{Re}\left(\frac{p_+^{i(1)}}{\bar{z}}\right) = \frac{\eta\xi\lambda\varepsilon}{2}, \tag{35}$$

$$B_1(\xi) = \frac{g\xi}{\nu}\sum_i \mathrm{Im}(a^i + b^i) = \frac{\xi\eta\lambda^2}{1+\varepsilon^2}\left[(\lambda+\varepsilon^2)\frac{N_a}{\Delta N} + \left(\frac{1}{\lambda}-1\right)(1+\varepsilon^2-\lambda)\frac{N_b}{\Delta N}\right], \tag{36}$$

$$B_2(\xi) = \frac{g\xi}{4\nu}\sum_i \mathrm{Im}(a^i - b^i) = \frac{\xi\eta}{8(1+\varepsilon^2)}\Big\{[(1+\varepsilon^2)^2 + \lambda(1-\varepsilon^2-2\varepsilon^4) + \lambda^2\varepsilon^2(\varepsilon^2-1) + 2\varepsilon^2\lambda^3]\frac{N_a}{\Delta N} + [(1-\varepsilon^2)^2 - \lambda(1+3\varepsilon^2 + 2\varepsilon^4) + \lambda^2\varepsilon^2(\varepsilon^2-1) + 2\varepsilon^2\lambda^3]\frac{N_b}{\Delta N}\Big\}. \tag{37}$$

The coefficients A_1 and A_2 are classical quantities—the active and the reactive parts of the radiation power; B_1 and B_2 determine the radial and azimuthal coefficients of the diffusion of the photons in the z-plane. Let us expand the oscillator distribution function in a Fourier series

$$\rho(z, t) = \frac{1}{\pi}\sum_{m=-\infty}^{+\infty}\rho_m(\xi, t)e^{-im\varphi}, \qquad \rho_m = \bar{\rho}_{-m}. \tag{38}$$

In this expansion $\rho_0(\xi, t)$ is the photon distribution function, in terms of which the mean values of quantities such as the density operator are expressed; these mean values contain equal numbers of creation and annihilation operators. Mean values of the operators of the type a^m, $a^{+l}a^{l+m}$, etc. are expressed in terms of $\rho_m(\xi, t)$.

4.2. PHOTON DISTRIBUTION FUNCTION

The average number of photons $\langle n\rangle$ and the dispersion $\langle(\Delta n)^2\rangle$ are connected with the distribution function $\rho_0(\xi, t)$, obviously, in the following manner:

$$\langle n\rangle = \mathrm{Sp}_a(a^+aR) = \int_0^\infty d\xi\,\xi\rho_0(\xi, t), \qquad \int_0^\infty d\xi\rho_0(\xi, t) = 1, \tag{39}$$

$$\langle(\Delta n)^2\rangle = \mathrm{Sp}_a((a^+a)^2R) - \langle n\rangle^2 = \int_0^\infty d\xi\,\xi^2\rho_0(\xi, t) + \langle n\rangle - \langle n\rangle^2. \tag{40}$$

Proceeding to the solution of Eq. (32) for the photon distribution function, we confine ourselves to an examination of the stationary state. Under stationary conditions there should be no probability flux; from this we get

$$A_1(\xi)\rho_0(\xi) + B_1(\xi)\frac{d\rho_0(\xi)}{d\xi} = 0. \tag{41}$$

The distribution function $\rho_0(\xi)$ assumes a simple form in the following three characteristic generation regions:

A. Below the threshold of classical generation ($g\tau \ll 1 - \eta \ll 1$). In this case the saturation effect does not play any role. Putting $\lambda = 1$ in (34) and (36) we obtain the Planck distribution

$$\rho_0(\xi) = \frac{1}{\langle n\rangle} e^{-\xi/\langle n\rangle}, \quad \langle n\rangle = \frac{N_a/\Delta N}{1-\eta}. \quad (42)$$

