INTERACTION BETWEEN PLASMA WAVES AND ACOUSTIC WAVES IN A TURBULENT PLASMA

V. G. MAKHAN'KOV and V. N. TSYTOVICH

Joint Institute for Nuclear Research

Submitted July 24, 1968

Zh. Eksp. Teor. Fiz. 56, 1872–1880 (June, 1969)

The interaction between plasma turbulence and hydrodynamic (acoustic) turbulence is considered. It is shown that if the plasma waves are intense and characterized by stationary broad spectra the interaction is of a decay nature.

AT the present time, the theory of nonlinear interactions in a plasma has been developed to a fairly great degree of detail.[1] However, up to the present time no consideration has been given to the effect of interactions between waves characterized by $\omega \gg \nu$ (ν is the frequency of binary collisions) with waves for which $\omega \ll \nu$. Interaction of plasma waves with low-frequency waves is of special interest because this interaction has been studied experimentally by a number of workers.[2,3] Below, we consider in detail as an example, the interaction of plasma waves and acoustic waves in a fully ionized plasma: specifically, we analyze the excitation of acoustic waves by the plasma waves. Here and in what follows we mean by the term "acoustic wave" an acoustic wave whose frequency is smaller than the electron-ion collision frequency ν_e and the ion-ion collision frequency ν_i. We first consider briefly the linear properties of acoustic waves.

1. DISPERSION PROPERTIES OF ACOUSTIC WAVES

If the frequency of an acoustic wave is lower than the frequency of binary collisions $\omega^s \ll \nu_{eff}$ the wave spectrum is determined by the hydrodynamic equations for the plasma as given, for example, in[4]. We denote by ν_e and ν_i the frequency of collisions between electrons and ions and all other plasma particles. It is important to note that in the region

$$\max(\omega^s, |\mathbf{k}_s| v_{Te}) \ll \nu_e, \qquad \max(\omega^s, |\mathbf{k}_s| v_{Ti}) \ll \nu_i \tag{1.1}$$

the acoustic waves can propagate in an isothermal plasma. Since $\omega^s \gtrsim |\mathbf{k}_s| v_{Ti}$ if (1.1) is to be satisfied it is sufficient that

$$|\mathbf{k}_s| v_{Te} \ll \nu_e, \quad \omega^s \ll \nu_i. \tag{1.2}$$

If $T_e/T_i \gtrsim 1$, then the first condition in (1.2) is more stringent. Assuming that the wave number $|\mathbf{k}_s|$ in a bounded plasma of dimension a is greater than $1/a$, we can obtain a bound on the system dimensions:

$$a \gg r_D N_D, \tag{1.3}$$

where $r_D = v_{Te}/\omega_{0e}$ is the plasma Debye radius while $N_D = (\omega_{0e}/\nu_e)$ is the order of the number of particles in a sphere given by the Debye radius: $N_D \sim n_0 (v_{Te}/\omega_{0e})^3$ (n_0 is the plasma density).

The condition in (1.3) is satisfied under experimental conditions that are frequently encountered in practice. For example with $n_0 \sim 10^{13}$ cm^{-3}, $v_{Te} \sim 10^8$ cm/sec, $r_D \sim 10^{-3}$ cm and $N_D \sim 10^4$ we find that $a \gg 10$ cm. This means that acoustic waves can propagate even in an isothermal plasma. Evidently, the excitation of such acoustic waves has been observed in a number of experiments (cf. for example[2]). At the present time experiments have been reported in which an investigation was made of the nonlinear interaction of acoustic waves with other kinds of plasma waves.[2,3] However, there is no theory available for the nonlinear interaction of acoustic waves.

The dispersion properties for "collisional" sound in an isotropic plasma can be obtained easily with the help of the dielectric constant of the plasma under collision-dominated conditions, as can be obtained from the equations given by Braginskiĭ.[4] For a fully ionized plasma we find

$$\varepsilon = 1 + \varepsilon_e + \varepsilon_i, \tag{1.4}$$

where

$$\varepsilon_e = i\omega_{0e}^2 / \varkappa\omega\omega_e, \tag{1.5}$$

$$\varepsilon_i = i\omega_{0i}^2 / \varkappa\omega\omega_i, \tag{1.6}$$

and

$$\varkappa = 1 + \left(0.51\nu_e + 1{,}22\frac{k^2 v_{Te}^2}{\Omega_e} - 1.73\frac{k^2 v_{Te}^2}{\Omega_e}\frac{\delta}{\Omega_e}\right)\left(\frac{1}{\omega_e} + \frac{m_e}{m_i}\frac{1}{\omega_i}\right),$$

$$\omega_e = -i\omega + i\frac{k^2 v_{Te}^2}{\omega}\left(1 - 0.97\frac{i\omega}{\nu_e} - 1.71\frac{i\omega}{\Omega_e} - 1.73\frac{i\delta\omega}{\Omega_e\Omega_i}\right),$$

