ELECTRON IN THE QUANTIZED FIELD OF A MONOCHROMATIC ELECTROMAGNETIC WAVE

I. BERSON

Institute of Physics, Latvian Academy of Sciences

Submitted November 16, 1968

Zh. Eksp. Teor. Fiz. 56, 1627–1633 (May, 1969)

The wave function is determined for a system consisting of an electron and the quantized field of a plane monochromatic electromagnetic wave interacting with it. Various representations of the obtained wave function are presented, and its relation to Volkov's solution is discussed.

As long ago as 1935 Volkov[1] found an exact solution of the Dirac equation for an electron in the field of a plane electromagnetic wave. Interest in this solution has increased appreciably with the availability of intense beams of light from lasers. A large number of articles have appeared (one can find a bibliography on this problem in articles[2,3]) in which the influence of the strong field of a plane electromagnetic wave on the behavior of different quantum processes is considered, for example, the effect on Compton scattering, on the production and annihilation of electron-positron pairs, etc. In the majority of cases the calculation of these phenomena is based on the Volkov solution. Those few cases in which a summation of a selected class of Feynman diagrams is carried out constitute an exception. It should, however, be noted that the Volkov solution is quasiclassical since it describes only the electron quantum mechanically. The electromagnetic potential is described classically as a function of the coordinates and time. In the present article it is shown that in the case of a monochromatic wave a solution of the present problem is possible when the field of the electromagnetic wave also is quantized, and the vector potential of this wave is an operator.

1. SOLUTION OF THE DIRAC EQUATION FOR AN ELECTRON IN THE QUANTIZED FIELD OF A MONOCHROMATIC WAVE

We shall start from the Dirac equation with the presence of an electromagnetic field taken into consideration[4]

$$\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-ieA_\mu\right)+m\right]\psi=0, \tag{1}$$

where e and m are the charge and mass of an electron, and the γ_μ are Hermitian matrices.[1)] The quantized field of a monochromatic electromagnetic wave may be described by an operator of the form[4]

$$A_\mu=\frac{e_\mu}{\sqrt{2k_0\Omega}}[ce^{i(kx)}+c^+e^{-i(kx)}], \tag{2}$$

where e_μ is a unit vector describing the polarization of the wave, and Ω is the normalization volume. It is assumed that the four-dimensional wave vector $k = (\mathbf{k}, ik_0)$ satisfies the condition $k^2 = \mathbf{k}^2 - k_0^2 = 0$, and it is also assumed that the Lorentz condition is satisfied so that $(ek) = 0$. With a view toward convenience during the various transformations of the wave functions obtained below, let us choose the "coordinate" representation[5,6] for the operators c^+ and c which create and annihilate photons with wave vector k and polarization e_μ:

$$c=\frac{1}{\sqrt{2}}\left(\xi+\frac{\partial}{\partial\xi}\right),\quad c^+=\frac{1}{\sqrt{2}}\left(\xi-\frac{\partial}{\partial\xi}\right). \tag{3}$$

If expression (2) is substituted into Eq. (1) then we obtain the Dirac equation, modified in the sense that the wave function ψ will now depend on both the electron's coordinates x_μ and on the field variable ξ.

We shall seek a solution of Eq. (1) with the operator potential (2) in the form

$$\psi=U_1\Phi;\quad U_1=\exp\left\{\frac{i(kx)}{2}\left(\frac{\partial^2}{\partial\xi^2}-\xi^2\right)\right\}. \tag{4}$$

If we substitute this function into Eq. (1), multiply the latter from the left by U_1^+ and take the operator identity

$$\exp\left\{-\frac{i(kx)}{2}\left(\frac{\partial^2}{\partial\xi^2}-\xi^2\right)\right\}\left(\xi\pm\frac{\partial}{\partial\xi}\right)\times\exp\left\{\frac{i(kx)}{2}\left(\frac{\partial^2}{\partial\xi^2}-\xi^2\right)\right\}=e^{\mp i(kx)}\left(\xi\pm\frac{\partial}{\partial\xi}\right); \tag{5}$$

into account, then we shall have the following equation for the unknown function Φ:

$$\left\{\gamma_\mu\left[\frac{\partial}{\partial x_\mu}-ib_\mu\xi+\frac{ik_\mu}{2}\left(\frac{\partial^2}{\partial\xi^2}-\xi^2\right)\right]+m\right\}\Phi=0; \tag{6}$$

where

$$b_\mu=ee_\mu/\sqrt{k_0\Omega}. \tag{7}$$

The transformation from Eq. (1) to Eq. (6) is a canonical transformation. In this connection it is of interest to note that if

$$U_1'=\exp\left\{-\frac{ik_0x_0}{2}\left(\frac{\partial^2}{\partial\xi^2}-\xi^2\right)\right\} \tag{8}$$

is used instead of U_1, and one also changes[4] from the matrices γ_μ to the Dirac matrices α and β, then instead of Eq. (6) we obtain the equation commonly used in quantum electrodynamics in the Hamiltonian formalism.[7] In this connection the total Hamiltonian will consist of the Hamiltonian for a free electron, the Hamiltonian for the

[1)] The following notation is used: $p_\mu = (\mathbf{p}, ip_0)$, $(pq) = \mathbf{p}\cdot\mathbf{q} - p_0q_0$, $\hat{p} = (\gamma p) = \gamma\cdot\mathbf{p} + i\gamma_0 p_0$ together with a system of units in which $e^2/4\pi = 1/137$, $\hbar = c = 1$.

free field of a monochromatic wave, and the time-independent interaction Hamiltonian.

Since Eq. (6) contains the variables x_μ only as derivatives, it is natural to seek its solution in the form

$$\Phi = e^{i(qx)}\varphi(\xi), \tag{9}$$

where q = (q, iq$_0$) is a constant, four-dimensional vector, and ϕ only depends on the variable ξ. As a result of the substitution of (9) into Eq. (6) we obtain

$$\left[\hat{q} - \hat{b}\hat{\xi} + \frac{\hat{k}}{2}\left(\frac{d^2}{d\xi^2} - \xi^2\right) - im\right]\varphi = 0. \tag{10}$$

In this equation it would be desirable to avoid the term containing $\hat{b}$. One can actually achieve this result with the aid of the following substitution:

$$\varphi = U_2 v, \quad U_2 = \exp\{l\hat{k}\hat{b}\xi\} = 1 + l\hat{k}\hat{b}\xi, \tag{11}$$

where for the time being l is an arbitrary constant. One can easily verify the last equation if one takes into account that $\hat{k}\hat{b} + \hat{b}\hat{k} = 2(bk)$, (bk) = 0, and $k^2 = 0$. Having substituted the function (11) into Eq. (10) and having multiplied the latter from the left by $U_2^+ = \exp\{-l\hat{k}\hat{b}\xi\}$ we arrive at the following equation for the function v:

$$\left\{\hat{q} - \hat{b}[1 - 2l(qk)]\xi + \frac{\hat{k}}{2}\left[\frac{d^2}{d\xi^2} - \xi^2 - 4l(qb)\xi - 4b^2l^2(qk)\xi^2 + 4b^2l\xi^2\right] - im\right\}v = 0. \tag{12}$$

If the arbitrary constant l is set equal to

$$l = 1/2(qk), \tag{13}$$

then Eq. (12) takes the form

$$\left\{\hat{q} + \frac{\hat{k}}{2}\left[\frac{d^2}{d\xi^2} - \xi^2 - \frac{2(qb)}{(qk)}\xi + \frac{b^2}{(qk)}\xi^2\right] - im\right\}v = 0. \tag{14}$$

Then we change to the variable

$$\eta = \tau(\xi + \varkappa), \tag{15}$$

where

$$\tau^4 = 1 - b^2/(qk), \quad \varkappa = (qb)/(qk)\tau^4. \tag{16}$$

Then we finally obtain the following equation:

$$\left[\hat{q} + \frac{\hat{k}\tau^2}{2}\left(\frac{d^2}{d\eta^2} - \eta^2 + \varkappa^2\tau^2\right) - im\right]v = 0. \tag{17}$$