Since $\langle n \rangle \gg 1$, we have for the photon-number fluctuation the usual expression

$$\langle (\Delta n)^2\rangle = \langle n\rangle^2. \quad (43)$$

B. Threshold of classical generation ($\eta = 1$). In this case it is impossible to neglect completely the saturation effect. Putting in (34) $\lambda \approx 1 - 4k(1+\epsilon^2)^{-1}$, we arrive at the Gaussian distribution

$$\rho_0(\xi) = \frac{2}{\pi\langle n\rangle} e^{-\xi^2/\pi\langle n\rangle^2}, \quad \langle n\rangle = \frac{\sqrt{(1+\varepsilon^2)N_a/\Delta N}}{2g\tau} \quad (44)$$

$$\langle (\Delta n)^2\rangle = \langle n\rangle^2(\pi/2 - 1). \quad (45)$$

Cases A and B pertain to the region of low radiation energies. As already noted, here $\tilde{\sigma}^i_\pm \approx \sigma^i_\pm$. Therefore, if we put in (42)–(45) $\epsilon = 0$ and $N_a = \Delta N$, then these formulas coincide with the corresponding results of[7].

C. Above the threshold of classical generation ($\eta - 1 \gg g\tau$). Here we deal with the region of classical generation. Since the relative fluctuations are small in this region, the diffusion coefficient $B_1(\xi)$ can be taken at the point $\xi = \langle n \rangle$. Then

$$\rho_0(\xi) = \frac{1}{\sqrt{2\pi\langle n\rangle d}} \exp\left\{-\frac{(\xi - \langle n\rangle)^2}{2d\langle n\rangle}\right\}, \quad \langle n\rangle = \frac{(\eta-1)(1+\varepsilon^2)}{4(g\tau)^2},$$
$$\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle} = d + 1, \quad (46)$$

$$d = \frac{(1/\eta + \varepsilon^2)N_a/\Delta N + (\eta-1)(1+\varepsilon^2 - 1/\eta)N_b/\Delta N}{(\eta-1)(1+\varepsilon^2)}. \quad (47)$$

When $N_b = 0$ we get

$$\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle} = \frac{\eta}{\eta - 1} - \frac{1}{\eta(1+\varepsilon^2)}. \quad (48)$$

This expression for the dispersion of the photon distribution function is smaller than that obtained in[7] by an amount equal to the second term of formula (48).

When $\eta \gg 1$ and $d \sim 1/\eta \ll 1$, the distribution function $\rho_0(\xi)$ becomes δ-like:

$$\rho_0(\xi) = \delta(\xi - \langle n\rangle), \quad (49)$$

and the photon distribution obeys the Poisson law.

Finally, when $N_a \approx N_b \gg \Delta N$ and $\eta \gg 1$, we have

$$d = N/2\Delta N \gg 1, \quad (50)$$

i.e., the dispersion of the photon distribution function increases as a result of the increase in the fluctuations in the occupation numbers of the atomic states. However, under the conditions when the inequality $\Delta N \gg \sqrt{N}$ holds, the relative fluctuations of the photons are small.

4.3. ATTENUATION OF AVERAGE FIELD

To find the harmonics $\rho_m(\xi, t)$ of the oscillator distribution function we can proceed in the following manner. Confining ourselves to the region of classical generation (case C) and assuming that the generation has become stationary, we seek an approximate solution of (32) in the form

$$\rho_m(\xi, t) \approx C_m \rho_0(\xi) e^{i\Delta\nu_m t}, \quad (51)$$

where the constant C_m is determined by the initial conditions. Then the time dependence of the mean value of the operator a^m is of the form

$$\langle a^m\rangle \sim e^{i\Delta\nu_m t}. \quad (52)$$

When $m = 1$ formula (52) determines the frequency shift and the attenuation of the average electric field in the laser.

When (51) is substituted in the diffusion equation (32), it suffices to take the coefficients A_2 and B_2 at $\xi = \langle n \rangle$. As a result we obtain the following expressions for the real and imaginary parts $\Delta\nu'_m$ and $\Delta\nu''_m$ of $\Delta\nu_n$:

$$\Delta\nu_m' = -m\nu\varepsilon, \quad (53)$$

$$\Delta\nu''_m = \frac{\eta\nu m^2}{4\langle n\rangle(1+\varepsilon^2)}\left\{\left[1 + \varepsilon^4 - \frac{2\varepsilon^2(1+\varepsilon^2)}{\eta} + \frac{\varepsilon^2(\varepsilon^2+1)}{\eta^2} + \frac{2\varepsilon^2}{\eta^3}\right]\frac{N}{\Delta N} + \varepsilon^2\left(2 + \frac{1}{\eta}\right) + \frac{1}{\eta}\right\}. \quad (54)$$

The generation frequency shift (53) has a purely classical origin and corresponds to the frequency pulling effect. The attenuation of the average field is due to quantum fluctuations of the phase and determines the spectral width of the laser emission line[7].

