THEORY OF INTERACTION OF ACOUSTIC WAVES IN SEMICONDUCTORS

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Submitted October 4, 1968

Zh. Eksp. Teor. Fiz. 56, 1881–1890 (June, 1969)

Some problems of the nonlinear interaction of acoustic waves in semiconductors are considered. Dynamic equations describing the space and time behavior of the interacting wave amplitudes are obtained from elasticity theory equations in which the spatial and temporal dispersion of the elastic stress in an arbitrary crystalline medium are taken into account. These equations extend the familiar Bloembergen equations to media in which the energy transfer from the waves to the medium is possible during the process of nonlinear interaction of the waves. The energy balance equations and the equation for the number of quanta are considered both for linear and nonlinear absorption. Nonlinear elastic responses of electronic origin are found for piezo-semiconductors with an arbitrary state of the electron subsystem. In the hydrodynamic approximation, an explicit form is found for the nonlinear admittances, with account of carrier drift in an external electric field. Under conditions of sound wave amplification by drifting supersonic carriers, second harmonic generation and the transformation of transverse waves into longitudinal waves (TTL process) are considered.

PROBLEMS of the theory of nonlinear interaction of acoustic waves in semiconductors have recently taken on a special interest in connection with the possibility of amplification and generation of acoustic waves by drifting supersonic carriers.[1,2] The fact is that there are two relatively strongly interacting subsystems in semiconductors and semimetals: the crystal lattice and the electron-hole plasma carriers. In a number of cases, the plasma contribution to the decrement (or increment) of the sound waves plays a decisive role. Simple estimates show that under these conditions, the principal nonlinearity mechanism is of electronic origin and is connected with the reorganization of the subsystem under the action of the electric field which accompanies the sound wave. Since significant dispersion is characteristic for the electron subsystem, it is necessary to consider the theory of interaction of acoustic waves from the very beginning, taking both spatial and temporal dispersion into account.

Under conditions of weak nonlinearity, the nonlinear elastic stress can be expanded in powers of the deformation and limited to the first few terms. Such an approximation corresponds to renormalization of the phase velocity and decrement (or increment) of the wave as a result of nonlinear interaction, and also to allowance for the radiation of waves at the combination frequencies. The dynamic equations describing the behavior of the amplitudes and phases of the interacting waves in space and time are obtained by the Bloembergen method of contracted equations.[3,4] The theory developed in the present paper is valid only for powers of the sound waves less than the power corresponding to saturation of the nonlinear plasma properties connected with the capture of carriers by the electric field of the wave.

The interaction of acoustic waves in solids has already been considered both theoretically[5] and experimentally.[6] However, the complete dynamic equations for the amplitudes, with account of spatial and temporal dispersion, were not derived in these researches. In the researches of [7], the electron "concentration" nonlinearity was investigated theoretically for piezo-semiconductors, but the authors limited themselves to the consideration of the established regime and did not solve the problem of the initial or boundary regime.

In the first section of the present paper, we set up the general dynamic equations of the elasticity theory of crystals for arbitrary spatial and temporal dispersion of the linear and nonlinear elastic-modulus tensors. The approximation of three interacting waves is then considered and the conservation laws for energy, number of quanta, and flux of acoustic phonons are then studied with account of nonlinear interaction.

The problem is considered for piezo-semiconductors, where the effects of nonlinear interaction can play a significant role, especially under conditions of amplification of the acoustic waves. The general connection is found between the nonlinear elastic responses of the system and the corresponding nonlinear conductivity tensors of the medium, which are computed in explicit form in the long-wave approximation, when the frequency of the waves is much less than the reciprocal of the characteristic relaxation time of the energy of the plasma carriers (electrons). The effects of harmonic generation and the transformation of transverse waves into longitudinal waves are then considered under the conditions of wave amplification.

1. NONLINEAR EQUATIONS FOR THE AMPLITUDES OF SOUND WAVES

The purpose of the present section is the formulation of nonlinear equations for the amplitudes of sound waves in an elastic medium with arbitrary spatial and temporal dispersion.

The equations of elasticity theory for the displacement vector have the form

$$\rho\partial^2 u_i / \partial t^2 = \partial\sigma_{ij} / \partial x_j, \tag{1.1}$$

where ρ is the density, σ_{ij} the elastic stress tensor.

For weakly nonlinear and homogeneous medium, the coupling between the deformation tensor u_{ij} and the stress tensor can be represented in the form

$$\sigma_{ij}=\sum_{n=1}^{\infty}\int d\mathbf{r}_1\ldots d\mathbf{r}_n\int_{-\infty}^{t}dt_1\ldots\int_{-\infty}^{t_{n-1}}dt_n$$

$$\times S^{(n)}_{iji_1j_1\ldots i_nj_n}(\mathbf{r}-\mathbf{r}_1,t-t_1,\ldots,\mathbf{r}_{n-1}-\mathbf{r}_n,t_{n-1}-t_n)$$

$$\cdot u_{i_1j_1}(\mathbf{r}_1,t_1)\ldots u_{i_nj_n}(\mathbf{r}_n,t_n). \qquad (1.2)$$

Equations (1.2) represent the material equations for weakly nonlinear, homogeneous crystalline medium; the nuclei $S^{(n)}$ are the corresponding nonlinear admittances of the system; here $S^{(1)}_{iji_1j_1}$ is the linear elastic-modulus tensor with account of spatial and temporal dispersion. The admittances introduced above are obviously symmetric relative to the permutation of any pairs of indices i_kj_k; furthermore, in crystals with definite symmetries, these tensors are invariant relative to the entire group of transformations permitted by the given symmetry class.