$$\omega_i = -i\omega + i\frac{k^2 v^2_{Ti}}{\omega}\cdot$$

$$\times\left\{1 - 1.28\frac{i\omega}{\nu_i} - \frac{i\omega}{\Omega_i}\left(1 + \frac{\delta^2}{\Omega_i\Omega_e} - 0.71\frac{\delta}{\Omega_e}\right) + \frac{i\omega}{\Omega_e}\left(0.71 - \frac{\delta}{\Omega_i}\right)\right\}, \tag{1.7}$$

$$\Omega_e = -\frac{3}{2}i\omega + \delta + 3.16\frac{k^2 v_{Te}^2}{\nu_e} - \frac{\delta^2}{\Omega_i}, \quad \nu_e = \frac{4}{3}\sqrt{\frac{2\pi}{m_e}}\frac{e^4 n_0 L}{T_e^{3/2}},$$

$$\Omega_i = -\frac{3}{2}i\omega + \delta + 3.9\frac{k^2 v_{Ti}^2}{\nu_i}, \quad \nu_i = \frac{4}{3}\sqrt{\frac{\pi}{m_i}}\frac{e^4 n_0 L}{T_i^{3/2}}, \quad \delta = 3\frac{m_e}{m_i}\nu_e,$$

where L is the Coulomb logarithm. We now consider the wave spectrum $\omega_{\mathbf{k}}^s$ and the damping rates $\gamma_{\mathbf{k}}^s$ for collisional sound in the general case of a nonisothermal plasma (T_e can be either larger or smaller than T_i).

1. If $\omega_{\mathbf{k}}^s \nu_e \gg k_s^2 v_{Te}^2$, then

$$\omega_{\mathbf{k}}^s = |\mathbf{k}_s| v_s, \quad v_s = v_{Te}\sqrt{\frac{10}{3}\frac{m_e}{m_i}} = v_{Ti}\sqrt{\frac{10}{3}}, \tag{1.8}$$

$$\gamma_{\mathbf{k}}^s(1) = \gamma_e + \gamma_i, \quad \gamma_e = 0.08\frac{k^2 v_{Te}^2}{\nu_e}, \quad \gamma_i = 0.9\frac{k^2 v_{Ti}^2}{\nu_i}, \tag{1.9}$$

where m_e is the electron mass and m_i is the ion mass.

2. In the opposite limit $\omega_k^s \nu_e \ll k_s^2 v_{Te}^2$

$$\omega_k{}^s = |\mathbf{k}_s| v_s, \quad v_s = v_{Te}\sqrt{\frac{m_e}{m_i}\left(1+\frac{5}{3}\frac{T_i}{T_e}\right)} = v_{Ti}\sqrt{\frac{5}{3}+\frac{T_e}{T_i}} \quad (1.10)$$

$$\gamma_k{}^s(2) = \gamma_e + \gamma_i, \quad \gamma_e = 0{,}16\frac{m_e}{m_i}\nu_e, \quad \gamma_i = \frac{1{,}92+0{,}64T_e/T_i}{{}^5/_3 + T_e/T_i}\frac{k^2 v_{Ti}^2}{\nu_i} \quad (1.11)$$

The difference between the electron temperature and the ion temperature can be introduced only in the case in which the frequencies of interest or the growth rates are faster than the reciprocal times for temperature equilibration between electrons and ions due to collisions. $\sim \nu_e m_e/m_i$ [Eqs. (1.10) and (1.11)]. In the opposite limit we make use of equations (1.10) and (1.11) taking $T_e = T_i$. We also note that in a narrow range $\omega_k^s \nu_e \sim k^2 v_{Te}^2$ the damping rate is of the order of the frequency. The region in which the acoustic wave can exist (1.8) and (1.10) lies on both sides of the critical wavelength for the acoustic waves $\lambda^* = 1/k^*$ where

$$\lambda^* \sim r_D N_D \sqrt{\frac{m_e}{m_i}}\left(1+\frac{T_i}{T_e}\right)^{-1}, \quad (1.12)$$

That is to say (1.10) holds when

$$r_D N_D \ll \lambda \ll r_D N_D \sqrt{\frac{m_i/m_e}{1+T_i/T_e}}, \quad (1.13)$$

and (1.8) when

$$a \gg \lambda \gg r_D N_D \sqrt{m_i/m_e}. \quad (1.14)$$

The frequency of the acoustic waves (1.10) is greater than the quantity $\nu_e m_e/m_i$ when

$$\lambda \ll r_D N_D \sqrt{\frac{m_i}{m_e}\left(1+\frac{5}{3}\frac{T_i}{T_e}\right)}. \quad (1.15)$$

It is then evident that satisfaction of (1.13) is a sufficient condition for introducing the difference in the temperatures T_e and T_i in the frequency of the acoustic waves.

In the region given by (1.14) the frequency of the acoustic waves in a nonisothermal plasma can be greater than $\nu_e m_e/m_i$ only when $T_i \gg T_e$ and

$$r_D N_D \sqrt{\frac{T_i m_i}{T_e m_e}} \gg \lambda \gg r_D N_D \sqrt{\frac{T_e m_i}{T_i m_e}}. \quad (1.16)$$

If (1.16) is not satisfied the plasma can be regarded as isothermal $T_e = T_i$.

