WIGHTMAN FORMULATION FOR A NONLOCALIZABLE FIELD THEORY. I.

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It is shown that in nonlocalizable theories with an exponential growth of the vacuum expectation values in momentum space, the Wightman functions retain a number of the properties characteristic of localizable field theories. It is found that the holomorphy domain of these functions in coordinate space includes real points—the Jost points. The existence of Jost points enables one to introduce the concept of quasilocality, and to prove theorems about the equivalence of weak local commutativity and CPT invariance and about the connection between spin and statistics. The existence of asymptotic states is proved for a special class of nonlocalizable theories.

1. INTRODUCTION

THE goal of this article is to investigate the properties of Wightman functions in quantum field theories in which the vacuum expectation values of the fields may increase exponentially in the momentum representation. For such theories it is already impossible to introduce the condition of locality or microcausality in the usual form, since the basis function spaces necessary for a formulation of these theoreies are in the x-representation spaces (quasi) of analytic functions and do not contain finite (vanishing outside a finite region) functions. Therefore theories of this type are essentially nonlocal or nonlocalizable.

There are many reasons for the interest in nonlocalizable field theories. In the first place one cannot exclude the possibility that the principle of microcausality is not valid over small distances. It is true that one can violate microcausality while remaining within the limits of localizable theories. Investigation of such theories[1] leads to the conclusion that one can apparently avoid the appearance of unphysical singularities in the interaction amplitudes only by introducing into the theory intermediate states with an indefinite metric which, in turn, produces certain unsolved problems. Secondly, in recent years considerable progress has been achieved in the development of an axiomatic theory of localizable fields,[2 3] equivalent to the principle of locality (Jaffe[3]) or microcausality (Meĭman[2]).[1)]

The mathematical apparatus developed in these articles may without difficulty be generalized to nonlocalizable theories. This, of course, opens new possibilities and removes a number of limitations inherent in localizable theories. For certain models the introduction of nonlocalizable fields permits one to construct, within the framework of perturbation theory, a Lagrangian formulation of S-matrix theory which is free of ultraviolet divergences (see the articles by Efimov[5,8]). In nonlocalizable theories, due to their leading to a broader space of functionals the theorem about the global nature of local commutativity[9] turns out to be invalid.[2)] As Voronov showed,[10] in nonlocalizable theories the well-known Lehmann–Symanzik-Zimmermann theorem about decrease of the vertex part does not hold which, in particular, enables one for a number of simple dispersion models to find nontrivial solutions without the vanishing of the renormalized charge. In addition, in nonlocalizable theories one can generalize the dispersion relations without encountering any contradiction with existing experimental data.[6] Finally, the whole series of essentially nonlinear unrenormalizable Lagrangians considered in articles [8,11,12] leads to a nonlocalizable type of matrix elements. From a purely heuristic point of view, it would be incautious to neglect such possibilities.

In Sec. 2 of the present article a formulation of the Wightman postulates for nonlocalizable fields is considered on the basis of an assignment in the p-representation of a topological space of basis functions on which the field operators (and the Wightman functions) are defined. The basis space is chosen so as to guarantee the formulation of the spectral conditions; however, in the coordinate representation the basis space already does not contain finite functions. In Sec. 3 the analytic properties of Wightman functions in a nonlocalizable theory are studied. Just as in the case of localizable theories, the key to the solution of the questions appearing here is the consecutive utilization of the spectral conditions and of invariance under the Poincaré group. It turns out that along with the essential differences from localizable theories (the principal one of which is associated with the absence of the usual concept of locality) there are many common features between both classes of theories; the most important being the existence of an analogue of the Bargmann-Hall-Wightman theorem which here also leads to the presence of real points of holomorphy for Wightman functions. From here arises the possibility to formulate the condition of quasilocality (or l-locality for brevity) for the nonlocalizable functionals under consideration. This condition in a natural way generalizes the usual locality

[1)]See articles [4-6] for attempts to generalize Meĭman's formulation to the 4-dimensional case. We also note the work of Solov'ev [7] where a more general class of spaces is considered than in [3], permitting a formulation of the spectral condition and microcausality.

[2)]This theorem, originally established for a theory of moderate growth, allows an immediate generalization to the case of strictly localizable fields considered by Jaffe. [9]

condition to the language of Wightman functions and goes over into it in the limit when the region of nonlocality vanishes.

Further, in Sec. 4 it is shown that starting from the analytic properties of Wightman functions one can prove a theorem about the equivalence of CPT invariance and weak local commutativity (WLC), and one can also prove a theorem about the connection between spin and statistics. Concluding remarks are collected together in Sec. 5. The validity of the reconstruction theorem[13] is mentioned. For a special class of nonlocalizable theories one is able to prove a theorem, important for the physical interpretation of the theory, about the existence of asymptotic states or the S-matrix (the Haag-Ruelle theorem;[14] in this regard also see the examples of nonlocalizable theories in [15]). An instructive example is analyzed, characterizing the analytic properties of the two-point Wightman function in a nonlocalizable theory.

One must emphasize two properties which are essential for nonlocalizable theories. The first—the exponential growth of the Wightman functions is, as follows from the results of this work, apparently the limitingly-admissible situation from the point of view of physical interpretation since in the opposite case these functions (and all other Green's functions in the x-representation) will not have real points of holomorphy, and also such important physical characteristics as CPT invariance and WLC, and the connection between spin and statistics lose their meaning. The physical interpretation of such a theory will be extremely difficult even if such a theory is possible in general. The second property is the essential and so far unsolved problem of all nonlocalizable theories (and nonlocal theories also)—the question of whether microcausality is satisfied.

We note that one of the first attempts at a Wightman formulation of unrenormalizable and, in particular, nonlocalizable theories in the language of analytic functionals was undertaken in the articles by Khoruzhiĭ.[16] However, in these articles the important question about the holomorphy domain of the Wightman functions in such theories was not raised.