We shall seek the solution of Eq. (1.1) in the form of a Fourier integral over time and space:

$$\mathbf{u}(\mathbf{r},t)=\int dk\,\mathbf{u}(k)\,e^{i(\mathbf{qr}-\omega t)}, \qquad (1.3)$$

where k denotes the set of quantities $(\omega, \mathbf{q})$. Using (1.3), we obtain an expression for the transformation of the Fourier equation (1.1):

$$L_{ij}(k)u_j(k)=iq_j\sigma^{(NL)}_{ij}(k), \qquad (1.4)$$

where

$$L_{ij}(k)\equiv-\rho\,\omega^2\delta_{ij}+{}^1/_2[S^{(1)}_{lijm}(k)+S^{(1)}_{ljim}(k)]\,q_lq_m \qquad (1.5)$$

is the operator of the linear dispersion equation; $\sigma^{(NL)}_{ij}$ is the nonlinear stress, which is determined by the second, third, and so forth terms of the expansion in (1.2). The operator $L_{ij}(k)$ is generally non-Hermitian and satisfies the condition $L_{ij}(k) = L_{ij}(-k^*)$

In contrast with the isotropic medium, the separation in crystals of waves into purely longitudinal and purely transverse is generally impossible: to each direction of the wave vector $\mathbf{q}$ there corresponds a characteristic wave whose displacement vector possesses both longitudinal and transverse displacement components. Therefore, we must consider the eigenvalues and eigenvectors of the operator $L_{ij}(k)$, which are defined by the equations

$$L_{ij}(k)\,b_j(k)=\lambda(k)\,b_i(k), \qquad \operatorname{Det}|L_{ij}(k)-\lambda(k)\delta_{ij}|=0 \qquad (1.6)$$

We shall consider the case of nondegenerate eigenvalues when all three values of $\lambda_\alpha(k)$ are different. The eigenvectors $\mathbf{b}^\alpha(k)$ in the general case can also be complex:[1)] their normalization conditions are written in the form $b_i^\alpha(k)\tilde{b}_i^{\alpha'}(k) = \delta_{\alpha\alpha'}$, where $\tilde{b}_i^\alpha$ is the vector that is the Hermitian conjugate of b_i^α. Multiplying Eq. (1.4) by $\tilde{b}_i^\alpha(k)$ and using the condition of the orthogonality of waves with different polarizations, we get an equation for the Fourier components of the amplitude of the eigenwave u_α:

$$\lambda_\alpha(k)\,u_\alpha(k)=\sum_{\alpha_1,\alpha_2}\int d\xi^{(2)}\Lambda^{(2)}_{\alpha\alpha_1\alpha_2}(k,k_1,k_2)\,u_{\alpha_1}(k_1)\,u_{\alpha_2}(k_2)$$

$$+\sum_{\alpha_1,\alpha_2,\alpha_3}\int d\xi^{(3)}\Lambda^{(3)}_{\alpha\alpha_1\alpha_2\alpha_3}(k,k_1,k_2,k_3)\,u_{\alpha_1}(k_1)\,u_{\alpha_2}(k_2)\,u_{\alpha_3}(k_3),$$

where $d\xi^{(2)} = dk_1dk_2\delta(k-k_1-k_2)$ and $d\xi^{(3)} = dk_1dk_2dk_3 \times \delta(k-k_1-k_2-k_3)$, and $\Lambda^{(2)}$ and $\Lambda^{(3)}$ are expressed in linear fashion in terms of the faltung of the tensors $S^{(2)}$ and $S^{(3)}$ with vectors $\mathbf{b}^\alpha$ and $\mathbf{q}$.

In the linear approximation, when the amplitudes of the waves are small, Eq. (1.7) reduces to the usual dispersion equation $\lambda_\alpha(k) = 0$. This equation determines the frequencies of the characteristic oscillations of the system and the decrement (or increment) of γ_α. For weak damping or growth of the wave ($|\gamma_\alpha/\omega_\alpha| \ll 1$), the equations for the determination of the eigenfrequencies and decrements will have the respective forms

$$\operatorname{Re}\lambda_\alpha(\omega_\alpha,\mathbf{q})\equiv\lambda_\alpha'(\omega_\alpha,\mathbf{q})=0, \qquad (1.8)$$

$$\gamma_\alpha(\mathbf{q})=-\left(\frac{\partial\lambda_\alpha'}{\partial\omega_\alpha}\right)^{-1}\operatorname{Im}\lambda_\alpha(\omega_\alpha,\mathbf{q}). \qquad (1.9)$$

In linear theory, the characteristic oscillations are damped or growing oscillations whose amplitude is determined by the initial and boundary conditions. When account is taken of the nonlinear properties of the medium, the amplitudes of the oscillations are no longer determined by the external conditions only, but also by the interaction between the oscillation modes. The sound field in this case is conveniently written in terms of a set of wave packets for which a small scale in time and space corresponds to rapid oscillations with the wave vector $\mathbf{q}$ and frequency $\omega_\alpha(\mathbf{q})$, while large scale corresponds to a slow change in the amplitude of the wave packets in time and space.