Its solution, finite for $\eta \to \infty$, is the product of a harmonic oscillator eigenfunction[8] and a constant bispinor u:

$$v = H_n(\eta)\exp(-\eta^2/2)u, \tag{18}$$

where $H_n(\eta)$ is the Hermite polynomial of order n. The constant bispinor u satisfies the same equation as in the case of a free electron,

$$(\hat{p} - im)u = 0, \tag{19}$$

where

$$p_\mu = q_\mu - k_\mu\tau^2[(n + 1/2) - \varkappa^2\tau^2/2], \tag{20}$$

Having collected together all substitutions (4), (9), (11), and (18), we obtain the result that the wave function which satisfies Eq. (1) with the potential (2), has the form

$$\psi_{qn} = N\exp\left\{i(qx) + \frac{i(kx)}{2}\left(\frac{\partial^2}{\partial\xi^2} - \xi^2\right)\right\} \times \left(1 + \frac{\hat{k}\hat{b}}{2(qk)}\xi\right)H_n(\eta)\exp\left\{-\frac{\eta^2}{2}\right\}u; \tag{21}$$

where N is a normalization factor.

We note that the solution of the Klein-Gordon equation with the potential (2) can be found in similar fashion. This solution coincides with the function (21) if in the latter the round bracket containing γ matrices and the constant bispinor u are set equal to unity.

2. DISCUSSION OF THE OBTAINED SOLUTION AND TRANSFORMATION OF THE WAVE FUNCTION

If the interaction between the electron and the field of a monochromatic wave is not present, i.e., if $b_\mu = 0$, then $\tau = 1$, $\kappa = 0$, the variable η turns into ξ and in place of the wave function (21) we have

$$\psi_{qn} = N\exp\{i(qx) - i(kx)(n + 1/2)\}H_n(\xi)\exp\left\{-\frac{\xi^2}{2}\right\}u. \tag{22}$$

The function (22) is an eigenfunction of the momentum operator $(1/2)k_\mu(\xi^2 - \partial^2/\partial\xi^2)$ for the monochromatic wave and the electron's momentum operator $-i\partial/\partial x_\mu$ with the corresponding eigenvalues $k_\mu(n + 1/2)$ and $p_\mu = q_\mu - k_\mu(n + 1/2)$. From here it is seen that the previously-introduced four-dimensional vector q_μ represents the total momentum of the system. This vector is conserved even in the case when the interaction is present. However, for $b_\mu \neq 0$ neither the electron's momentum nor the momentum of the monochromatic wave is separately conserved, but the "quasimomentum"

$$\bar{k}_\mu = k_\mu\tau^2[(n + 1/2) - \varkappa^2\tau^2/2], \tag{23}$$

of the monochromatic wave is conserved; the "quasimomentum" is an eigenvalue of the operator

$$L = \frac{k_\mu\tau^2}{2}U_1U_2\left(\eta^2 - \frac{\partial^2}{\partial\eta^2} - \varkappa^2\tau^2\right)U_2^+U_1^+ = \frac{k_\mu}{2}\left[\xi^2 - \frac{\partial^2}{\partial\xi^2} + \frac{2e(qA)}{(qk)} - \frac{e^2A^2}{(qk)} - i\frac{e}{2(qk)}\gamma_\nu\gamma_\sigma F_{\nu\sigma}\right],$$

where $F_{\nu\sigma} = (\partial A_\sigma/\partial x_\nu) - (\partial A_\nu/\partial x_\sigma)$ is the electromagnetic field tensor, and the potential A_μ is defined as the operator expression (2). The "quasimomentum" p_μ of the electron, which is equal to $q_\mu - \bar{k}_\mu$, is also conserved. The operator corresponding to it is equal to the difference between the operator for the total momentum of the system and the operator L. If we return to Eq. (19), then it is seen that the same relation holds for the "quasimomentum" of the electron as for the momentum of a free electron:

$$p^2 + m^2 = 0. \tag{24}$$

Now let us note the following property of the functions ψ_{pn}: since (qk) = (pk) and (qe) = (pe) the wave function does not change its form if instead of the total momentum q_μ one substitutes the electron's "quasimomentum" p_μ. In this connection only an additional factor $e^{i(\bar{k}x)}$ appears.