2. EQUATIONS FOR THE NONLINEAR INTERACTION OF PLASMA WAVES AND ACOUSTIC WAVES

We start from the equation

$$\frac{\partial}{\partial t}\hat{\varepsilon}E = -4\pi\int S(k,k_1,k_2)E_{k_1}E_{k_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\cdot \times\delta(\omega-\omega_1-\omega_2)\,dk\,d\omega\,dk_1\,d\omega_1\,dk_2\,d\omega_2 e^{i(\mathbf{kr}-\omega t)}, \quad (2.1)$$

where $\hat{\epsilon}$ is a linear integral operator for the dielectric constant. Since we are only interested in the interaction between acoustic waves (s) and plasma waves (l), we write the equations for the E^l and E^s fields separately:[1)]

$$\frac{\partial}{\partial t}\hat{\varepsilon}^l E^l = -8\pi\int S(k,k_1,k_2)E_{k_1}{}^s E_{k_2}{}^l e^{i(\mathbf{kr}-\omega t)}\cdot \times\delta(k-k_1-k_2)\,dk\,dk_1\,dk_2, \quad (2.2)$$

$$\frac{\partial}{\partial t}\hat{\varepsilon}^s E^s = -4\pi\int S(k,k_1,k_2)E_{k_1}{}^l E_{k_2}{}^l e^{i(\mathbf{kr}-\omega t)}\cdot \times\delta(k-k_1-k_2)\,dk\,dk_1\,dk_2; \quad k=\{\mathbf{k},\omega\}. \quad (2.3)$$

We write the fields in the form

$$E^\alpha = \int E(\mathbf{k},t)\exp\{i(\mathbf{kr}-\mathrm{Re}\,\omega_k{}^\alpha t)\}\,dk \quad (\alpha = l, s), \quad (2.4)$$

where $\mathrm{Re}\,\omega_k^l$ is the solution of the linear dispersion equation $\mathrm{Re}\,\epsilon^l(\omega,\mathbf{k}) = 0$ and $\mathrm{Re}\,\omega_k^s$ is the solution of the linear dispersion equation $\mathrm{Re}\,\epsilon^s(\omega,\mathbf{k}) = 0$.

We consider the case in which the interaction between ss and ll waves can be neglected, in which case ϵ^l and ϵ^s represent the linear dielectric constants. The excitation and interaction of l and s waves can only be discussed in the case in which the correction to the frequency associated with this interaction is small compared with the frequency itself. If this condition is not satisfied there is a substantial change in the dispersion properties of the plasma at the frequencies being considered. We shall not consider this latter case here. In accordance with the foregoing considerations, in the left sides of (2.2) and (2.3) we have

$$\mathrm{Re}\,\varepsilon^l(\mathrm{Re}\,\omega_\mathbf{k},\mathbf{k}) = 0, \quad \frac{1}{E^\sigma}\left|\frac{\partial E^\sigma}{\partial t}\right| \ll \mathrm{Re}\,\omega_\mathbf{k}{}^\sigma,$$

so that we need only consider the first term in the expansion in the small parameter

$$\left|\frac{1}{E^\sigma}\frac{\partial E^\sigma}{\partial t}\right| \Big/ \mathrm{Re}\,\omega_\mathbf{k}{}^\sigma \ll 1, \quad \sigma = l, s.$$

A similar expansion applies for the right sides of Eqs. (2.2) and (2.3), in which we neglect terms of order $(\partial E^\sigma/\partial t)/E^\sigma \mathrm{Re}\,\omega_k^s$ since taking account of these terms introduces corrections beyond the accuracy of the expansion being used. In view of these remarks, we can now write Eqs. (2.2) and (2.3) and the approximate form

$$\left(\frac{\partial}{\partial t}+\gamma^l\right)E^l(\mathbf{k},t) = -\frac{8\pi}{\xi^l(\mathbf{k})}\int E^l(\mathbf{k}_1,t)E^s(\mathbf{k}_2,t)\cdot \times S_1(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)e^{i\Delta\omega t}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,d\mathbf{k}_1 d\mathbf{k}_2, \quad (2.5)$$

$$\left(\frac{\partial}{\partial t}+\gamma^s\right)E^s(\mathbf{k}_2,t) = -\frac{4\pi}{\xi^s(\mathbf{k})}\int E^l(\mathbf{k},t)E^l(-\mathbf{k}_1,t)\cdot \times S_2(\mathbf{k}_2,\mathbf{k},-\mathbf{k}_1)^{-i\Delta\omega t}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}, \quad (2.6)$$

where $\gamma^{l,s}$ is the linear damping rate for the plasma waves (l) and the acoustic waves (s)

$$\xi^\alpha(\mathbf{k}) = \omega\frac{\partial}{\partial\omega}\mathrm{Re}\,\varepsilon^\alpha(\omega,\mathbf{k})\Big|_{\omega=\mathrm{Re}\,\omega_k{}^\alpha} \quad (2.7)$$