Using the representation of wave packets, it is not difficult to obtain an equation for the slow changes in amplitude of the sound waves $u_\alpha(\mathbf{q}, \mathbf{r}, t)$. Multiplying Eq. (1.7) here (written in terms of the variables q' and ω) by $\exp\{i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r} - i[\omega'-\omega_\alpha(\mathbf{q}')]t\}$ and integrating over $d\mathbf{q}'$ and $d\omega'$ in the vicinity of the center of gravity of the packet, we get, after successive transformations, the nonlinear dynamic equation for the acoustic waves:

$$\frac{\partial u_\alpha}{\delta t}+\mathbf{v}_\alpha\frac{\delta u_\alpha}{\partial\mathbf{r}}=\gamma_\alpha u_\alpha+i\left(\frac{\partial\lambda_\alpha'}{\partial\omega_\alpha}\right)^{-1}\sum_{\pm}\sum_{\alpha_1,\alpha_2}\sum_{\mathbf{q}_1,\mathbf{q}_2}\Lambda^{(2)}_{\alpha\alpha_1\alpha_2}(k,\pm k_1,\pm k_2)$$

$$\times\,u_{\alpha_1}(\pm\mathbf{q}_1,\mathbf{r},t)\,u_{\alpha_2}(\pm\mathbf{q}_2,\mathbf{r},t)\,e^{i(\Delta\mathbf{qr}-\Delta\omega t)}+\ldots, \qquad (1.10)$$

where $\mathbf{v}_\alpha = -(\partial\lambda'_\alpha/\partial\mathbf{q})(\partial\lambda'_\alpha/\partial\omega_\alpha)^{-1}$ is the group velocity of the corresponding characteristic wave and $\Delta k = k \pm k_1 \pm k_2$. The summation in (1.10) extends over all wave vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ for which the conditions $k \approx \pm k_1 \pm k_2$ are satisfied. These conditions determine the selection rule or conservation law for the nonlinear interaction of the waves. The sign ± in the sum in (1.10) means that it is necessary to take the sum of all terms with different signs in front of the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. Similarly, nonlinear terms were taken into account in (1.10) that describe the interaction of a large number of waves.

Equations of the type (1.10) are the ordinary dynamic equations for the amplitudes of the interacting waves,

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[1)] For example, in the presence of gyrotropy in an elastic medium, the $\mathbf{b}^\alpha$ are complex and their imaginary parts describe the rotation of the plane of polarization.

which are well known both in nonlinear optics and in the theory of nonlinear effects in a plasma.[3,4,8] However, the theory of the nonlinear interaction of acoustic waves in crystals is seen to be more complicated in comparison with the corresponding theory for electromagnetic waves. The latter is connected not only with the crystalline properties of the medium with relatively strong absorption but, as will be shown below, fundamentally with the absence of symmetry in the matrix elements $\Lambda^{(2)}_{\alpha\alpha_1\alpha_2}$ (see Eq. (2.11)), which in turn leads to the laws of conservation of energy of the interacting waves and of the number of quanta, which differ materially from the case of plasma and nonlinear optics.[3,4,8]

2. THREE INTERACTING WAVES

We limit ourselves to the consideration of processes in which only three waves interact with one another. This means that we shall consider the transformation of waves of the type $\alpha_1 + \alpha_2 \rightleftharpoons \alpha_3$, where α_i characterizes the types of wave. The conservation laws in elementary acts will evidently be

$$\omega_3 - \omega_1 - \omega_2 = \Delta\omega, \qquad \mathbf{q}_3 - \mathbf{q}_1 - \mathbf{q}_2 = \Delta\mathbf{q}. \tag{2.1}$$

Here $\Delta\omega$ and $\Delta\mathbf{q}$ are the frequency and wave vector deviations satisfying the conditions $|\Delta\omega/\omega_{\min}| \ll 1$, $|\Delta\mathbf{q}/q_{\min}| \ll 1$. Under the conditions of exact synchronism, $\Delta\omega = 0$ and $\Delta\mathbf{q} = 0$.