2. SPACE OF BASIS FUNCTIONS. WIGHTMAN FUNCTIONS

Let us consider for simplicity the theory of one scalar neutral field with mass $m > 0$. The field is described by operator-valued generalized functions $\tilde{A}(\tilde{\varphi})$ (in symbolic form $\tilde{A}(\tilde{\varphi}) = \int d^4p\tilde{A}(p)\tilde{\varphi}(p)$) over the space of basis functions $\mathfrak{M}(R^4)$ (the basis space is defined below). (Here and below a tilde denotes the Fourier transform of the corresponding quantity.) It is assumed that all of the Wightman axioms are satisfied except locality.[13] In particular, it is assumed that the operator $\tilde{A}(\tilde{\varphi})$ is defined on a dense domain D of the Hilbert space H of states which include the unique cyclic vacuum state vector Ψ_0, i.e., the set of vectors

$$\Psi = \tilde{A}(\tilde{\varphi}_1)\ldots\tilde{A}(\tilde{\varphi}_n)\Psi_0 \tag{1}$$

is dense in H.

Let us denote by $\mathfrak{M}(R^{4n})$ and $C(R^{4n})$ the countably normed, complete, linear kernel spaces of basis functions in the p- and x-representations, respectively. We assume that convergence (or topology) in $\mathfrak{M}(R^{4n})$ is defined by the following family of norms:

$$\|\tilde{\varphi}\|_k = \sup_{p;\, m\leqslant k} g(k\|p\|^2)\,|D^m\tilde{\varphi}(p)|, \tag{2}$$

where k and m are integers,

$$D^m = \prod_{i=1}^{n}\frac{\partial^{m_i}}{\partial p_{i0}^{m_{i0}}\ldots\partial p_{i3}^{m_{i3}}},\quad m=\sum_{i=1}^{n} m_i, m_i = m_{i0}+\ldots m_{i3};$$
$$\|p\|^2 = \sum_{i=1}^{n}\|p_i\|^2,\quad \|p_i\|^2 = p_{i0}^2 + \mathbf{p}_i^2,$$
$$\overline{\lim_{\nu\to\infty}}\sup_n\,[c_{\nu+n}2(\nu+n)!]^{1/2(\nu+n)} = \rho, \tag{3}$$

where $\overline{\lim}$ denotes the limit of the sequence from above.[3)] The function $g(t^2) = \sum_{\nu=0} c_\nu t^{2\nu}$; $c_\nu \geq 0$, $c_0 > 0$ is an entire analytic function of first order growth and type ρ with respect to t.

The space $\mathfrak{M}(R^{4n})$ consists of all $\tilde{\varphi}(p)$ for which the family of norms (1) is finite:

$$\mathfrak{M}(R^{4n}) = \{\tilde{\varphi}(p) : \|\tilde{\varphi}\|_k < \infty,\quad k = 1,\ldots\}. \tag{4}$$

It is obvious that the space of finite functions $K \subset \mathfrak{M}$. The space $C(C^{4n})$ is defined as follows:

$$C(C^{4n}) = \{\varphi(z) : \varphi(z) = F[\tilde{\varphi}](z), \tilde{\varphi}(p) \in \mathfrak{M}(R^{4n})\}, \tag{5}$$

where F denotes the Fourier-Laplace transform.

As Jaffe showed,[3] the space $C(R^{4n})$ contains a dense set of finite functions if and only if

$$\int_0^\infty \frac{\ln g(t^2)\,dt}{1+t^2} < \infty, \tag{6}$$

or

$$\sum_{\nu=0}^{\infty} |c_\nu|^{1/\nu} < \infty. \tag{7}$$

In our case these conditions are not satisfied (owing to (2)); the space $C(C^{4n})$ consists of entire analytic functions. Since it does not contain finite functions, locality may not be formulated in the usual way.[4)]

Invariance under the Poincaré group means

$$U(a,\Lambda)\tilde{A}(\tilde{\varphi})U^*(a,\Lambda) = \int \tilde{A}(\tilde{p})e^{ipa}\tilde{\varphi}(\Lambda p)\,d^4p. \tag{8}$$

Here $U(a, \Lambda)$ is a unitary representation of this group in H; a is the 4-vector of the translation; $\Lambda \in L_+^\uparrow$ ($L_+^\uparrow$ is the restricted Lorentz group).

The Wightman functions

$$\tilde{\mathscr{W}}_n(\tilde{\varphi}_1,\ldots,\tilde{\varphi}_n) = (\Psi_0, \tilde{A}(\tilde{\varphi}_1)\ldots\tilde{A}(\tilde{\varphi}_n)\Psi_0),\quad \tilde{\varphi}_i(p)\in\mathfrak{M}(R^4) \tag{9}$$

are continuous with respect to each argument of the polylinear functionals, by virtue of the nuclear theorem these functionals can be uniquely extended to generalized functions over the space $\mathfrak{M}(R^{4n})$:

$$\tilde{\mathscr{W}}_n(\tilde{\varphi}) = \langle\tilde{\mathscr{W}}_n(p_1,\ldots,p_n)\quad \tilde{\varphi}(p_1,\ldots,p_n)\rangle. \tag{10}$$

According to the general theory,[17] a Wightman function admits a representation in the form

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[3)]This means that $\lim\limits_{\|p\|\to\infty}\dfrac{\ln g(k\|p\|^2)}{\|p\|} = \rho\sqrt{k}$.