$$\Delta\omega = \mathrm{Re}(\omega_\mathbf{k}{}^l - \omega_{\mathbf{k}_1}{}^l - \omega_{\mathbf{k}_2}{}^s), \quad (2.8)$$

while S_1 and S_2 are obtained from $S(k, k_1, k_2)$ by integration of Eqs. (2.2) and (2.3) with respect to frequency, making use of (2.4):

$$S_1(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = S(\mathbf{k},\mathrm{Re}(\omega_{\mathbf{k}_1}{}^l+\omega_{\mathbf{k}_2}{}^s);\ \mathbf{k}_1,\mathrm{Re}\,\omega_{\mathbf{k}_1}{}^l;\ \mathbf{k}_2,\mathrm{Re}\,\omega_{\mathbf{k}_2}{}^s),$$

$$S_2(\mathbf{k}_2,\mathbf{k},-\mathbf{k}_1) = S(\mathbf{k}_2\ \mathrm{Re}(\omega_\mathbf{k}{}^l-\omega_{-\mathbf{k}_1}{}^l);\ \mathbf{k},\ \mathrm{Re}\,\omega_\mathbf{k}{}^l;-\mathbf{k}_1,-\mathrm{Re}\,\omega_{-\mathbf{k}_1}{}^l).$$

Let us consider a rather intense packet of l-waves. Let $\mathbf{k}_0$ be the mean wave number for the packet. We shall be interested in the dispersion relation for s-waves characterized by $\mathbf{k} = \mathbf{k}_s$. We assume that the

[1)]In obtaining Eqs. (2.2) and (2.3) we have neglected terms that do not satisfy the condition $\mathbf{k} - \mathbf{k}_2 - \mathbf{k}_1 = 0$.

vector $\mathbf{k}_0 \pm \mathbf{k}_s$ does not fall within the range of wave vectors of the intense packet, that is to say, we assume that this vector characterizes a weak plasma wave. On the right side of Eq. (2.6) we then have the product of the amplitude of the strong wave $E^{l(0)}(\mathbf{k})$ and the amplitude of the weak wave (designated by δE^l)

$$\delta E^l(\mathbf{k}, t) E^{l(0)}(-\mathbf{k}_1, t) + E^{l(0)}(\mathbf{k}, t)\,\delta E^l(-\mathbf{k}_1, t), \tag{2.9}$$

while Eq. (2.5) allows us to determine $\delta E^l(\mathbf{k})$ from $E^{l(0)}(\mathbf{k})$ if, on the right side, we neglect the contribution of δE^l as compared with $E^{l(0)}$ and if we neglect $\partial/\partial t$ as compared with the damping of the wave on the left side. As a result we obtain an equation that describes the excitation of s waves by strong plasma waves (this is discussed in greater detail in[5]):

$$\left(\frac{\partial}{\partial t} + \gamma^s\right) E^s(\mathbf{k}_2, t) = \frac{4(4\pi)^2}{\xi^s(\mathbf{k}_2)} \int \frac{d\mathbf{k}_1\, d\mathbf{k}_1'}{\gamma^l(\mathbf{k}_1 + \mathbf{k}_2)} [E^{l(0)}(\mathbf{k}_1)]^* E^{l(0)}(\mathbf{k}_1')$$
$$\times E^s(\mathbf{k}_2 + \mathbf{k}_1 - \mathbf{k}_1') \exp\{i \operatorname{Re}(\omega_{\mathbf{k}_1}{}^l - \omega^l_{\mathbf{k}_1'} + \omega_{\mathbf{k}_2}{}^s - \omega^s_{\mathbf{k}_2 + \mathbf{k}_1 - \mathbf{k}_1'})t\}$$
$$\times \frac{1}{\xi^l(\mathbf{k}_1 + \mathbf{k}_2)} S_2(\mathbf{k}_2, \mathbf{k}_1 + \mathbf{k}_2, -\mathbf{k}_1) S_1(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_1', \mathbf{k}_2 + \mathbf{k}_1 - \mathbf{k}_1'). \tag{2.10}$$

The equation that has been obtained is suitable for analyzing the effect of a infinitesimally narrow packet of l-waves:

$$E^l(\mathbf{k}, t) = E^l(t)\,\delta(\mathbf{k} - \mathbf{k}_0) + E^{l*}(t)\,\delta(\mathbf{k} + \mathbf{k}_0). \tag{2.11}$$

In this case, if rapidly oscillating terms are neglected Eq. (2.10) becomes

$$\frac{\partial}{\partial t} \ln E^s(\mathbf{k}_2, t) + \gamma^s$$
$$= \frac{4(4\pi)^2 |E^l(t)|^2}{\xi^s(\mathbf{k}_2)} \left\{ \frac{S_2(\mathbf{k}_2, \mathbf{k}_0 + \mathbf{k}_2, -\mathbf{k}_0) S_1(\mathbf{k}_0 + \mathbf{k}_2, \mathbf{k}_0, \mathbf{k}_2)}{\xi^l(\mathbf{k}_0 + \mathbf{k}_2)\gamma^l(\mathbf{k}_0 + \mathbf{k}_2)} - \frac{S_2(\mathbf{k}_2, \mathbf{k}_2 - \mathbf{k}_0, \mathbf{k}_0) S_1(\mathbf{k}_2 - \mathbf{k}_0, -\mathbf{k}_0, \mathbf{k}_2)}{\xi^l(\mathbf{k}_0 - \mathbf{k}_2)\gamma^l(\mathbf{k}_0 - \mathbf{k}_2)} \right\}. \tag{2.12}$$