Simple study shows[4,5] that under the conditions in which the characteristic oscillations can be regarded approximately as longitudinal (L) and transverse (T), processes allowed by the conservation laws (2.1) are

$$T + T \rightleftharpoons L, \qquad T + L \rightleftharpoons L. \tag{2.2}$$

Moreover, interaction between oscillations of one type with parallel wave vectors $\mathbf{q}$ is also allowed within the range of limits on $\Delta\mathbf{q}$. For three interacting waves $1 + 2 \rightleftharpoons 3$, Eqs. (1.10) have the form

$$\hat{D}_1 u_{\alpha_1} = \gamma_{\alpha_1} u_{\alpha_1} + V_{\alpha_1\alpha_2\alpha_3}(k_1, -k_2, k_3)\, u_{\alpha_2}^* u_{\alpha_3} e^{i(\Delta\mathbf{qr} - \Delta\omega t)}, \tag{2.3}$$

where the operator $\hat{D}_1 = \partial/\partial t + v_{\alpha_1}\partial/\partial r$ and

$$V_{\alpha_1\alpha_2\alpha_3}(k_1, -k_2, k_3) = -2i\left(\frac{\partial\lambda_{\alpha_1}'}{\partial\omega_{\alpha_1}}\right)^{-1} \Lambda^{(2)}_{\alpha_1\alpha_2\alpha_3}(k_1, -k_2, k_3). \tag{2.4}$$

Equations for the amplitudes u_{α_2} and u_{α_3} have a form similar to (2.3) and can be obtained from (2.3) with the aid of the formal substitutions $1 \to 2$, $2 \to 1$, $3 \to 3$ for u_{α_2}, and $1 \to 3$, $2 \to 1$, $3 \to 2$, $-\mathbf{k}_2 \to \mathbf{k}_1$, $u^*_{\alpha_2} \to u_{\alpha_1}$, $\Delta\mathbf{q} \to -\Delta\mathbf{q}$, $\Delta\omega \to -\Delta\omega$ for u_{α_3}. Usually, the dynamic equation of the interacting waves can be written in the form of equations for the modulus and phase of the amplitude, putting the complex amplitudes in the form $u_\alpha = A_\alpha \exp(i\varphi_\alpha)$. Carrying out the separation in the set (2.3), we get

$$\hat{D}_1 A_1 = \gamma_1 A_1 + W_1 A_2 A_3 \cos(\theta + \chi_1), \tag{2.5a}$$

$$A_1 \hat{D}_1 \varphi_1 = W_1 A_2 A_3 \sin(\theta + \chi_1), \tag{2.5b}$$

wnere $\theta(\mathbf{r}, t) = \varphi_3 - \varphi_1 - \varphi_2 + \Delta\mathbf{q}\cdot\mathbf{r} - \Delta\omega t$ and W_1 and t, are the modulus and phase of the relation (2.4).

The equations for the amplitudes A_2 and A_3 and phases φ_2 and φ_3 have a similar form and can be obtained from (2.5) by means of the formal substitution $1 \to 2$, $2 \to 1$, $3 \to 3$ for A_2 and φ_2, and $1 \to 3$, $2 \to 1$, $3 \to 2$ and $\theta \to -\theta$ for A_3 and φ_3.

The dynamic equations (2.3) and (2.5), in contrast to the known equations in nonlinear optics (see the book of Bloembergen[4]), contain different phases χ_i. Precisely because these phases are different (their equality is obviously possible only in extraordinary cases), there are no solutions similar to the optical case for the theory of nonlinear interaction of elastic waves.

In particular, (2.3) leads to equations that describe the process of the self-action of the sound wave, leading to the emission of a second harmonic:

$$\begin{gathered} \hat{D}_\omega u_\omega = \gamma_\omega u_\omega + V_{\alpha\alpha\alpha}(k, -k, 2k)\, u_\omega^* u_{2\omega} e^{i(\Delta\mathbf{qr} - \Delta\omega t)}, \\ \hat{D}_{2\omega} u_{2\omega} = \gamma_{2\omega} u_{2\omega} - {}^1/_2 V_{\alpha\alpha\alpha}(2k, k, k)\, u_\omega^2 e^{-i(\Delta\mathbf{qr} - \Delta\omega t)}. \end{gathered} \tag{2.6}$$

We now obtain the conservation laws for the total values of the energy of the waves and the number of quanta. For this purpose, we introduce the definition for the energy, energy flux, number of quanta and quantum flux:

$$\mathscr{E}_\alpha(\mathbf{q}, \mathbf{r}, t) = -\omega_\alpha \frac{\partial\lambda_\alpha'}{\partial\omega_\alpha} |u_\alpha(\mathbf{q}, \mathbf{r}, t)|^2, \qquad \mathbf{S}_\alpha(\mathbf{q}, \mathbf{r}, t) = \mathbf{v}_\alpha \mathscr{E}_\alpha(\mathbf{q}, \mathbf{r}, t);$$

$$N_\alpha(\mathbf{q}, \mathbf{r}, t) = \frac{\mathscr{E}_\alpha(\mathbf{q}, \mathbf{r}, t)}{\hbar\omega_\alpha}, \qquad \mathbf{P}_\alpha(\mathbf{q}, \mathbf{r}, t) = \mathbf{v}_\alpha N_\alpha(\mathbf{q}, \mathbf{r}, t). \tag{2.8}$$