Let us consider several limiting cases for the attenuation frequency. When $N = \Delta N$ and $\omega = 0$ we have

$$\Delta\nu''_m = \frac{\nu m^2(1+\eta)}{4\langle n\rangle}. \quad (55)$$

Near the generation threshold ($\eta \approx 1$) formula (55) coincides with that obtained in[7]:

When $\epsilon \gg 1$ and $N = \Delta N$

$$\Delta\nu''_m = \frac{\nu m^2}{\langle n\rangle}\frac{\varepsilon^2(\eta-1)^2}{4\eta}; \quad (56)$$

If $N \gg \Delta N$ and $\epsilon = 0$, then

$$\Delta\nu''_m = \frac{\nu m^2\eta}{4\langle n\rangle}\frac{N}{\Delta N}. \quad (57)$$

5.1. CASE $\nu\tau \gtrsim 1$

We shall analyze here the case of high density of the excited atoms, when the photon mean free path becomes smaller than the length of the electromagnetic train in spontaneous emission. Under these conditions, effects of the self-consistent field become significant in the spontaneous emission of the atoms. Effects of this kind, as is well known[17-19], are important in the theory of diffusion of resonant radiation in a gas. However, in the theory of radiation diffusion one is usually not interested in the photon distribution function in a single mode, and furthermore the radiation field is assumed as a rule to be weak, so that the saturation effect can be disregarded.

In the description of the laser radiation in the first approximation, one can confine oneself to a consideration of only one mode of the electromagnetic field, but the saturation effect, generally speaking, must be taken into account in all orders.

As already noted above, the criterion for the collective effects in the diffusion of laser emission is $\nu\tau \gtrsim 1$. The same condition can be written in the usual form as the criterion for the density of the excited atoms, if account is taken of the fact that the cross section for the absorption (emission) of a resonant photon is $\lambda\!\!\!^{-2}$ ($2\pi\lambda\!\!\!^{-}$—radiation wavelength):

$$\frac{N}{V}\lambda^3\omega_0\tau \gtrsim 1.$$

Here it is assumed for simplicity that the lifetime of the atom is determined by the spontaneous emission at the transition frequency $\tau \sim \lambda\!\!\!^{-3}/d_{ab}^2$, and that the emission energy is not too high ($\eta \sim 1$). In a strong radiation field the photon absorption cross section decreases like $1/\eta$.

We obtain below the stationary distribution function of the photons in the region of laser parameters where the classical generation is stable. Near the boundary of the instability region, the photon fluctuations become very strong. We also determine the damping decrement of the mean field in the quasistationary approximation.

5.2. INITIAL EQUATIONS

We proceed to calculate the photon diffusion coefficient with allowance for the "overlap" of the processes of emission and absorption of photons by various atoms. Whereas in the case $\nu\tau \ll 1$ it was sufficient to find only the correlation in the distributions of the spins and the photons, it is now necessary to take into account also the correlation in the distributions of the different spins. In other words, it is necessary already to know the irreducible part of δr_{ij} of the joint distribution function of the spins r_{ij}. It is clear that the equation for the distribution function $\rho(z, t)$ no longer reduces to the Fokker-Planck equation, since the relaxation in the spin system occurs more slowly than the relaxation of the oscillator.

To facilitate further calculations, we make the following simplifying assumptions. We confine ourselves to a consideration of the stationary generation regime under conditions of exact resonance ($\omega = 0$). In addition, we assume that in the absence of radiation all the spins are turned upward ($N = \Delta N$). Since all the spins are under identical conditions, the summation over the spins reduces now to multiplication by N. We note also that in the stationary state the distribution function ρ_0 depends only on $\xi = |z|^2$. Under these conditions it follows from (14′) that

$$v_+ = v_- = 0. \tag{58}$$

The integral of motion (58) denotes that there is no probability flux in the stationary case.