The nonlinear interaction described by Eq. (2.12) corresponds to the nonlinear excitation of acoustic waves. By virtue of the relation[2)]

$$\left| \frac{\partial}{\partial t} \ln E^s(\mathbf{k}_2) \right| \ll \omega_{\mathbf{k}_2}{}^s$$

we find that the nonlinear growth rate cannot be large and cannot vary rapidly with time. This means that the field associated with the intense packet varies rather slowly. A situation of this kind is reasonable for intense waves under conditions in which the waves are subject to excitation which is essentially equal to the damping, that is to say, when $\gamma^l(\mathbf{k}_0) \ll \omega^s$. The indicated limitations determine the region of applicability of (2.12) if the intense waves exhibit a very narrow spectrum. We note that the nonlinear interaction between the s-waves themselves can be neglected in the initial stage of the nonlinear instability described by (2.12). As far as the ll interaction is concerned we find that in obtaining (2.12) this interaction can be of importance only in the equation that describes weak waves. But an interaction of this kind, linear in δE, will correspond to the interaction of weak waves and strong waves for which $\omega_- \ll \nu_e, \nu_i$. As has been shown by the authors in[6] the nonlinear growth rates under conditions $\omega_- \ll \nu_e, \nu_i$ are smaller than $\gamma^l(\mathbf{k})$ over a wide range of parameters.

Let us now consider the case of a rather wide frequency spectrum for the plasma waves, with the phases being random. Again we can isolate a region in which there are intense l-waves and regions in which there are weak l-waves, retaining only those terms that are linear in the weak l-waves required in (2.9) along with the terms that are written, and taking account of terms such as $E^{l(0)}(\mathbf{k}, t) E^{l(0)}(\mathbf{k}_1, t)$. However, when averages are taken by means of the relation

$$\langle E^{l(0)*}(\mathbf{k}, t) E^{l(0)}(\mathbf{k}_1, t)\rangle = |E^{l(0)}(\mathbf{k}, t)|^2 \delta(\mathbf{k} - \mathbf{k}_1) \tag{2.13}$$

we obtain an additional source of s-waves which is proportional to $\delta(\mathbf{k}_2)$. If we are interested in s-waves characterized by $\mathbf{k}_2 \neq 0$, the result can be obtained by averaging Eq. (2.10), making use of Eq. (2.13). The equation assumes the form

$$\left(\frac{\partial}{\partial t} + \gamma^s\right) E^s(\mathbf{k}_2, t) = \frac{4(4\pi)^2 E^s(\mathbf{k}_2, t)}{\xi^s(\mathbf{k}_2)} \int d\mathbf{k}_1 |E^{l(0)}(\mathbf{k}_1, t)|^2 \cdot$$
$$\times \frac{S_1(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_2) S_2(\mathbf{k}_2, \mathbf{k}_1 + \mathbf{k}_2, -\mathbf{k}_1)}{\gamma^l(\mathbf{k}_1 + \mathbf{k}_2)\xi^l(\mathbf{k}_1 + \mathbf{k}_2)}. \tag{2.14}$$

Results obtained by means of Eq. (2.14) are similar to those which have been obtained from Eq. (2.12). This can be understood because at low frequencies the interaction of the high frequency waves is automatically averaged over the high frequency. For the slow change in the l-wave in the case being considered, a wide spectrum is possible in general if the excitation and damping are balanced over a wide range of $\mathbf{k}_1$ vectors.

We now consider the case in which the characteristic time for the nonlinear interaction of ll-waves is smaller than for the ls-waves. We assume that there exists, for these waves, a region of generation in which the excitation of waves is stronger than the damping and a region of absorption in which the damping predominates. Let the nonlinear effect of the ll-interaction be responsible for the spectral transfer of energy of the l-wave from the region of generation into the region of absorption. Furthermore, we assume that the distribution of l-waves remain stationary. This means that in the region of generation the excess of excitation over damping compensates for the loss of waves due to spectral transfer while in the region of absorption the damping of the waves is compensated by the influx due to the spectral transfer. Thus, in the absence of the ls-interaction the spectrum of l-waves is stationary. The presence of the ls-interaction leads to a slow change of this spectrum. Taking account of the ll-interaction in (2.2) and writing out the quantity $(\partial E^l/\partial t)/E^l \operatorname{Re} \omega^l_{\mathbf{k}}$ we can obtain the nonlinear equation for the field E^l which describes both the ll and ls interaction.