[4)]First we shall in general avoid conversion to the x-representation. In Sec. 3 we see in what sense one can talk about the quantities of the theory in x-space.

$$\widetilde{\mathscr{W}}_n(\tilde{\varphi}) = \int (dp) g(k\|p\|^2) \sum_{m\leq k} \tilde{f}_m(p) \tilde{\varphi}(p), \quad \tilde{\varphi} \in \mathfrak{M}(R^{4n}),$$

$$(dp) = d^4p_1 \ldots d^4p_n, \tag{11}$$

where $\tilde{f}_m(p)$ are bounded, measurable functions. Below for convenience instead of the functionals $\widetilde{\mathscr{W}}_n(\tilde{\varphi})$ given by expression (11) we shall use the generalized functions $\widetilde{\mathscr{W}}_n(p)$ and write

$$\widetilde{\mathscr{W}}_n(\tilde{\varphi}) = \int \widetilde{\mathscr{W}}_n(p_1; \ldots; p_n) \tilde{\varphi}(p_1; \ldots; p_n)\,(dp). \tag{11'}$$

Let us note certain basic properties of $\widetilde{\mathscr{W}}_n(p)$. From the property of translational invariance (see Eq. (8)) for $\Lambda = 1$ it follows that

$$\widetilde{\mathscr{W}}_n(\tilde{\varphi}) = \widetilde{\mathscr{W}}_n(\tilde{\varphi} e^{ipa}) = \int \widetilde{\mathscr{W}}_n(p_1, \ldots, p_n) \exp\left\{i \sum_{i=1}^n p_i a\right\} \tilde{\varphi}(p_1, \ldots, p_n)\,(dp).$$

Or, in the language of the generalized functions $\widetilde{\mathscr{W}}_n(p)$,

$$\widetilde{\mathscr{W}}_n(p_1, \ldots, p_n) = (2\pi)^4 \delta\Big(\sum_{i=1}^n p_i\Big) \widetilde{W}_{n-1}(p_1, p_1 + p_2, \ldots, p_1 + \ldots + p_{n-1})$$

$$\equiv (2\pi)^4 \delta\Big(\sum_{i=1}^n p_i\Big) \widetilde{W}_{n-1}(q_1, \ldots, q_{n-1}), \quad q_j = \sum_{i=1}^j p_i. \tag{12}$$

From the spectral conditions it follows that

$$\operatorname{supp} \widetilde{W}_{n-1}(q_1, \ldots, q_{n-1}) \subset (\Gamma_+)^{n-1}, \tag{13}$$

where

$$(\Gamma_+)^{n-1} = \{q : q_i^2 > 0, \quad q_{i0} > 0, \quad i = 1, \ldots, n-1\}.$$

3. ANALYTIC PROPERTIES, JOST POINTS, AND QUASILOCALITY

It is best of all to formulate the restrictions on the growth of Wightman functions in the language of the generalized functions $\widetilde{W}_{n-1}(q_1, \ldots, q_{n-1})$. Using (12) one can write the functions $\widetilde{\mathscr{W}}_n(\tilde{\varphi})$ in the form

$$\widetilde{\mathscr{W}}_n(\tilde{\varphi}) = \int \widetilde{W}_{n-1}(q_1, \ldots, q_{n-1}) \tilde{\psi}(q_1, \ldots, q_{n-1})(dq) \equiv \langle \widetilde{W}_{n-1}, \tilde{\psi} \rangle \equiv \widetilde{W}_{n-1}(\tilde{\psi}). \tag{14}$$

Here

$$\tilde{\psi}(q_1, \ldots, q_{n-1}) = (2\pi)^4 \int \tilde{\varphi}(p_1, \ldots, p_{n-1}) \delta\Big(\sum_{i=1}^n p_i\Big) \prod_{j=1}^{n-1} \delta\Big(q_i - \sum_{i=1}^j p_i\Big)\,(dp)$$

$$= (2\pi)^4 \tilde{\varphi}(q_1, q_2 - q_1, \ldots\ q_{n-1} - q_{n-2}, -q_{n-1})$$

is a basis function belonging to $\mathfrak{M}(R^{4(n-1)})$. Therefore $\widetilde{W}_{n-1}(\tilde{\psi})$ admits the representation (11):

$$\widetilde{W}_{n-1}(\tilde{\psi}) = \int (dq) g(k\|q\|^2) \sum_{m\leq k} F_m(q) D^m \tilde{\psi}(q), \tag{15}$$

where k and ρ are in general different than in (11) or in (2), and the $\tilde{F}_m(q)$ are bounded, measurable functions. By the growth of $\widetilde{W}_{n-1}(q)$ we shall understand the type of growth $\rho\sqrt{k}$ of the functions $g(k\|q\|^2)$. The quantities ρ_n and k_n may be different for different $\widetilde{W}_{n-1}(\psi)$. We shall assume that

$$\overline{\lim_n}\, \rho_n \sqrt{k_n} = \bar{l} > 0. \tag{16}$$

In the opposite case the domain of nonlocality, which is characterized by $\bar{l}$, will grow with an increase in the number of variables in a Wightman function, which hardly has any physical meaning.

The spectral conditions (12) allow us to extend the functional $\widetilde{\mathscr{W}}(\tilde{\psi})$, and consequently also $\widetilde{\mathscr{W}}(\tilde{\varphi})$, to a broader class of functions. Let us consider the extension of the functional

$$\widetilde{W}(e^{-iq\zeta}) \equiv W(\zeta) = \mathscr{W}(z) = \int \widetilde{W}(q) e^{-iq\zeta}(dq), \tag{17}$$

where the $\zeta_i = \xi_i - i\eta_i = z_i - z_{i-1}$ are chosen so that

$$\eta_{i0} > |\eta_i| + l, \quad i = 1, \ldots, n-1. \tag{18}$$

An extension of the functional exists since it follows from Eq. (15) that as $\|q_i\| \to \infty$

$$g(k\|q\|^2) < \exp(\bar{l} + \varepsilon) \sum_i \|q_i\|$$

with arbitrary $\epsilon > 0$; consequently the integral in (15) converges with $\tilde{\psi}(q) = e^{-iq\zeta}$ if $l\|q_i\| \leq \eta_i q_i$, which is satisfied for

$$(\eta_{i0} - |\eta_i|)^2 \geqslant 2\bar{l}^2 \equiv l^2.$$

It follows from the relativistic invariance of $W(\zeta)$ that the functional (17) also exists for

$$\eta \in V_\Lambda{}^l = \{\eta \,|\, (\Lambda\eta_i)_0 > |\Lambda\eta_i| + l; \quad i = 1, \ldots, n-1\}, \ \Lambda \in L_+^\uparrow.$$