We write down Eqs. (15) and (20) for the macroscopic quantities:

$$P_\alpha = \sum_i \langle\sigma_\alpha{}^i r_i\rangle_i, \quad \tilde{P}_\alpha{}^- = \sum_i \langle\sigma_\alpha{}^i\tilde{\sigma}_-{}^i r_i\rangle_i, \quad \tilde{P}_\alpha{}^+ = \sum_i \langle\tilde{\sigma}_+{}^i\sigma_\alpha{}^i r_i\rangle_i, \tag{59}$$

$$\delta P_{\alpha\beta} = \sum_{i\neq j}\langle\sigma_\alpha{}^i\sigma_\beta{}^j\delta r_{ij}\rangle_{ij}, \quad \alpha,\beta = +,-,3. \tag{60}$$

The two-particle correlation matrix is obviously symmetrical: $\delta P_{\alpha\beta} = \delta P_{\beta\alpha}$. We then get from (15)

$$\frac{P_\alpha}{\tau} = \frac{P_\alpha^{(0)}}{\tau} + \frac{1}{i}\sum_\beta \mathscr{H}_{\alpha\beta}P_\beta - \frac{g}{i}\frac{d\rho_0}{\rho_0 d\xi}[(\tilde{P}_\alpha{}^- + \delta P_{\alpha-})\bar{z} - (\tilde{P}_\alpha{}^+ + \delta P_{\alpha+})z], \tag{61}$$

where $P_3^{(0)} = N$ and $P_\pm^{(0)} = 0$. The matrix $\mathscr{H}_{\alpha\beta}$ is given by

$$\mathscr{H}_{+3} = g\bar{z},\ \mathscr{H}_{-3} = -gz,\ \mathscr{H}_{3+} = 2gz,\ \mathscr{H}_{3-} = -2g\bar{z}, \tag{62}$$

and the remaining matrix elements are equal to zero.

From (20) we have, accurate to $1/N$,

$$\frac{2\delta P_{\alpha\beta}}{\tau} = \frac{1}{i}\sum_{\alpha'\beta'}\mathscr{H}_{\alpha\beta,\alpha'\beta'}\delta P_{\alpha'\beta'} - \left(\frac{g}{i}\right)\{\nabla P_\alpha(\delta P_{\beta-} + \tilde{P}_\beta{}^-) + \nabla P_\beta(\delta P_{\alpha-} + \tilde{P}_\alpha{}^-) - \overline{\nabla}P_\alpha(\delta P_{\beta+} + \tilde{P}_\beta{}^+) - \overline{\nabla}P_\beta(\delta P_{\alpha+} + \tilde{P}_\alpha{}^+)\}. \tag{63}$$

The matrix $\mathscr{H}_{\alpha\beta,\alpha'\beta'}$ results from the Hamiltonian $\mathscr{H}_{0i} + \mathscr{H}_{0j}$ and has a somewhat cumbersome form; we shall not write it out in explicit form. We use further the following symmetry conditions, which follow, as can be readily seen, from (63):

$$\delta P_{++}/\bar{z}^2 = \delta P_{--}/z^2, \quad \delta P_{3+}/\bar{z} = -\delta P_{3-}/z. \tag{64}$$

The matrix $\delta P_{\alpha\beta}$ has thus only four independent matrix elements.

Using the condition (58), we can represent the system of equations (63) in the following form:

$$\begin{pmatrix} \frac{1}{\tau}+\nu & -\frac{g}{i}\bar{z} & 0 & 0 \\ -2\frac{g'}{i}z & \frac{2}{\tau}+\nu & \frac{2g'}{i}\bar{z} & -\frac{g\bar{z}}{i} \\ 0 & \frac{gz}{i} & \frac{1}{\tau}+\nu & 0 \\ 0 & -\frac{4g'z}{i} & 0 & \frac{1}{\tau} \end{pmatrix}\begin{pmatrix}\delta P_{++} \\ \delta P_{3+} \\ \delta P_{+-} \\ \delta P_{33}\end{pmatrix} = \begin{pmatrix} -\nu\tilde{P}_+{}^+ \\ -\nu\tilde{P}_3{}^+ + \frac{g}{i}\frac{dP_3}{d\xi}(z\tilde{P}_+{}^+ - \bar{z}\tilde{P}_+{}^-) \\ -\nu\tilde{P}_+{}^- \\ 2\frac{g}{i}\frac{dP_3}{d\xi}z\tilde{P}_3{}^+ \end{pmatrix} \tag{65}$$

where $g' = g(1 + (1/2)dP_3/d\xi)$.