From the dynamic equations (2.3), we obtain the balance equation for the total energy of the three waves in the medium:

$$\begin{gathered} \frac{\partial\mathscr{E}}{\partial t} + \frac{\partial\mathbf{S}}{\partial\mathbf{r}} = 2\sum_\alpha \gamma_\alpha \mathscr{E}_\alpha - 4\,\mathrm{Im}\{[\omega_{\alpha_1}\Lambda^{(2)}_{\alpha_1\alpha_2\alpha_3}(k_1, -k_2, k_3) \\ + \omega_{\alpha_2}\Lambda^{(2)}_{\alpha_2\alpha_1\alpha_3}(k_2, -k_1, k_3) - \omega_{\alpha_3}\Lambda^{(2)*}_{\alpha_3\alpha_1\alpha_2}(k_3, k_1, k_2)]\, u_{\alpha_1}^* u_{\alpha_2}^* u_{\alpha_3} e^{i(\Delta\mathbf{qr} - \Delta\omega t)}\}, \end{gathered} \tag{2.9}$$

where

$$\mathscr{E} = \sum_\alpha \mathscr{E}_\alpha, \qquad \mathbf{S} = \sum_\alpha \mathbf{S}_\alpha.$$

The balance equation for the number of quanta of the interacting waves follows in the same way from (2.3):

$$\begin{gathered} \frac{\partial}{\partial t}(N_{\alpha_1} + N_{\alpha_3}) + \frac{\partial}{\partial\mathbf{r}}(\mathbf{P}_{\alpha_1} + \mathbf{P}_{\alpha_3}) = 2(\gamma_{\alpha_1}N_{\alpha_1} + \gamma_{\alpha_3}N_{\alpha_3}) \\ -\frac{4}{\hbar}\,\mathrm{Im}\{[\Lambda^{(2)}_{\alpha_1\alpha_2\alpha_3}(k_1, -k_2, k_3) - \Lambda^{(2)*}_{\alpha_3\alpha_1\alpha_2}(k_3, k_1, k_2)]\, u_{\alpha_1}^* u_{\alpha_2}^* u_{\alpha_3} e^{i(\Delta\mathbf{qr} - \Delta\omega t)}\} \end{gathered} \tag{2.10}$$

and similar relations for $N_{\alpha_2} + N_{\alpha_3}$, which can be obtained from (2.10) with the help of the formal substitution of indices $1 \to 2$, $2 \to 1$, $3 \to 3$.

If the properties of the medium are such that the matrix elements that characterize the nonlinear interaction of the waves in the medium satisfy the conditions

$$\Lambda^{(2)}_{\alpha_1\alpha_2\alpha_3}(k_1, -k_2, k_3) = \Lambda^{(2)}_{\alpha_2\alpha_1\alpha_3}(k_2, -k_1, k_3) = \Lambda^{(2)*}_{\alpha_3\alpha_1\alpha_2}(k_3, k_1, k_2), \tag{2.11}$$

then Eqs. (2.9) and (2.10) give the usual laws of conservation of energy and number of quanta with account of linear damping. In the case considered here, the relation (2.11) cannot be fulfilled, and therefore additional terms appear in Eqs. (2.9)–(2.11), describing the transfer of energy from the waves to the medium by way of the various nonlinear mechanisms. Account of these terms is necessary in the low-frequency case when $\omega\tau \ll 1$, where τ is the characteristic relaxation time; in the given case, this is simply the relaxation time of the energy (temperature) of the electrons.

We note that in nonlinear optics[4] and collision-free plasma, when the frequencies of the interacting waves are much greater than the reciprocals of the relaxation times, the symmetry rule (2.11) is satisfied. Absence of this symmetry for interacting acoustic waves leads to new qualitative characteristics of the solution of the equations for three-wave interaction; in particular, the system (2.5) no longer has three first integrals (in our notation, $\mathbf{S}$, $\mathbf{P}_{\alpha_1} + \mathbf{P}_{\alpha_3}$, $\mathbf{P}_{\alpha_2} + \mathbf{P}_{\alpha_3}$), on the basis of the the existence of which the solution of the equations given by Bloembergen[4] is constructed. Furthermore, a simple investigation shows that the stationary solutions (of the type of sinusoidal stationary waves) in the system (2.5) are shown to be impossible, while the fundamental reason for this lies in the fact that the phase of the waves $\theta(\mathbf{r}, t)$ is unstable when the time or coordinate is increased.

3. CALCULATION OF NONLINEAR ELASTIC ADMITTANCES. THE CASE OF THE PIEZO-SEMICONDUCTOR

In semiconductors and semimetals there are several possible mechanisms for nonlinear interaction, both of lattice and of electronic origin. In the present paper, we consider the nonlinear effects of electronic origin, which play the decisive role for piezo-semiconductors in the majority of cases [2)] We shall consider that frequency region for which one cannot take into account the spatial and temporal dispersion of the lattice constants; then the equations of motion will be

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \lambda_{ilmj}\frac{\partial u_{mj}}{\partial x_l} - \beta_{l,ij}\frac{\delta E_l}{\delta x_j}. \tag{3.1}$$