One can write the domain V^l_Λ in the following equivalent form:

$$V_\Lambda{}^l = \{\eta \,|\, \eta_i - lu \in \Gamma_+; \ i = 1, \ldots, n-1; \ u^2 = 1; \ u \in \Gamma_+\}. \tag{19}$$

And what is more, according to a theorem of Vladimirov[18] (Chapter V, p. 272, subsection 2), $W(\zeta)$ is a holomorphic function of ζ in a tube which is defined in the following way:

$$T^l_{n-1} = \{\zeta \,|\, \operatorname{Im}\zeta = (\operatorname{Im}\zeta_1, \ldots, \operatorname{Im}\zeta_{n-1}) \in \bigcup_{(\Lambda)} V_\Lambda{}^l \equiv V^l; \ \Lambda \in L_+^\uparrow\}. \tag{20}$$

The latter assertion follows from the fact that

$$\widetilde{W}(q) e^{-q\eta} \in S^*, \quad \eta \in V^l,$$

i.e., it is a functional of temperate growth. It is not difficult to verify that the domain V^l is convex.

In similar fashion it can be shown that $\mathscr{W}(z)$ is holomorphic in the domain

$$\sigma_n{}^l = \{z \,|\, (\operatorname{Im} z_1, \operatorname{Im}(z_2 - z_1), \ldots, \operatorname{Im}(z_n - z_{n-1})) \in V^l\}.$$

Application of the Bargmann-Hall-Wightman theorem [13,14] to the function $W(\zeta)$ gives: from the invariance of the function $W(\zeta)$ under Lorentz transformations $\Lambda \in L_+^\uparrow$ in the domain T^l_{n-1} it follows that it admits a single-valued $L_+(C)$-invariant analytic continuation into a wider domain

$$T^{l'}_{n-1} = \bigcup \Lambda T^l_{n-1}, \text{ where } \Lambda \in L_+(C),$$

i.e., it is holomorphic for all

$$\zeta' \in T^{l'}_{n-1} = \{\zeta' : \zeta_i' = \Lambda\zeta_i; \ \Lambda \in L_+(C); \ \zeta \in T^l_{n-1}; \ i = 1, \ldots, n-1\}. \tag{21}$$

Here $L_+(C)$ is the complex Lorentz group.

The function $\mathscr{W}_n(z)$ correspondingly admits a single-valued holomorphic continuation into the region

$$\sigma^{l'}_n = \bigcup_{\Lambda \in L_+(C)} \Lambda\sigma_n{}^l = \{z' : z_i' = \Lambda z_i; \ z \in \sigma_n{}^l, \ \Lambda \in L_+(C)\}. \tag{22}$$

A remarkable consequence of this extension is the presence of real points of holomorphy (Jost points) for $W_{n-1}(\zeta)$ and $\mathscr{W}_n(z)$. However, in the first place this domain is narrower than the domain of real points of holomorphy in a localizable theory and, in the second place, in contrast to a localizable theory it is impossible to directly determine the generalized functions $\mathscr{W}_n(x)$ and

$\mathscr{W}_{n-1}(\xi)$ for all x and ξ as the boundary values of the corresponding holomorphic functions $\mathscr{W}_n(z)$ and $W_{n-1}(\zeta)$ since in the present case the real domain does not lie on the boundary of the primitive domains of holomorphy σ_n^l and T_{n-1}^l.

Now let us prove a theorem providing criteria as to which real point lies in the extended tube $T_{n-1}^{l'}$.

Theorem. In order for a real point $\rho = (\rho_1, \ldots, \rho_{n-1})$ to be a point of holomorphy of $W_{n-1}(z)$, it is necessary and sufficient that a transformation $\Lambda_1 \in L_+^\uparrow$ be found which satisfies the inequalities

$$|(\Lambda_1\rho_i)_0 - l\,\mathrm{sh}\,t| < (\Lambda_1\rho_i)_1 - l\,\mathrm{ch}\,t, \quad i = 1, \ldots, n-1 \tag{23}$$

with a certain t.

Proof of sufficiency. Let (23) be satisfied. The transformation

$$\Lambda = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L_+(C) \tag{24}$$

translates a point ρ into the domain T_{n-1}^l. Actually $\mathrm{Im}\,\Lambda\Lambda\,\rho_i \equiv \eta_i = (\rho_{i1}, \rho_{i0}, 0, 0)$. Using (23) and (19) we obtain the result that $\eta - lu \in \Gamma_+$, $i = 1, \ldots, n-1$, where $u = (\cosh t, \sinh t, 0, 0)$.