The population difference between the upper and lower states $P_3(\xi)$ decreases with increasing field, i.e., $dP_3/d\xi < 0$, so that a situation in which $g'/g < 0$ is possible. This produces, as it were, effective attraction forces between the spins, and the correlation between them can become very strong. Recognizing that the generation parameter η now takes the form

$$\eta = g^2\tau N/\nu \tag{66}$$

and using (58), we write the system (61) in the following form:

$$\eta q = 1 + A\frac{d\rho_0}{\rho_0 d\xi}, \quad q = \frac{P_3}{N}, \tag{67}$$

$$\eta q = \eta - 4\xi(g\tau)^2 + B\frac{d\rho_0}{\rho_0 d\xi}, \tag{68}$$

$$A = \frac{\eta}{N}\left[\tilde{P}_+{}^- + \delta P_{+-} - \frac{z}{\bar{z}}(\tilde{P}_+{}^+ + \delta P_{++})\right], \quad B = \frac{2g\tau z\eta}{iN}(\tilde{P}_3{}^+ + \delta P_{3+}). \tag{69}$$

Equations (67) and (68) constitute a system of differential equations of first order for the two unknown functions $\rho_0(\xi)$ and $q(\xi)$, where q is the excess population for one spin.

The coefficients A and B depend on ξ, $q(\xi)$, and $dq(\xi)/d\xi$. To find A and B it is necessary to solve the system (65) and express $\tilde{P}_\alpha^+$ and $\tilde{P}_\alpha^-$ in terms of P_α. According to the definition (59), we have

$$\tilde{P}_+{}^+ = -P_+{}^2/N = \nu\tau\bar{z}^2/\eta,$$
$$\tilde{P}_3{}^+ = i\nu\bar{z}(1+q)/g, \tag{70}$$
$$\tilde{P}_+{}^- = \tilde{P}_-{}^+ = {}^1\!/_2 N(1+q) - \nu\tau\xi/\eta.$$

5.3. PHOTON DISTRIBUTION FUNCTION

Since the terms containing the gradient of the photon distribution function in Eqs. (67) and (68) have the character of quantum corrections ($A \sim B \sim 1$), we shall solve these equations by perturbation theory. As will be shown later, perturbation theory is valid everywhere with the exception of a certain region of the parameters η and $\nu\tau$, in which the stationary generation becomes unstable.

The solution of the equations for $\rho_0(\xi)$ and $q(\xi)$ is in general quite cumbersome, and we shall consider immediately two limiting cases: generation in the vicinity of the threshold and generation above the threshold.

1) $|\eta - 1| \ll 1$. In this case the saturation effect is weak and $g\tau|z| \ll 1$. The only component of the vector $\widetilde{P}_\alpha$ which is not small is the component $\widetilde{P}_+^- \approx N$, and in the matrix $\delta P_{\alpha\beta}$ the only large element is δP_{+-}, for which we get from (65)

$$\delta P_{+-} = -\frac{\nu\tau}{1+\nu\tau} P_+^-. \tag{71}$$

Thus, in this limit we obtain ultimately

$$q = 1, \quad A = \frac{1}{1+\nu\tau}, \quad B = 0, \tag{72}$$

$$\frac{d\rho_0}{\rho_0 d\xi} = (1+\nu\tau)[\eta - 1 - 4\xi(g\tau)^2]. \tag{73}$$

When $\nu\tau \ll 1$, the distribution function $\rho_0(\xi)$ coincides with that obtained in the preceding case (formulas (42)–(48) with $N = \Delta N$). Below the threshold ($g\tau \ll 1 - \eta \ll 1$) the function $\rho_0(\xi)$ is of the form

$$\rho_0(\xi) = \frac{1}{\langle n\rangle} e^{-\xi/\langle n\rangle}, \quad \langle n\rangle = \frac{1}{(1-\eta)(1+\nu\tau)}. \tag{74}$$