Here $\beta_{l,ij}$ in the piezo "deformation" tensor,[9] $\mathbf{E}$ is the electric field accompanying the sound wave in the piezomedium; it satisfies the Maxwell equations

$$\varepsilon_0\frac{\partial E_i}{\partial t} - 4\pi\beta_{i,jl}\frac{\partial u_{jl}}{\partial t} = -4\pi j_i(\mathbf{r}, t), \qquad \operatorname{rot}\mathbf{E} = 0, \tag{3.2}$$

where $\mathbf{j}(\mathbf{r}, t)$ is the current induced in the plasma medium by the sound wave. Transforming to the Fourier representation and limiting ourselves only to terms of third order, we get an integral equation for the longitudinal wave $E(k)$:

$$E(k) = \frac{4\pi i}{\omega\varepsilon_{\|}(k)}\Big\{\omega q\boldsymbol{\beta}(\mathbf{q})\mathbf{u}(k) + \int d\xi^{(2)}\sigma_{\|}^{(2)}(k, k_1, k_2, E(k_1)E(k_2)$$
$$- \int d\xi^{(3)}\sigma_{\|}^{(3)}(k, k_1, k_2, k_3)E(k_1)E(k_2)E(k_3)\Big\}. \tag{3.3}$$

Here $\beta_i(\mathbf{q}) = \beta_{l,im}q_l q_m/q^2$, $\sigma_{\|}^{(S)}$ is the nonlinear longitudinal conductivity of s-th order, and $\epsilon_{\|}(k) = \epsilon_0 + 4\pi i\omega^{-1}\sigma_{\|}^{(1)}(k)$ is the dielectric constant of the crystal. Expressing the electric field $\mathbf{E}$ in terms of the displacement vector $\mathbf{u}$ from Eq. (3.3) and substituting it in (3.1), we obtain the explicit form of the dispersion operator and the nonlinear elastic admittances for the piezo-semiconductor:

$$L_{ij} = -\rho\omega^2\delta_{ij} + \left[\lambda_{ilmj}\frac{q_l q_m}{q^2} + \frac{4\pi\beta_i(\mathbf{q})\beta_j(\mathbf{q})}{\varepsilon_{\|}(k)}\right]q^2;$$

$$\Lambda^{(m)}_{\alpha\alpha_1\ldots\alpha_m}(k, k_1, \ldots, k_m) = \frac{(4\pi)^{m+1}}{\omega}\frac{q\boldsymbol{\beta}(\mathbf{q})\mathbf{b}^{\alpha^*}(k)}{\varepsilon_{\|}(k)}\left(\prod_{l=1}^{m}\frac{q_l\boldsymbol{\beta}(\mathbf{q}_l)\mathbf{b}^{\alpha_l}(k_l)}{\varepsilon_{\|}(k_l)}\right)F_m. \tag{3.5}$$

In (3.5), F is determined exclusively by the plasma properties and is expressed in terms of the dielectric constant and nonlinear conductivities of the carrier plasma.

For the response of second order in the displacement of the lattice, we have

$$F_2(k, k_1, k_2) = \sigma_{\|}^{(2)}(k, k_1, k_2), \tag{3.6}$$

and for the third-order response

$$F_3(k, k_1, k_2, k_3) = \sigma_{\|}^{(3)}(k, k_1, k_2, k_3) - i\frac{8\pi}{3}\sum_{s=1}^{3}M_s, \tag{3.7}$$

where

$$M_1 = \frac{\sigma_{\|}^{(2)}(k, k_1, k_2 + k_3)\,\sigma_{\|}^{(2)}(k_2 + k_3, k_2, k_3)}{(\omega_2 + \omega_3)\varepsilon_{\|}(k_2 + k_3)}$$

while M_2 and M_3 are obtained from M_1 by cyclic permutation of the indices 1, 2, and 3 in these expressions.

Equations (3.5)–(3.7) show that the nonlinear elastic properties of a piezoelectric medium are completely determined by the corresponding nonlinear responses of the electron subsystem. These expressions are applicable to a plasma medium with arbitrary spatial and temporal dispersion; in particular, they can be used successfully both for the case of low frequencies, when $q\delta \ll 1$, and also for high frequencies, when $q\delta \gg 1$, where δ is the mean free path of the carriers.

As an example, we set down the expressions for the longitudinal admittances in the case of a solid-state electron plasma in the low-frequency region, where the equations of the hydrodynamics of a charged liquid are applicable for the description of the collective motions in such a system:

$$\sigma_{\|}^{(1)}(k) = \frac{\sigma_0}{\Delta(k)}, \qquad \sigma_0 = \frac{e^2 n_0}{m\nu}; \tag{3.8}$$

$$\sigma_{\|}^{(2)}(k, k_1, k_2) = -\frac{e^3 n_0}{2m^2\nu^2}\frac{1}{\Delta(k)}\left\{\frac{q_1}{\omega_1}\frac{1}{\Delta(k_1)}\frac{(\mathbf{q}\mathbf{q}_2)}{qq_2} + \frac{q_2}{\omega_2}\frac{1}{\Delta(k_2)}\frac{(\mathbf{q}\mathbf{q}_1)}{qq_1}\right\}; \tag{3.9}$$

$$\sigma_{\|}^{(3)}(k, k_1, k_2, k_3) = -\frac{e}{m\nu}\frac{1}{\Delta(k)}\sum_{s=1}^{3}B_s. \tag{3.10}$$