We will assume that the interaction with s-waves only changes the correlation of a small number of l-waves. This is equivalent to the assumption that the s-waves interact effectively only with a small number of resonance l-waves (similar to the case in ordinary decay processes). As we shall see below, the results of the calculation verify this assumption.

Averaging the equation for the s-waves over the ensemble of plasma oscillations and assuming that the spectrum of l-waves is quasistationary[7] we have (for details see[5])

$$\Omega + i\gamma^s = \frac{2(4\pi)^2}{\xi^s(\mathbf{k}_2)} \int \frac{d\mathbf{k}\, d\omega'}{\Omega + \Delta\omega_1} \left\{ \frac{|E(\mathbf{k} - \mathbf{k}_2, \omega')|^2}{\xi^l(\mathbf{k})} \right.$$

[2)]If this is not the case the notion of acoustic waves is not meaningful.

$$-\frac{|E(\mathbf{k},\Delta\omega_1+\omega')|^2}{\xi^l(\mathbf{k}_1-\mathbf{k}_2)}\Big\} S_1'(\mathbf{k},\mathbf{k}-\mathbf{k}_2,\mathbf{k}_2,\omega',\Omega)S_2'(\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k},\omega'), \tag{2.15}$$

$$S_1'(\mathbf{k},\mathbf{k}-\mathbf{k}_2,\mathbf{k}_2,\omega',\Omega)$$
$$= S_1'(\mathbf{k}, \mathrm{Re}\,\omega^l_{\mathbf{k}-\mathbf{k}_2}-\omega'; \mathbf{k}-\mathbf{k}_2, \mathrm{Re}\,\omega^l_{\mathbf{k}-\mathbf{k}_2}-\omega'; \mathbf{k}_2, \mathrm{Re}\,\omega_{\mathbf{k}_2}{}^s+\Omega),$$
$$S_2'(\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k},\omega')$$
$$= S_2'(\mathbf{k}_2, \mathrm{Re}\,\omega_{\mathbf{k}}{}^l-\mathrm{Re}\,\omega^l_{\mathbf{k}-\mathbf{k}_2}; \mathbf{k}, \mathrm{Re}\,\omega_{\mathbf{k}}{}^l-\omega'; \mathbf{k}_2-\mathbf{k}, -\mathrm{Re}\,\omega^l_{\mathbf{k}-\mathbf{k}_2}+\omega'),$$

where $\Delta\omega_1 = \mathrm{Re}\,(\omega^l_{\mathbf{k}-\mathbf{k}_2} + \omega^s_{\mathbf{k}_2} - \omega^l_{\mathbf{k}}) \ll \Omega$ while $[E^s(\mathbf{k}_2, t) = \int E2\mathbf{k}_2, \Omega)e^{-i\Omega t}dt]$. In obtaining Eq. (2.15) we have made use of the fact that in the approximation being used here

for $S_1(\mathbf{k},\mathbf{k}-\mathbf{k}_2,\mathbf{k}_2) = S_1^*(\mathbf{k}-\mathbf{k}_2,\mathbf{k},-\mathbf{k}_2)$,

which follows directly from the general expressions for the nonlinear second-order field polarizability.[6]

We now write the functions $S_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_-)$ and $S_2(\mathbf{k}_-, \mathbf{k}_1, -\mathbf{k}_2)$ in explicit form:[6]

$$S_1(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_-) = i\frac{n_0e^3|\mathbf{k}_-|(\mathbf{k}_1\mathbf{k}_2)}{m_e^2\omega_1\omega_-\omega_e(\mathbf{k}_-)\varkappa|\mathbf{k}_1||\mathbf{k}_2|}, \tag{2.16}$$

$$S_2(\mathbf{k}_-,\mathbf{k}_1,-\mathbf{k}_2) = i\frac{1{,}71\nu_e n_0 e^3|\mathbf{k}_-|(\mathbf{k}_1\mathbf{k}_2)}{m_e^2\Omega(\mathbf{k}_-)\Omega_e(\mathbf{k}_-)\omega_{0e}{}^2|\mathbf{k}_1||\mathbf{k}_2|}. \tag{2.17}$$

Here, e is the electron charge

$$\Omega(k_-) = 0{,}51\nu_e + i\frac{|\mathbf{k}_-{}^2|v_{Te}{}^2}{\omega_-}\left(1-i\,2{,}96\frac{\omega_-}{\Omega_e}\right)$$

and $\omega_- = \omega_1 - \omega_2$, $\mathbf{k}_- = \mathbf{k}_1 - \mathbf{k}_2$. For further simplification of the nonlinear dispersion equation (2.15) we assume that $|\mathbf{k}_2|/|\mathbf{k}_1| \ll 1$. Expanding Eq. (2.15) to first-order in $|\mathbf{k}_2|/|\mathbf{k}|$ and using Eqs. (2.16) and (2.17) we find