At the threshold ($\eta = 1$) we have

$$\rho_0(\xi) = \frac{2}{\pi\langle n\rangle} \exp\left\{-\frac{\xi^2}{\pi\langle n\rangle^2}\right\}, \quad \langle n\rangle = \frac{1}{2g\tau\sqrt{1+\nu\tau}}, \tag{75}$$

$$\langle(\Delta n)^2\rangle = \langle n\rangle^2(\pi/2 - 1).$$

Above the threshold ($g\tau \ll \eta - 1 \ll 1$) we have

$$\rho_0(\xi) = \frac{1}{\sqrt{2\pi d\langle n\rangle}} \exp\left\{-\frac{(\xi - \langle n\rangle)^2}{2d\langle n\rangle}\right\}, \tag{76}$$

$$\langle n\rangle = \frac{\eta - 1}{4(g\tau)^2}, \quad d = \frac{1}{(\eta-1)(1+\nu\tau)}, \quad \langle(\Delta n)^2\rangle = (1+d)\langle n\rangle. \tag{77}$$

We see thus that the effect of overlap of the trains in spontaneous emission leads to the suppression of the radiation fluctuations. When $\nu\tau > 1$ a "dragging" of the radiation takes place: the photon spontaneously emitted by one atom is either absorbed by the other atom or leaves the resonator even before it has time to be formed.

The fact that the correlation function δP_{+-} turned out to be negative can be explained in a simplified manner as follows: let us consider two classical radiators, dipoles 1 and 2. If the first dipole has emitted a wave with phase φ_1, then the second dipole radiates under the influence of such a wave, in the case of an exact resonance, a wave with phase $\varphi_1 + \pi/2$. Under the influence of the wave with phase $\varphi_1 + \pi/2$, dipole 1 emits a wave with phase $\varphi_1 + \pi$. The resultant radiation of the first dipole is thus attenuated. It is clear that this reasoning pertains only to spontaneous emission, since stimulated emission leaves all the dipoles in a state with equal phase. In a strong radiation field, when multiple re-radiation processes must be taken into account, the sign of the correlation function may change, and the radiation fluctuations can accordingly increase.

2) $\eta - 1 \gg g\tau$. Solving Eqs. (67) and (68) by perturbation theory in the principal order (the classical limit), we get

$$q_0 = \frac{1}{\eta}, \quad \xi = \langle n\rangle = \frac{\eta - 1}{4(g\tau)^2}. \tag{78}$$

In this region the distribution function of the photons has a Gaussian form (76), so that

$$\frac{d\rho_0}{\rho_0 d\xi} = -\frac{1}{d}\delta\xi, \quad \delta\xi = \frac{\xi - \langle n\rangle}{\langle n\rangle}. \tag{79}$$

Since $\delta\xi$ is a small quantity ($\delta\xi \sim \langle n\rangle^{-1/2} \ll 1$), it follows that by putting

$$q = q_0 + q_1\delta\xi, \quad \xi = \langle n\rangle(1+\delta\xi), \tag{80}$$

we obtain in the first order in $\delta\xi$ a system of two algebraic equations for d and q_1:

$$\eta q_1 = -A/d, \tag{67'}$$

$$-\eta q_1 = \eta - 1 + B/d. \tag{68'}$$

In the calculation of the coefficients A and B we shall write q_0 and $\langle n\rangle$ in lieu of q and ξ, and replace $\xi dq/d\xi$ by q_1. In this approximation, the coefficients A and B take the form

$$A = \frac{\lambda}{1+\nu\tau}\left[\frac{\eta-1}{2}(k\eta - 1) + \eta\left(1 + \frac{\nu\tau}{2}\right)\right], \tag{81}$$

$$B = \frac{\eta - 1}{2}\lambda\left(\frac{k\eta}{1+\nu\tau} + 1\right), \tag{82}$$

$$\frac{1}{\lambda} = 1 + \frac{\nu\tau}{2} + \frac{\eta-1}{2}\,\frac{2+\nu\tau}{1+\nu\tau}k, \quad k = 1 + \frac{2\nu\tau\eta}{\eta - 1}q_1. \tag{83}$$

We note that $2(1+\nu\tau)\tau^{-4}ƛ^{-1}$ is the determinant of the system (65). The function $\lambda(\eta)$ determines the magnitude of the spin correlation. When $\nu\tau \ll 1$, the quantity $\lambda(\eta) \approx 1/\eta$ coincides with a previously introduced function, which describes the usual effect of saturation of the populations of the atom by the field (see formula (30)).