Proof of necessity. Let $\rho \in T_{n-1}^{l'}$, i.e., let a transformation $\Lambda \in L_+(C)$ exist such that

$$\rho_i = \Lambda\zeta_i; \quad (\zeta_1, \ldots, \zeta_{n-1}) \in T_{n-1}^l; \quad \zeta_i = \xi_i + i\eta_i.$$

Any arbitrary complex Lorentz transformation $\Lambda \in L_+(C)$ is equivalent to one of two normal forms,[14] i.e., $\Lambda = \Lambda_2\Lambda^N\Lambda_1$ where $\Lambda_{1,2} \in L_+^\uparrow$. Only a transformation equivalent to

$$\Lambda^N(\varphi, \chi) = \begin{pmatrix} \cos\varphi & -i\sin\varphi & 0 & 0 \\ -i\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \mathrm{ch}\,\chi & i\,\mathrm{sh}\,\chi \\ 0 & 0 & -i\,\mathrm{sh}\,\chi & \mathrm{ch}\,\chi \end{pmatrix} \tag{25}$$

(φ and χ are real) can transform a point $\zeta \in T_{n-1}^l$ into a real point. Condition (19) that

$$\eta = (\mathrm{Im}\,\Lambda^N\Lambda_1\rho_1, \ldots, \mathrm{Im}\,\Lambda^N\Lambda_1\rho_{n-1}) \in V_\Lambda{}^l,$$

with some $\Lambda \in L_+^\uparrow$ in terms of the components of ρ, takes the form

$$\{(\Lambda_1\rho_i)_1 \sin\varphi - lu_0; \quad (\Lambda_1\rho_i)_0 \sin\varphi - lu_1;$$
$$(\Lambda_1\rho_i)_3\,\mathrm{sh}\,\chi - lu_2; \quad (\Lambda_1\rho_i)_2\,\mathrm{sh}\,\chi - lu_3\} \in \Gamma_+.$$

Hence, denoting $u_0 = \sin\varphi \cdot \cosh t$, $u_1 = \sin\varphi \cdot \sinh t$, we obtain the result that

$$\rho_{i1} - l\,\mathrm{ch}\,t \geqslant |\rho_{i0} - l\,\mathrm{sh}\,t|; \quad i = 1, \ldots, n-1,$$

i.e., condition (23) is valid for ρ.

As $l \to 0$ condition (23) turns into, as is not difficult to verify, a condition which is equivalent to the usual condition for Jost points in a local theory:

$$\left(\sum \lambda_i\rho_i\right)^2 < 0, \quad \sum \lambda_i \neq 0, \quad \lambda_i \geqslant 0.$$

The presence of real points of holomorphy for Wightman functions makes it possible to formulate an analogue of the locality conditions in a nonlocalizable theory. We shall call a theory quasilocal (or l-local) if all $\mathscr{W}_n(z)$ are symmetric functions in the domain of holomorphy. It is obvious that the condition of l-locality substantially extends the holomorphy domain of $\mathscr{W}(z)$.

At real points of holomorphy the condition of l-locality gives

$$\mathscr{W}_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathscr{W}_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$
$$\text{for all } x = (x_1, \ldots, x_n) \in \mathrm{Re}\,\sigma_n{}^{l'}. \tag{26}$$

We note that for the two-point function the condition of l-locality is automatically satisfied:

$$\mathscr{W}_2(x_1; x_2) - \mathscr{W}_2(x_2; x_1) = 0, \quad (x_1 - x_2)^2 < -l^2. \tag{27}$$

An explicit example of the two-point Wightman function is given in subsection 5 of Sec. 5.

Since there are no finite basis functions in coordinate space, it is impossible to formulate the condition of l-locality in terms of the usual condition that the commutator of two operators vanishes in a certain region. However, as $l \to 0$ the condition of l-locality reduces to the locality condition for localizable theories in terms of Wightman functions, and in that case when the function g specifying the topology of the basis space satisfies condition (6), the symmetry of the Wightman functions is equivalent to the microcausality condition in terms of the commutator.

The theorem about the global nature of local commutativity,[9,19] from which it follows that for localizable theories local commutativity follows from l-locality, does not remain valid in the case of nonlocalizable theories in view of the transition to a broader class of functionals (see subsection 5 of Sec. 5).

Qualitatively one can say that the principle of microcausality is violated in an l-local theory in the region $-l^2 < \xi^2 < 0$ and "signals" can propagate with a velocity greater than c. Therefore one can say that the quantity l characterizes the region of nonlocalizability of the interaction.

In concluding this section we note that the condition of l-locality enables us even in the case of nonlocalizable theories to distinguish two classes of theories: a class of quasilocal theories, and a class for which condition (11) is not satisfied.[5)]

4. CPT INVARIANCE AND WEAK LOCAL COMMUTATIVITY. THE CONNECTION BETWEEN SPIN AND STATISTICS

The holomorphic property of $\mathscr{W}_n(z)$ and $W_{n-1}(\zeta)$ ensures the validity of an analogue of the CPT theorem in a nonlocalizable theory. It is convenient to formulate CPT invariance in the p-representation. We shall say that the theory of a scalar field is CPT invariant if

$$\widetilde{\mathscr{W}}_n(p_1, \ldots, p_n) = \widetilde{\mathscr{W}}_n(-p_n, \ldots, -p_1) \tag{28}$$

or

$$\widetilde{W}(q_1, \ldots, q_{n-1}) = \widetilde{W}(q_{n-1}, \ldots, q_1). \tag{29}$$

It is obvious that if one changes from the functionals $\widetilde{\mathscr{W}}_n(\widetilde{\varphi})$ to their Fourier transforms and associates with the functionals

$$\mathscr{W}_n(\varphi) = \int \mathscr{W}_n(x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n)\,(dx) \tag{30}$$

5)In the general case a nonlocalizable theory (not quasilocal) goes over into a nonlocal (but localizable) theory as $l \to 0$.

the generalized functions $\mathscr{W}_n(x_1, \ldots, x_n)$, then in terms of these functions the conditions for CPT invariance, (28) and (29), are written in the usual way:

$$\mathscr{W}_n(x_1, \ldots, x_n) = \mathscr{W}_n(-x_n, \ldots, -x_1) \tag{31}$$

or

$$W_{n-1}(\xi_1, \ldots, \xi_{n-1}) = W_{n-1}(\xi_{n-1}, \ldots, \xi_1). \tag{32}$$

The condition of weak local commutativity (WLC) means:

$$\mathscr{W}_n(x_1, \ldots, x_n) = \mathscr{W}_n(x_n, \ldots, x_1),$$
$$(x_1, \ldots, x_n) \in \operatorname{Re} \sigma_n^{l'}, \tag{33}$$

or

$$W_{n-1}(\xi_1, \ldots, \xi_{n-1}) = W_{n-1}(-\xi_{n-1}, \ldots, -\xi_1),$$
$$(\xi_1, \ldots, \xi_{n-1}) \in \operatorname{Re} T_{n-1}^{\sim} \tag{34}$$

Theorem. If the WLC condition is satisfied in a (real) neighborhood of a point $x \in \operatorname{Re} \sigma_n^{l'}$, then relation (28) (or (31)) is valid, i.e., the theory possesses CPT symmetry. Conversely, conditions (33) and (34) for WLC follow from conditions (28) and (32) for CPT invariance.