It is easiest of all to consider the orthogonality conditions for the functions ψ_{pn} in a coordinate system whose 3-axis coincides with the direction of propagation of the electromagnetic wave. In this coordinate system $p_1 = q_1$, $p_2 = q_2$, $p_3 - p_0 = q_3 - q_0$, $(pk) = k_0(p_3 - p_0)$, and the product (pb) only depends on the components p_1 and

p_2. In this coordinate system the functions ψ_{pn} are orthogonal in the following sense:

$$\int \psi^+_{p'n'}\psi_{pn}\,dx_1\,dx_2\,dx_3'\,d\xi = N^2\pi^{1/2}2^n n!\tau^{-1}\delta_{n'n}\delta_{p_1'p_1}\delta_{p_2'p_2}\delta_{p_3'-p_0'},\ p_3-p_3, \quad (25)$$

where the variable $x_3' = x_3 + x_0$. Here the orthogonality condition for harmonic oscillator eigenfunctions with identical arguments has been taken into account, the normalization volume has been set equal to unity, and the fact that $u^+u = 1$ has been used. If the factor in front of the Kronecker delta symbols on the right-hand side of Eq. (25) is chosen equal to unity, then the normalization factor is given by

$$N = (\pi^{1/2}2^n n!\tau^{-1})^{-1/2}. \quad (26)$$

Expression (21) for the function ψ_{pn} is not convenient in the following respect: the effect of the operator U_1 on the function standing behind it is not explicitly calculated. But this is not difficult to do. In the first place one can transpose the operator U_1 behind the round bracket standing behind it. If the identity (5) is taken into account, then as a result of such a transposition one should replace the factor $\hat{b}\xi$ in the second term inside the round brackets by $e\hat{A}$, if the operator A_μ is defined according to Eq. (2). In the second place, one can calculate the effect of U_1 on an oscillator wave function of the variable η if the latter is expanded with respect to a system of unperturbed oscillator functions of the variable ξ. As a result

$$\psi_{pn} = Ne^{i(qx)}\left(1 + \frac{e\hat{k}\hat{A}}{2(pk)}\right)uF_n, \quad (27)$$

where

$$F_n = U_1 H_n(\eta)\exp\left\{-\frac{\eta^2}{2}\right\} = \sum_{m=0}^{\infty}\exp\left\{-i(kx)\left(m+\frac{1}{2}\right)\right\}\beta_{m^n}H_m(\xi)\exp\left\{-\frac{\xi^2}{2}\right\}, \quad (28)$$

$$\beta_{m^n} = \frac{1}{\pi^{1/2}2^m m!}\int d\xi\, H_n(\eta)H_m(\xi)\exp\left\{-\frac{\xi^2}{2}-\frac{\eta^2}{2}\right\}. \quad (29)$$

One can evaluate the integral (29) with the aid of a generating function.[8] Comparing expressions (27), (28), and (22), one can conclude that if, in the absence of any interaction between the electron and the quantized wave, the oscillator is found in the state n, then the interaction leads to the result that all states of an unperturbed oscillator are present in the wave function of the system. If one does not pay any attention to the second term in the round brackets of the function (27), which is completely absent for a spinless electron described by the Klein-Gordon equation, then it is clear that the probability of finding the oscillator in a state m is proportional to $|\beta_m^n|^2$.

It is of interest to note that the scalar function F_n may be transformed into a form which is more closed. In order to do this, in formula (28) it is necessary to interchange the order of summation and integration, as a result of which the sum inside the integral sign can be evaluated with the aid of Mehler's formula[9] for Hermite polynomials. Having performed the integration[9] after this, we finally obtain

$$F_n = \left\{\frac{2z[1+z^2-\tau^2(1-z^2)]^n}{[1+z^2+\tau^2(1-z^2)]^{n+1}}\right\}^{1/2} H_n\left(\frac{\tau[\varkappa(1+z^2)+2z\xi]}{\{(1+z^2)^2-\tau^4(1-z^2)^2\}^{1/2}}\right) \times\exp\left\{\frac{\xi^2[z^2-1-\tau^2(1+z^2)]-4\tau^2\varkappa z\xi-\tau^2\varkappa^2(1+z^2)}{2[1+z^2+\tau^2(1-z^2)]}\right\}, \quad (30)$$

where $z = e^{-i(kx)}$.