Solving the system (67′) and (68′), we obtain the dispersion of the distribution function of the photons

$$d = -\frac{A}{\eta q_1} = \frac{(1+\nu\tau/2)\lambda(\eta)}{(1+\nu\tau)(\eta - 1)}, \tag{84}$$

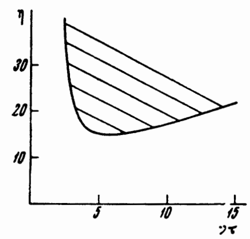
$$\lambda(\eta) = \frac{(1+\nu\tau)[2+\nu\tau+2\nu\tau\eta(\eta-1)]}{(1+\nu\tau/2)[(1+\nu\tau)(2+\nu\tau)+(2-\nu\tau)(\eta-1)]}. \tag{85}$$

When $\eta - 1 \ll 1$ we return to the previously considered case (formula (77)).

When $\nu\tau \gg \eta$ we obtain for the photon dispersion

$$\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle} = 1 + \frac{(2\eta - 1)\eta}{(\eta - 1)\nu\tau} \tag{86}$$

and far from the generation threshold, when $\eta - 1$ is not small, we have $\langle(\Delta n)^2\rangle/\langle n\rangle \to 1$, i.e., we obtain a Poisson distribution function, and the state of the quantum oscillator is a set of coherent states with $|z| = \sqrt{\langle n\rangle}$ and with a uniform phase distribution.



In the region $\eta \gtrsim \nu\tau$, the fluctuations increase. On approaching the curve (see the figure) defined by the equation

$$\eta - 1 = \frac{(1+\nu\tau)(2+\nu\tau)}{\nu\tau - 2} \tag{87}$$

λ and d increase strongly, and the fluctuations become anomalously large. As shown in the appendix, the region bounded by the curve (87) (shown shaded in the figure), coincide with the region of instability of the stationary generation regime. It is clear that in a small vicinity $\sim 1/\sqrt{\langle n \rangle}$ near the shaded region, $\delta\xi$ can no longer be regarded as small, and the perturbation theory employed by us is no longer valid. In this small vicinity, the spin correlation also increases appreciably; the irreducible part δr_{ij} of the joint distribution function r_{ij} is of the order of $1/\sqrt{\langle n \rangle}$, whereas in the region where we have the usual fluctuations we get $\delta r_{ij} \sim 1/\langle n \rangle$. Inside the shaded region there is no stationary solution, since the generation occurs in a pulsating ("spike") regime.

5.4. ATTENUATION OF AVERAGE FIELD

The attenuation of the average field, connected with the quantum fluctuations of the phase, can be regarded as a slow quasistationary process whose characteristic time is much longer than τ. Assuming that the radial function $\rho_0(\xi)$ has already relaxed to its stationary value, we represent the distribution function ρ in the form $(z = \sqrt{\xi}\, e^{i\varphi})$

$$\rho(\xi, \varphi, t) \approx \rho_0(\xi)\rho_1(\varphi, t). \tag{88}$$

We also write down the vector P_α in the form

$$P_\alpha = P_\alpha^{(\text{ст})} + P_\alpha^{(1)}, \quad |P_\alpha^{(1)}| \ll P_\alpha^{(\text{ст})}, \tag{89}$$

where the stationary value $P_\alpha^{(\text{st})}$ was obtained in the preceding section. Retaining in (61) both the radial and the azimuthal parts, we obtain

$$\frac{P_\alpha^{(1)}}{\tau} = -\frac{1}{i}\sum_\beta \mathscr{H}_{\alpha\beta}P_\beta^{(1)} + \frac{g}{2\xi}\frac{\partial\rho_1}{\rho_1\partial\varphi}[\bar{z}(\bar{P}_{\alpha-} + \delta P_{\alpha-}) + z(\bar{P}_{\alpha+} + \delta P_{\alpha+})]. \tag{90}$$

Substituting here the previously obtained stationary values $\tilde{P}_\alpha^+$, $\tilde{P}_\alpha^-$, $\delta P_{\alpha+}$, and $\delta P_{\alpha-}$, we obtain a solution of this equation:

$$P_+^{(1)} = \frac{\nu\bar{z}(\eta+1)}{4g|z|(1+\nu\tau)}\frac{\partial\rho_1}{\rho_1\partial\varphi}, \quad P_3^{(1)} = 0. \tag{91}$$