First let us prove the last assertion. From Eq. (28) we quickly deduce that

$$\mathscr{W}_n(z_1, \ldots, z_n) = \int \mathscr{W}_n(p_1, \ldots, p_n) \exp\left\{-i \sum p_i z_i\right\} (dp)$$
$$= \mathscr{W}(-z_n, \ldots, -z_1) \tag{35}$$

everywhere in σ_n^l and, consequently, over the entire domain of holomorphy $\sigma_n^{l'}$. Carrying out the transformation $z_i \to z_i' = \Lambda z_i = -z_i$, $\Lambda \in L_+(C)$ in $\mathscr{W}_n(-z_n, \ldots, -z_1)$ we obtain

$$\mathscr{W}_n(z_1, \ldots, z_n) = \mathscr{W}_n(z_n, \ldots, z_1)$$

in the domain of holomorphy $\sigma_n^{l'}$ and, in particular, in the real domain $\operatorname{Re} \sigma_n^{l'}$ and so forth. Now let expression (33) be valid in the real neighborhood of a point $(x_1, \ldots, x_n) \in \operatorname{Re} \sigma_n^{l'}$. From the uniqueness of the analytic continuation we conclude that

$$\mathscr{W}_n(z_1, \ldots, z_n) = \mathscr{W}_n(z_n, \ldots, z_1) = \mathscr{W}_n(-z_n, \ldots, -z_1)$$

over the entire domain of holomorphy $\sigma_n^{l'}$. Hence, taking (35) into consideration, we may write down

$$\int (dz) \int [\widetilde{\mathscr{W}}(p_1, \ldots, p_n) - \widetilde{\mathscr{W}}(-p_n, \ldots, -p_1)]$$
$$\times \exp\left\{-i \sum p_i z_i\right\} \varphi(z_1, \ldots, z_n)\, (dp) = 0. \tag{36}$$

Here $\varphi(z_1, \ldots, z_n) \in C(C^{4n})$ and the integration is carried out along an arbitrary contour within the limits of the domain of holomorphy σ_n^l. Now let us choose the integration contour in (36) so that $\operatorname{Im} z_i = 0$, and $\operatorname{Im}(z_{i+1}^0 - z_i^0) = a > l$, $i = 1, \ldots, n-1$, and let us consider

$$J = \int (dz) \exp\left\{-i \sum p_i z_i\right\} \varphi(z_1, \ldots, z_n). \tag{36'}$$

One can deform the integration contour up to the real domain. Then we obtain

$$\int \widetilde{\mathscr{W}}(p_1, p_2) \tilde{\varphi}(p_1, p_2)\, (dp) = 0, \quad \tilde{\varphi} \in \mathfrak{M}(R^{4\cdot 2}), \tag{37}$$

Substituting (37) into (36) we finally find

$$\int [\widetilde{\mathscr{W}}_n(p_1, \ldots, p_n) - \widetilde{\mathscr{W}}_n(-p_n, \ldots, -p_1)] \tilde{\varphi}(p_1, \ldots, p_n)\, (dp) = 0 \tag{38}$$

for arbitrary $\tilde{\varphi} \in \mathfrak{M}(R^{4n})$, which is equivalent to (28). In terms of the functions $\mathscr{W}(x_1, \ldots, x_n)$ this means that (31) and (32) are valid.

Let us prove the connection between spin and statistics in an l-local theory. Let a scalar field be quantized so that $\mathscr{W}_2(z_1, z_2)$ is antisymmetric with respect to interchange of the arguments in the domain of holomorphy, i.e., it is quantized according to Fermi-Dirac statistics. Let us show that in such a theory all $\mathscr{W}_n(z_1, \ldots, z_n) = 0$. By assumption $\mathscr{W}_2(z_1, z_2) = -\mathscr{W}_2(z_2, z_1)$ in the domain $\sigma_2^{l'}$. Therefore $W_1(x_1 - x_2) = -W_1(x_2 - x_1)$ for $(x_1 - x_2)^2 < -l^2$. Comparing this condition with (27) we find that $W_1(x_1 - x_2) = 0$ at points of holomorphy on the real axis. Hence, due to the uniqueness of the analytic continuation, by repeating the arguments used for the proof of (38), we find that

$$J = \int (dx) \exp\left\{-i \sum p_i x_i\right\} \varphi(x_1, \ldots, x_n) = \tilde{\varphi}(p_1, \ldots, p_n).$$

or, in particular,

$$\int \widetilde{\mathscr{W}}(p_1, p_2) \tilde{\varphi}^*(p_1) \tilde{\varphi}(p_2)\, (dp) = \|\tilde{A}(\tilde{\varphi}) \Psi_0\|^2 = 0,$$
$$\tilde{\varphi}(p) \in \mathfrak{M}_1(R^4). \tag{39}$$

From (39) it follows that $\tilde{A}(\tilde{\varphi})\Psi_0 = 0$ and $\mathscr{W}_n(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n) = 0$. In the case when other fields, l-local or l-antilocal with respect to $\tilde{A}(\tilde{\varphi})$, are present one can show that $\tilde{A}(\tilde{\varphi}) = 0$ by using the Federbush-Johnson theorem.[13-19] All of these results can be generalized without difficulty to more complicated cases associated with the presence of spins.