3. RELATION TO THE VOLKOV SOLUTION

The wave function (21) or (27) and (30) describes a system consisting of an electron and n photons interacting among themselves. Since any linear combination of the functions ψ_{pn} will also be a solution of Eq. (1), then it is possible to find such a wave function which will describe the motion of an electron in the field of a wave characterized by a given distribution with respect to the occupation numbers of the photons. In particular, one can find the wave function describing the motion of an electron in a so-called coherent light wave. In contrast to the completely undetermined phase of a wave consisting of n photons,[7] the phase of a coherent wave is completely determined. In analogy to the formation of a coherent state,[10] let us put together the following linear combination of the functions ψ_{pn}:

$$\psi_{p\bar{n}} = e^{-\bar{n}/2}\sum_{n=0}^{\infty}\frac{\bar{n}^{n/2}}{\sqrt{n!}}e^{i\theta n}\psi_{pn}, \quad (31)$$

where $\bar{n}^{1/2}e^{i\theta}$ is equal to the complex number α introduced by Glauber,[10] $\bar{n}$ is the average number of photons, and θ is the phase of the light wave which we henceforth set equal to zero. If expressions (27), (30), and (20) are taken into account, then with the aid of the generating function it is not difficult to evaluate the sum over n in the wave function (31).

Now let us consider the limit of the function (31) when the average number of photons $\bar{n}$ and the normalization volume Ω tend to infinity, but in such a way that the average number of photons per unit volume, $\bar{n}/\Omega$, remains finite. We still consider that[8,10] a coherent state describes a wave packet with its center at the point $\xi_0 = (2\bar{n})^{1/2}$. Therefore the wave function (31) will differ from zero appreciably only for $\xi \sim (2\bar{n})^{1/2}$ or for $\zeta = \xi - (2\bar{n})^{1/2}$ of the order of unity. If one now carries out the above-indicated summation and the limiting transition, then the wave function (31) takes the form

$$\psi_p = \frac{1}{\pi^{1/4}}\left[1+\frac{e\hat{k}\hat{a}}{2(pk)}\cos(kx)\right]u\cdot\exp\left\{-\frac{\zeta^2}{2}+i(px) - i\frac{e^2a^2}{4(pk)}(kx)+i\frac{e(ap)}{(pk)}\sin(kx)-i\frac{e^2a^2}{8(pk)}\sin 2(kx)\right\}, \quad (32)$$

where the notation $a_\mu = e_\mu(2\bar{n}/k_0\Omega)^{1/2}$ has been introduced.

The function (32) coincides with the Volkov solution[1-3] for a monochromatic wave to within a factor $\pi^{-1/4}\exp\{-\zeta^2/2\}$, representing the eigenfunction of an oscillator for the ground state. Therefore the Volkov solution describes the motion of an electron in a coherent light wave filling all space, with an average energy density $k_0^2a^2/2 = k_0\bar{n}/\Omega$.

In conclusion the author expresses his gratitude to A. Gaĭlitis, M. Gaĭlitis, and Ya. Valdmanis for a discussion of the work and valuable remarks.

[1]D. M. Volkov, Z. Physik 94, 250 (1935); Zh. Eksp. Teor. Fiz. 7, 1286 (1937).
[2]H. R. Reiss and J. H. Eberly, Phys. Rev. 151, 1058 (1966).
[3]F. Ehlotzky, Z. Physik 203, 119 (1967).
[4]A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya élektrodinamika (Quantum Electrodynamics), Fizmatgiz, 1959 (English Transl., Interscience Publishers, 1965).
[5]A. S. Davydov, Kvantovaya mekhanika (Quantum Mechanics), Fizmatgiz, 1963 (English Transl., Pergamon Press, 1965).
[6]E. M. Henley and W. Thirring, Elementary Quantum Field Theory, McGraw-Hill, 1962 (Russ. Transl., IIL, M. 1963).
[7]W. Heitler, The Quantum Theory of Radiation, Oxford, 1953 (Russ. Transl., IIL, M. 1956).
[8]L. I. Schiff, Quantum Mechanics, McGraw-Hill, 1955 (Russ. Transl., IIL, M. 1959).
[9]Higher Transcendental Functions, edited by A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, Vol. II, McGraw-Hill, 1953 (Russ. Transl., Nauka, 1966).
[10]R. J. Glauber, "Optical Coherence and Photon Statistics" in the book Quantum Optics and Quantum Radiophysics (Russ. Transl., Mir, 1966).

Translated by H. H. Nickle
185