We see therefore that the diffusion coefficient for the phase has no singularity near the instability region defined by Eq. (87). It now follows from (14′) that

$$\frac{\partial\rho_1}{\partial t} = \Delta\nu\frac{\partial^2\rho_1}{\partial\varphi^2}, \quad \Delta\nu = \frac{\nu(1+\eta)}{4\langle n\rangle(1+\nu\tau)}. \tag{92}$$

For the Fourier components ρ_{1m} (see formula (38)) we get

$$\rho_{1m} \sim e^{-m^2\Delta\nu t}. \tag{93}$$

When $\nu\tau \ll 1$ $\quad \Delta\nu = \nu(1+\eta)/4\langle n\rangle$. (94)

When $\nu\tau \gg 1$ $\quad \Delta\nu = (1+\eta)/4\langle n\rangle\tau$, (95)

i.e., the width of the spectral line is determined in this case not by the Q of the resonator, but by the lifetime of the atom.

6. CONCLUSION

Thus, the problem of a quantum oscillator resonantly interacting with a system of N spins, acquires in the coherent-state representation a relatively simple and lucid character.

In the quasiclassical limit, the problem reduces to an investigation of a closed system of equations of the hydrodynamic type. The accuracy of the obtained equations is of the order of $1/\sqrt{\langle n \rangle}$. Depending on the ratio of the number of spins and the characteristic number of photons, there are two limiting cases. When $N/\langle n \rangle \sim \nu\tau \ll 1$, the problem can be considered in the "given field" approximation; the collective effects do not play any role in this region. When $\nu\tau \gtrsim 1$, the problem must be considered in the self-consistent field approximation. In the present paper we investigated both limiting cases.

In the calculation of the quantum corrections connected with the diffusion of the photons to the classical current above the generation threshold, it is necessary to take into account the dependence of the diffusion coefficient on the radiation power. The use of $\tilde{\sigma}_i$ in place of σ_i in Eqs. (26) and (27) leads in this case to a decrease of the diffusion coefficient. When $\nu\tau > 1$, the fluctuations of the radiation are determined essentially by the collective effects. The overlap of the electromagnetic trains in the spontaneous emission at low radiation energies leads to a decrease of the fluctuations. In the region of intermediate radiation energies ($\eta < \nu\tau$), the photons have in practice a Poisson distribution. Finally at high energies, the fluctuations increase strongly near the instability region (see the figure). On the boundary of the instability region, the relative fluctuations are of the order of unity. However, the phase diffusion coefficient has no singularity near this boundary.

In conclusion we note that the simplest laser model considered by us can be generalized relatively simply to include allowance for the thermal motion of the atoms, for the difference between the relaxation times in the spin system, etc.

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APPENDIX

Let us consider the problem of the stability of the stationary generation regime in the classical limit. The self-consistent system of Maxwell's equations for the classical field z(t) and the equations of motion of the medium are

$$\frac{dz(t)}{dt} + \nu z(t) = -igP_-(t),$$

$$\frac{dP_-}{dt} + \frac{P_-}{\tau} = igz(t)P_3,$$

$$\frac{dP_3}{dt} + \frac{P_3 - N}{\tau} = -2gi(z(t)P_+ - \text{c.c.}).$$

Linearizing this system of equations near the stationary solution $P_3 = N/\eta$, $|z|^2 = (\eta - 1)/4(g\tau)^2$ and separating the time dependence in exponential form $e^{xt/\tau}$, we obtain the following dispersion equation:

$$x^3 + ax^2 + bx + c = 0,$$

where $a = 2 + \nu\tau$, $b = \eta + \nu\tau$, $c = 2\nu\tau(\eta - 1)$. The boundary of the instability region is obtained from the condition Re x = 0, from which we get

$$c = ab,$$

which coincides with formula (87).

Note added in proof (7 May 1969). Quantum fluctuations in a gas laser were recently considered by a semiphenomenological method in a paper by Yu. L. Klimontovich and P. S. Landa (Zh. Eksp. Teor. Fiz. 56, 275 (1969) Soviet Phys. JETP 29, 151 (1969)).

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