$$\Omega + i\gamma^s = \frac{4\pi}{\xi^s(\mathbf{k}_2)}\int\frac{d\mathbf{k}\,d\omega'}{\Omega+\Delta\omega_1}\Big\{|E(\mathbf{k},\omega')|^2-|E(\mathbf{k},\omega'+\Delta\omega_1)|^2$$
$$-\left(\mathbf{k}_2\frac{\partial}{\partial\mathbf{k}}\right)|E(\mathbf{k},\omega')|^2\Big\}\frac{1{,}71\nu_e n_0 e^4(\mathrm{Re}\,\omega_{\mathbf{k}-\mathbf{k}_2}-\omega')^{-1}}{m_e^4 v_{Te}^2\varkappa\Omega(k_-)\Omega_e(k_-)\omega_{0e}}\left(\frac{\mathbf{k},\mathbf{k}-\mathbf{k}_2}{|\mathbf{k}||\mathbf{k}-\mathbf{k}_2|}\right)^2 \tag{2.18}$$

3. INVESTIGATION OF CERTAIN PARTICULAR CASES

As we have noted in Sec. 1, there are two regions of transparency for the acoustic waves, these regions being separated by a region of strong absorption. The further simplification of Eq. (2.18) is associated with the investigation of excitation of acoustic waves in the transparency regions. In the high-frequency region $\omega_-\nu_e \ll k_2^2 v_{Te}^2$ we obtain the following nonlinear dispersion equation for the acoustic waves:

$$\Omega + i\gamma_{\mathbf{k}_2}{}^s(2) = -0{,}27\omega_{0e}\frac{\nu_e^2}{k_2{}^2 v_{Te}{}^2}\left(1+\frac{5}{3}\frac{T_i}{T_e}\right)^{-1}\int d\mathbf{k}\,d\omega'\,Q\frac{\Delta\omega_1+\mathrm{Re}\,\omega_{\mathbf{k}_2}{}^s}{\Omega+\Delta\omega_1}. \tag{3.1}$$

Here, $W_{\mathbf{k},\omega} = |E(\mathbf{k},\omega)|^2/4\pi n_0 T_e$ is the relative spectral density of the plasma waves while $\gamma^s_{\mathbf{k}_2}$ is determined from Eq. (1.11)

$$Q = \left(\mathbf{k}_2\frac{\partial}{\partial\mathbf{k}}\right)W_{\mathbf{k},\omega'}. \tag{3.2}$$

As we have already noted above, in the limit $\Omega \ll \Delta\omega$ Eq. (3.1) describes the kinetic decay of the instability with a damping rate

$$\gamma(2) \approx 0{,}27\pi\omega_{0e}\frac{\nu_e^2\omega_{\mathbf{k}_2}{}^s}{|\mathbf{k}_2|^2 v_{Te}{}^2}\int d\mathbf{k}\,d\omega' Q\delta(\Delta\omega_1). \tag{3.3}$$

If $\gamma(2) > \gamma^s_{\mathbf{k}_1}(2)$ the damping or excitation of the high-frequency acoustic waves will be determined by the nonlinear interaction.[3)] It follows from Eq. (3.3) that a knowledge of the nonlinear growth rate $\gamma(2)$ depends on the sign of Q and when $Q > 0$ Eq. (3.3) describes the excitation of high-frequency acoustic waves by the plasma waves.

In similar fashion we can obtain the dispersion equation in the other transparency region, the low-frequency region $\omega_-\nu_e \gg k_2^2 v_{Te}^2$. However, in this case the quantity S_1S_2 (to a high degree of accuracy) is imaginary[4)] so that an equation of the form of (3.1) describes the nonlinear real correction to the frequency $\delta\omega$. However, if we take account of the small real part in S_1S_2 there is a weak nonlinear instability. From the condition $\gamma(1) > \gamma^s_{\mathbf{k}}(1)$ it follows that $\delta\omega \gtrsim \omega^s_{\mathbf{k}}$, that is to say, there must be a strong change in the dispersion properties of the plasma at the frequency $\omega^s_{\mathbf{k}}$; this change cannot be described in the framework of the analysis given here.

We now consider Eq. (2.12) and investigate the excitation of acoustic waves by a narrow packet of plasma waves. Using Eqs. (2.16) and (2.17) and neglecting small terms of order $\omega^s_{\mathbf{k}}/\omega_{0e}$, in the high-frequency region we obtain the acoustic waves

$$\frac{\partial}{\partial t}\ln E_{\mathbf{k}_2}{}^s(t) + \gamma_{\mathbf{k}_2}{}^s = \frac{3{,}42}{3{,}16}\frac{\nu_e\omega_{0e}{}^3}{k_2{}^4 v_{Te}{}^4}\frac{|E^l|^2}{4\pi n_0 T_e}\frac{1}{\xi^s(\mathbf{k}_2)} \tag{2.7}$$
$$\times\left\{\left(\frac{\mathbf{k}_0+\mathbf{k}_2,\mathbf{k}_0}{|\mathbf{k}_0+\mathbf{k}_2|\mathbf{k}_0}\right)^2(\omega^l_{\mathbf{k}_0+\mathbf{k}_2}-\omega_{\mathbf{k}_0}{}^l)-\left(\frac{\mathbf{k}_0-\mathbf{k}_2,\mathbf{k}_0}{|\mathbf{k}_0-\mathbf{k}_2|\mathbf{k}_0}\right)^2(\omega_{\mathbf{k}_0}{}^l-\omega^l_{\mathbf{k}_0-\mathbf{k}_2})\right\}. \tag{3.4}$$