Here

$$B_1 = \frac{(\mathbf{q}\mathbf{q}_1)}{qq_1}\frac{|\mathbf{q}_2 + \mathbf{q}_3|}{\omega_2 + \omega_3}\sigma_{\|}^{(2)}(k_2 + k_3, k_2, k_3),$$

and B_2 and B_3 are obtained from B_1 by cyclic permutation of the indices 1, 2, 3 in the quantities $k_i(\omega_i, \mathbf{q}_i)$. Here n_0 is the equilibrium concentration of carriers (electrons), ν is the effective collision frequency, m the effective mass of the carrier,

$$\Delta(k) = 1 - \frac{(\mathbf{q}\mathbf{v}_d)}{\omega} + i\frac{v_{Te}^2 q^2}{\omega\nu},$$

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2) Nevertheless, it should be noted that lattice mechanisms of nonlinearity, for example, electrostriction, can be reduced formally to the same consideration. In the presence of electrostriction, the free energy of the crystal is written in the form

$$F = F_0 + \frac{1}{2}\lambda_{ilmj}u_{il}u_{mj} + \frac{1}{8\pi}\hat{\varepsilon}_{ij}E_iE_j + \beta_{l,ij}E_lu_{ij} + a_{ij,lm}E_iE_ju_{lm}.$$

As a consequence of the piezoeffect or the presence of an external constant field, electrostriction leads to the result that the equations of motion of the lattice and Maxwell's equations become nonlinear. Solving them, it is not difficult to obtain equations of the type (1.10). In dielectrics, especially in ferroelectrics, where there is an appreciable electrostriction constant in addition to the piezoeffect, such a nonlinear mechanism can appreciably surpass the nonlinear effect associated with anharmonism and it should, in principle, just as in optics [3,4] guarantee the possibility of parametric amplification of the sound waves.

where v_{Te} is the thermal velocity, and $\mathbf{v}_d = -eE_d/m\nu$ the drift velocity.[3)] Substituting Eqs. (3.8)–(3.10) in Eqs. (3.5)–(3.7), we get the explicit form of the nonlinear admittances for the case of a piezo-semiconductor. It is seen from them, in particular, that the symmetry relations (2.11) do not hold. Physically, this means that nonlinear absorption of the energy of the wave takes place in the medium; this is connected with the presence of collisions.

Similarly, general expressions can be obtained for the nonlinear admittances in the case of semiconductors with deformation interaction.

4. GENERATION OF THE SECOND HARMONIC. TTL PROCESS

Let us consider the simplest solutions of the set (2.6), which describe the second-harmonic generation and the process of the formation of longitudinal waves by coalescence of two transverse (TTL process) under the conditions of sound-wave amplification.

We consider the problem in the half-space $x > 0$ and assume that at $x - 0$ the amplitude of the fundamental is $A_\omega = A_0$ the amplitude of the second harmonic is $A_{2\omega} = 0$. It then follows from the set (2.6) that for small x the current $|W_\omega A_{2\omega}/\gamma_\omega| \ll 1$, will be

$$A_\omega(x) = A_0 \exp(\gamma_\alpha x / v_s). \tag{4.1}$$

The solution (4.1) corresponds to the general linear theory of the amplification of a wave with frequency ω. Substituting (4.1) in the second Eq. (2.6) and using the condition on the phase $\cos(-\theta_0 + \chi_{2\omega}) = 1$, we find the amplitude of the second harmonic:

$$A_{2\omega}(x) = \frac{1}{2} W_{2\omega} A_0^2 \frac{1}{\gamma_{2\omega} - 2\gamma_\omega}\left[\exp\left(\frac{\gamma_{2\omega} x}{v_s}\right) - \exp\left(\frac{2\gamma_\omega x}{v_s}\right)\right]. \tag{4.2}$$

Using the value of the nonlinear conductivities, found above in the hydrodynamic approximation, it is not difficult to obtain the explicit form of the admittance

$$W_{2\omega} = \eta^2 \frac{(4\pi)^2 e n_0 \mu^2 \beta\omega}{2\varepsilon_0^2 v_s^2} \Bigg| \left[1 - \frac{v_d}{v_s} + i\frac{v_{Te}^2}{v_s^2}\frac{\omega}{\nu}\right] \times \left[1 - \frac{v_d}{v_s} + i\frac{4\pi\sigma_0}{\varepsilon_0\omega}(1 + q^2 r_D^2)\right]^{-2}\left[1 - \frac{v_d}{v_s} + i\frac{2\pi\sigma_0}{\varepsilon_0\omega}(1 + 4q^2 r_D^2)\right]^{-1}\Bigg|, \tag{4.3}$$

where $\mu = e/m\nu$ is the mobility of the carriers, v_s the velocity of the corresponding sound wave, β the effective piezomodulus (for longitudinal waves, this is $\beta_{x,xx}$; for transverse ones, $\beta_{x,xy}$ or $\beta_{x,xz}$, depending on the direction of the polarization vector, $\eta^2 = 4\pi\beta^2/\epsilon_0\rho v_s^2$ is the corresponding electromechanical coupling constant, r_D the Debye radius. In the derivation of (4.3), the dispersion of the sound waves has not been taken into account; therefore, $v_s(\omega) = v_s(2\omega)$. If we now substitute (4.3) in (4.2), then at small x, corresponding to the initial stage of the process of second-harmonic generation, the result obtained by Tell[11] follows from (4.2). It is not difficult to determine the characteristics growth length of second harmonic from Eq. (4.2).