5. CONCLUDING REMARKS

1. In the nonlocalizable theory under consideration one can prove the reconstruction theorem.[13] As a preliminary we note that in an l-local theory the cluster decomposition theorem is valid:

$$\langle \Phi, T(a) \Psi \rangle - \langle \Phi, \Psi_0 \rangle \langle \Psi_0, \Psi \rangle = t(a) \in S(R^3), \tag{40}$$

where $T(a) = U(a, 1)$. This follows from the results of article [20] where the specific properties of the basis function spaces were not used in the proof of formula (40). Therefore, starting from the given properties of $\widetilde{\mathscr{W}}_n(\tilde{\varphi})$ (including l-locality), one can construct a separable Hilbert space H, a continuous unitary representation $U(a, \Lambda)$ of the Poincaré group in H, a unique state Ψ_0 invariant under $U(a, \Lambda)$, and a Hermitian scalar field $\tilde{A}(\tilde{\varphi})$ such that

$$\langle \Psi_0, \tilde{A}(p_1) \ldots \tilde{A}(p_n) \Psi_0 \rangle = \widetilde{\mathscr{W}}_n(p_1, \ldots, p_n)$$

with prescribed properties. The proof is carried out in the same way as in [13,14], but only in the p-representation.

2. For a narrower class of nonlocalizable theories one can without difficulty prove the existence of asymptotic fields and the S-matrix (the Haag-Ruelle theorem[14]).[6)] For this purpose let us consider the truncated vacuum expectation values $\widetilde{\mathscr{W}}_n^T(\tilde{\varphi})$ instead of $\tilde{W}_n(\tilde{\varphi})$ (for their definition see, for example, [13,14]). The representation (11) is also valid for them. One can transform this representation to the form

[6)]A general proof of this theorem for a nonlocalizable field theory will be treated in a different place.

$$\widetilde{\mathscr{W}}_n{}^T(\tilde{\varphi}) = \int (dp) \sum_{m \leqslant k} \tilde{f}_m{}'(p) D^m \{g(k\|p\|^2)\tilde{\varphi}(p)\}, \tag{41}$$

where the $\tilde{f}'_m(p)$ are measurable bounded functions. Or, introducing an obvious notation, we obtain

$$\widetilde{\mathscr{W}}_n{}^T(\tilde{\varphi}) = \int (dp)\, \widetilde{\mathscr{W}}_{0n}{}^T(p_1, \ldots, p_n) g(k\|p\|^2)\tilde{\varphi}(p_1, \ldots, p_n), \tag{42}$$

where $\widetilde{\mathscr{W}}^T_{0n}(p_1, \ldots, p_n)$ is a generalized function of moderate growth, satisfying the spectral conditions (13) and translational invariance (12). In the general case the usual locality conditions for the generalized functions $\mathscr{W}^T_{0n}(x_1, \ldots, x_n) = F[\mathscr{W}^T_{0n}(p)] \cdot (x_1, \ldots, x_n)$ do not follow from l-locality. However, under the additional assumption that $\mathscr{W}_{0n}(x_1, \ldots, x_n)$ in (42) satisfies the local commutativity property

$$(x_i - x_{i+1})^2 < 0 \tag{43}$$

or decreases sufficiently rapidly in a space-like direction (with respect to the distances $(x_i - x_j)^2$), the proof of this theorem practically does not differ at all from the usual proof.

We note that the conjecture about the locality properties (43) of the functions $\mathscr{W}_{0n}(x_1, \ldots, x_n)$ entering into (42) does not by any means imply that the nonlocalizable theory under consideration is constructed by means of a trivial generalization of a local theory of moderate growth, since in symbolic notation

$$\mathscr{W}_n{}^T(x_1, \ldots, x_n) = \hat{g}\mathscr{W}_{0n}{}^T(x_1, \ldots, x_n),$$

where g is an integral operator formally obtained by taking the Fourier transform of the operator $g(k\|p\|^2)$; the functions $\mathscr{W}_{0n}(x_1, \ldots, x_n)$ cannot be identified with the Wightman functions of a local theory; in general they do not satisfy the requirements of positive definiteness and relativistic invariance.

3. The holomorphic properties of $\mathscr{W}_n(z_1, \ldots, z_n)$ permit one to manipulate with these functionals in the domain of holomorphy in the same way as with the usual functions. In particular, one can formulate the l-locality condition in a form which is very similar to the microcausality condition for a local theory:

$$\int [\mathscr{W}_n(x_1, \ldots, x_n) - \mathscr{W}_n{}^\pi(x_1, \ldots, x_n)]\, \varphi(x_1, \ldots, x_n)\,(dx) = 0, \tag{44}$$

if $\varphi(x_1, \ldots, x_n) \in K$ — the space of finite functions, and its support is concentrated in the region $\mathrm{Re}\,\sigma^{l\prime}_n$ where all $(x_i - x_j)^2 < -l^2$, $i \neq j = 1, \ldots, n$. Here the index π on $\mathscr{W}_n$ denotes an arbitrary permutation of the arguments.