It then follows that when $|\mathbf{k}_2| \ll |\mathbf{k}_0|$

$$\frac{\partial}{\partial t}\ln E_{\mathbf{k}_2}{}^s(t) + \gamma_{\mathbf{k}_2}{}^s = \frac{1{,}62}{1+{}^5/_3T_i/T_e}\nu_e\frac{W^l}{n_0T_e}. \tag{3.5}$$

As we have already noted above, the growth rate (3.5) must be smaller than $\omega^s_{\mathbf{k}_2}$ but larger than $\gamma^s_{\mathbf{k}_2}$ so that

$$\max\left\{0{,}16\frac{m_e}{m_i}\nu_e;\ \frac{k^2 v_{Ti}{}^2}{\nu_i}\right\} < \frac{|E^l|^2}{4\pi n_0 T_e} \ll \frac{|\mathbf{k}|v_s}{\nu_e}. \tag{3.6}$$

In the low-frequency region, as before, the primary feature is the real correction to the frequency $\omega^s_{\mathbf{k}_2}$.

4. DISCUSSION OF RESULTS

A number of features of the effects considered here should be emphasized.

1. First, the generation of acoustic waves can depend in a sensitive way on the correlation of the high-frequency waves, this being an important distinction between these interactions and collisionless interactions. This feature follows from Eq. (2.18). In addition to the term $(\mathbf{k}_2\partial/\partial\mathbf{k})/E(\mathbf{k},\omega)|^2$, which is analogous to the term that arises in collisionless generation when $\Delta\omega \ll \omega'$, there can be a term of the form $(\Delta\omega\partial/\partial\omega)|E(\mathbf{k},\omega|^2$. Thus, the generation or damping of the low-frequency waves can depend on the frequency correlation of the *ll* waves or (in the usual sense) on the relative correlation time for two fields.

2. A kinetic instability arises when $\Omega \ll \Delta\omega$. In addition to this, in accordance with Eq. (2.18) there can also exist a hydrodynamic instability when $\Delta\omega \ll \Omega$. However, in accordance with the assumptions that have been made, the indicated instability can only appear at short

3)We emphasize again that the inequality $\gamma(2) \ll \omega k_2^s$ must be satisfied.

4)The ratio $\mathrm{Re}\,S_1S_2/\mathrm{Im}\,S_1S_2 \sim \gamma^s_{\mathbf{k}}(1)/\omega^s_{\mathbf{k}}(1)$.

wavelengths. Hence, in order for this instability to appear it is necessary that, in addition to the intense stationary background, there exist an isolated beam of plasma waves whose spectrum does not overlap the primary spectrum. Such a beam of Langmuir waves must have an intensity much smaller than the intensity of the primary background, and the hydrodynamic instability describes the generation of low-frequency waves due to this beam alone. We note, in passing, that a beam of this kind can actually arise in the case of an instability due to beams of transverse waves in a plasma.[1,8]

3. We emphasize that the excitation of low-frequency waves by plasma waves has an analogy with the two-stream instability of charged particles in the plasma. Attention has been called to this analogy in[9,10]. It may be assumed that the production of stationary spectra in turbulence due to excitation, for example, by the two-stream instability, proceeds in several stages: 1) excitation of plasma waves by the beam; 2) excitation of ion-acoustic waves by the plasma waves if $T_e \gg T_i$; 3) excitation of acoustic waves by ion-acoustic waves and plasma waves. The last stage of this process in a plasma of large dimensions with no magnetic field must always be the acoustic waves. We emphasize again that acoustic waves having the lowest possible frequency exist in any plasma (also where $T_e \approx T_i$). Thus, the spectrum of the final stationary turbulence must include the development of acoustic turbulence.

4. In conclusion we note that the growth rate for the excitation of acoustic waves is $\nu_e^2/k^2v_{Te}^2$ times greater than that characteristic of collisionless excitation. This feature arises from the fact that in the region of frequent Coulomb collisions the basic contribution to the excitation of acoustic waves[11] is associated with a dissipative term of the form $n_0 m_e \nu_e (V_e - V_i)^2$, which appears in the equation for heat transfer, whereas in the collisionless plasma the important term is $(V_e \nabla)V_e$, which appears in the equation for momentum transfer. The ratio of these two terms is of order $\nu_e^2/k^2v_{Te}^2$, so that either one or the other plays the important role, depending on the limit being considered.[5)] Thus the instability being treated here might be called the nonlinear dissipative heat instability.

In conclusion the authors wish to thank L. I. Rudakov for valuable comments.

[5)]This result can be obtained qualitatively another way. This feature has been called to our attention by L. I. Rudakov, who also proposed the name for the instability.

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Translated by H. Lashinsky
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