In piezosemiconductors, when the nonlinear interaction between the waves is significant, the process of the coalescence of two transverse quanta into one longitudinal is possible, according to the scheme $T(\omega') + T(\omega'') \rightleftharpoons L(\omega' + \omega'')$. It follows from the conservation laws that the TTL process is possible for $\cos(\theta/2) \approx v_T(\omega)/v_L(2\omega)$, where θ is the angle between the wave vectors of the transverse waves. We shall assume that the amplitude of the transverse waves is much greater than the amplitude of the longitudinal ones, $|u_T| \gg |u_L|$. Then, we can neglect the nonlinear terms in the equation for transverse waves if the characteristic distance at which the amplitude of the transverse wave changes, because of the TTL process, is much greater than the characteristics scale l_x at which this phenomenon is considered. The solution of the stationary dynamical system (2.3) with boundary conditions $u_T(x = 0) = u_0$ and $u_L(x = 0) = 0$ for the longitudinal wave in the direction $\mathbf{q}_L \| \mathbf{E}_d$ will be

$$u_L(x) = V_{LTT} u_0^2 \left[\exp\left(\frac{\gamma_L x}{v_L}\right) - \exp\left(\frac{2\gamma_T x}{v_T}\right)\right] \Big/ \left(\gamma_L - 2\gamma_T \frac{v_L}{v_T}\right), \tag{4.4}$$

where the "matrix element" V_{LTT} of the process is equal to

$$V_{LTT} = -i\frac{(4\pi)^3 e n_0 \mu^2 \omega \cos(\theta/2)}{4\rho\varepsilon_0^3 v_L v_T^3} \beta_{x,xx}\left[\beta_{x,ix} b_i^T \cos^2\frac{\theta}{2} + \beta_{x,iy} b_i^T \sin^2\frac{\theta}{2}\right]^2 \left(1 - \frac{v_d}{v_T}\cos\frac{\theta}{2} + i\frac{v_{Te}^2}{v_T^2}\frac{\omega}{\nu}\right) \cdot \times \left[1 - \frac{v_d}{v_T}\cos\frac{\theta}{2} + i\frac{4\pi\sigma_0}{\varepsilon_0\omega}(1 + q^2 r_D^2)\right]^{-2}\left[1 - \frac{v_d}{v_L} + i\frac{2\pi\sigma_0}{\varepsilon_0\omega}(1 + 4q^2 r_D^2)\right]^{-1} \tag{4.5}$$

Here $\mathbf{b}^T$ is the unit polarization vector of transverse waves. It follows from (4.5) that for the observation of the TTL process, the orientation of the crystal and the polarization of the transverse waves ought to be such that $\beta_{x,xx} \neq 0$ and $\beta_{x,ix} b_i^T \neq 0$, simultaneously.

The theory of nonlinear interaction of elastic waves in solids, considered above, can be applied with success also to electromagnetic waves close to the absorption and emission lines, when the dispersion plays a significant role.

In conclusion, we express our sincere gratitude to L. B. Keldysh for useful discussions.

[3)]Nonlinear conductivities were obtained in the region $\omega \ll \delta\nu$, where δ is the fraction of the transferred energy for each elastic collision [10].

[1] A. R. Huston, J. H. McFee, and L. D. White, Phys. Rev. Lett. 7, 237 (1961).

[2] M. E. Gertsenshtein and V. I. Pustovoĭt, PTÉ No. 7, 1009 (1962).

[3] S. A. Akhmanov and R. B. Khokhlov, Problemy nelineinoĭ optiki (Problems of Nonlinear Optics) (Press of the All-Union Institute of Scientific and Technical Information), 1964.

[4] N. Bloembergen, Nonlinear Optics (Benjamin, 1965).

[5] L. H. Taylor and F. R. Rollins, Phys. Rev. 136, A591 (1964).

[6] L. K. Zarembo and V. A. Krasil'nikov, Vvedenie v nelineinuyu akustiku (Introduction to Nonlinear Acoustics) (Nauka, 1966).

[7] V. L. Gurevich, Fiz. Tverd. Tela 5, 1222 (1963) [Sov. Phys.-Solid State 5, 892 (1963)]; V. L. Gurevich, B. D. Laĭkhtman, Zh. Eksp. Teor. Fiz. 46, 598 (1964) [Sov. Phys.-JETP 19, 407 (1964)].

[8] V. N. Tsytovich, Usp. Fiz. Nauk **90**, 435 (1966) [Sov. Phys.-Uspekhi **9**, 805 (1967)].
[9] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media) (Gostekhizdat, 1957) [Addison-Wesley, 1960].
[10] V. L. Ginzburg and A. V. Gurevich, Usp. Fiz. Nauk **70**, 201 (1960) [Sov. Phys.-Uspekhi **3**, 115 (1960)].
[11] B. Tell, Phys. Rev. **136**, A772 (1964).

Translated by R. T. Beyer
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