4. In obtaining all results of the present article, nowhere have we resorted to the explicitly relativistically invariant form of the generalized functions $\widetilde{\mathscr{W}}_n(p_1, \ldots, p_n)$

5. It is of interest to see how the singularities of a nonlocalizable theory appear in the example of a simple model for the two-point Wightman function. Let

$$\tilde{W}_2(q) = \theta(q_0) \exp(l\sqrt{q^2})\, \theta(q^2)\, (q^2)^{-1/2}, \quad l > 0.$$

In this case the function

$$W_2(z) = \int \tilde{W}_2(q) e^{-iqz} d^4q$$

For $\mathrm{Im}\, z \in \Gamma^l_+$ (see (17)) can be explicitly calculated and is equal to

$$W_2(z) = \frac{1}{2\pi^2(z^2 + l^2)} \left\{ \frac{2l}{z^2} + \frac{1}{\sqrt{z^2 + l^2}} \ln \frac{l + \sqrt{z^2 + l^2}}{l - \sqrt{z^2 + l^2}} \right\}. \tag{45}$$

The branches of the radical and the logarithm in (45) are chosen in the following way:

for $\mathrm{Re}\, z^2 = x^2 < -l^2$, $\mathrm{Im}\, z^2 = 0$

$$\sqrt{z^2 + l^2} = i\sqrt{-(l^2 + \mathrm{Re}\, z^2)};$$

for $\mathrm{Re}\, z^2 > 0$, $\mathrm{Im}\, z^2 = \epsilon$

$$W_2(x^2 \pm i\varepsilon) = \frac{1}{2\pi^2(x^2 + l^2)} \left\{ \frac{2l}{x^2 \pm i\varepsilon} + \frac{1}{\sqrt{l^2 + x^2}} \cdot \ln \left| \frac{l + \sqrt{x^2 + l^2}}{l - \sqrt{x^2 - l^2}} \right| \pm \frac{i\pi}{\sqrt{l^2 + x^2}} \right\}.$$

It is easy to see that

A. This function is holomorphic everywhere in the plane of the complex variable z^2 except for the following singularities on the real axis: a pole at $z^2 = -l^2$; a pole and a logarithmic branch point at $z^2 = 0$.

B. We have

$$W_2(x^2 - i\varepsilon x_0) - W_2(x^2 + i\varepsilon x_0) = 0, \quad x^2 < -l^2,$$

i.e., $W_2(x^2)$ automatically satisfies quasilocality. It is quite clear that the theorem about the global nature of local commutativity is violated.

C. One can formally define the generalized function $W_2(x)$ everywhere on the real axis as the following limit:

$$W_2(x) = \lim_{z^2 = x^2 - i\varepsilon x_0;\ \varepsilon \to 0} W_2(z).$$

But, in this connection it loses the property of relativistic invariance in the region $-l^2 \le x^2 < 0$. However, the functional

$$\int dz \int_{ia-\infty}^{ia+\infty} dz_0\, W_2(z) \varphi(z) = (2\pi)^4 \int \tilde{W}_2(q) \tilde{\varphi}(q)\, d^4q = (2\pi)^4\, \tilde{W}_2(\tilde{\varphi}), \quad \tilde{\varphi}(q) \in \mathfrak{M}(R^4)$$

(the path of integration over z_0 may be arbitrary only it should pass above the point $-\mathrm{Im}\, z_0 = l$) preserves the property of relativistic invariance.

D. As $l \to 0$ the function $W_2(z)$ goes over into the Wightman function

$$W_2(z) \to \frac{1}{2\pi} \frac{1}{(z^2)^{3/2}} \tag{46}$$

for a localizable theory of moderate growth. The properties of $W_2(z)$ will be considered in more detail in another article.

[1] D. A. Kirzhnits, Doctoral Dissertation, FIAN (1966); Uspekhi Fizicheskikh Nauk **90**, 129 (1966) [Sov. Phys.-Uspekhi **9**, 692 (1967)].

[2] N. N. Meĭman, Zh. Eksp. Teor. Fiz. **47**, 1966 (1964) [Sov. Phys.-JETP **20**, 1320 (1965)]; Institute of Theoretical and Experimental Physics Preprint No. 254 (1964).

[3] A. M. Jaffe, Phys. Rev. **158**, 1454 (1967).

[4] Nguyen van Hieu, Ann. Phys. (N.Y.) **33**, 428 (1965).

[5] G. V. Efimov, Commun. math. Phys. **5**, 42 (1967).

[6] V. Ya. Feinberg, FIAN Preprint No. 137 (1967).

[7] M. A. Solov'ev, FIAN Preprint (1968).

[8] G. V. Efimov, Doctoral Dissertation, Dubna (1968).

[9] A. S. Wightman, "Quantum Field Theory and Analytic Functions of Several Complex Variables," Proc. Indian Math. Soc. **24**, 625 (1960). D. Ya. Petrina, UMZh **13**, 109 (1961).

[10] B. L. Voronov, Zh. Eksp. Teor. Fiz. **54**, 1816 (1968) [Sov. Phys.-JETP **27**, 976 (1968)].

[11] E. S. Fradkin, Nucl. Phys. **49**, 624 (1963).

[12] G. V. Efimov, Zh. Eksp. Teor. Fiz. **44**, 2107 (1963) [Sov. Phys.-JETP **17**, 1417 (1963)].

[13] R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That, Benjamin, 1964 (Russ. Transl., Nauka, 1966).

[14] Res Jost, The General Theory of Quantized Fields, American Mathematical Society, 1965 (Russ. Transl., Izd. Mir, 1967).

[15] M. Z. Iofa and M. A. Lifshits, FIAN Preprint No. 125 (1968).

[16] S. S. Khoruzhiĭ, JINR Preprints R2-3085 and R2-3086 (1966).

[17] I. M. Gel'fand and G. E. Shilov, Obobshchennye funktsii i deĭstviya nad nimi (Generalized Functions and Operations on Them), Fizmatgiz, 1959 (English Transl., Academic Press, 1964).

[18] V. S. Vladimirov, Metody teorii funktstiĭ mnogikh kompleksnykh peremennykh (Methods of the Theory of Functions of Many Complex Variables), Nauka, 1964 (English Translation by Scripta Technica, Inc., M.I.T. Press, 1966).

[19] P. G. Federbush and K. A. Johnson, Phys. Rev. **120**, 1926 (1960).

[20] R. Jost and K. Hepp, Helv. Phys. Acta **35**, 34 (1962).

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188