INELASTIC SCATTERING OF AN ELECTRON BY A PARAMAGNETIC IMPURITY IN A METAL

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We consider the question of inelastic scattering of an electron in a metal by a paramagnetic impurity. An integral equation is obtained for the processes of the coalescence of an electron-hole pair and an electron (hole) into one particle, the solution of which depends only on the elastic scattering amplitude. This equation is used to investigate the influence of inelastic processes on the unitarity condition for elastic scattering, and it is shown that the role of the inelastic processes reduces to a renormalization of the exchange-interaction constant and of the Kondo energy. In addition, it is shown that the paramagnetic impurities cause a peculiar interaction between the electrons.

1. INTRODUCTION

THE problem of the scattering of an electron in a metal by a paramagnetic impurity was solved by the method proposed by Suhl[1,2], based on the use of the unitarity conditions and the analytic properties of the scattering amplitude (see the papers of Suhl and Wong[3] and of the authors[4,5].

This solution method is apparently based on two assumptions. First, the contribution from the many-particle intermediate states is neglected in the unitarity condition for the scattering amplitude, and second, it is assumed in the determination of the function satisfying the unitarity conditions that there are no Castillejo-Dalitz-Dyson poles (CDD-ambiguity, for details see[3,4]). The second assumption seems to us exceedingly reasonable, but the proof of its validity is possible only within the framework of the solution of the dynamic problem.

The present paper is devoted to a clarification of the question of many-particle intermediate states. We confine ourselves to the case of a pointlike scatterer and zero temperature, so that this paper is a direct continuation of an earlier paper[4], which is henceforth cited as I.

The present paper deals in detail with the question of the role of three-particle states, and it is shown that allowance for these states leads in the main to a renormalization of the exchange-interaction constant, and consequently to a renormalization of the Kondo energy. A qualitative discussion of the role of states with a large number of particles also leads to the conclusion that they can be taken into account with the aid of renormalization.

In addition, it is shown here that impurities with spins give rise to a peculiar interaction between the electrons. The presence of such an interaction was already noted earlier by Solyom and Zawadowski[6]. This interaction should be responsible for the change of the superconducting-transition temperature in the presence of impurities (see the papers of Abrikosov and Gor'kov[7] and Ginzburg[8]), but a detailed discussion of this question is beyond the scope of the present article.

2. UNITARITY CONDITIONS

We shall use the retarded scattering amplitude F, introduced in[5] (henceforth cited as II), for which the following unitarity conditions hold when T = 0 (I, II):

$$i(F^+ - F)_{\alpha'\alpha}^{M'M} = \frac{1}{2}\sum_{nM_1} \langle M'|j_{\alpha'}|nM_1\rangle\langle M_1 n|j_\alpha^+|M\rangle\delta(E_{n0}-E),\quad E > E_F;$$
$$i(F^+ - F)_{\alpha'\alpha}^{M'M} = \frac{1}{2}\sum_{nM_1} \langle M'|j_\alpha^+|nM_1\rangle\langle M_1 n|j_{\alpha'}|M\rangle\delta(E_{0n}-E),\quad E < E_F. \tag{1}$$

In the intermediate states n, we can have here, besides an electron ($E > E_F$) or a hole ($E < E_F$), also any number of electron-hole pairs. The total number of electrons and holes will be called simply the number of particles. Separating the single-particle terms in (1), using the connection obtained in II between the corresponding matrix elements j and the scattering amplitudes, and also recognizing that F = A + BR, where $R = \mathbf{S}\cdot\boldsymbol{\sigma}$, we can represent the unitarity conditions in the form

$$\operatorname{Im} A = k\left\{|A|^2 + |B|^2 S(S+1) + \frac{1}{2k}\Delta_A(\zeta)\right\},$$
$$\operatorname{Im} B = k\left\{AB^* + A^*B - |B|^2\varepsilon(\zeta) + \frac{1}{2k}\Delta_B(\zeta)\right\}, \tag{2}$$

where $\zeta = E - E_F$, and $\Delta_{A,B}(\zeta)$ is determined by the equations

$$(\Delta_A + \Delta_B R)_{\alpha'\alpha}^{M'M} = \frac{1}{2}\sum_{nM_1}\langle M'|j_{\alpha'}|nM_1\rangle\langle M_1 n|j_\alpha^+|M\rangle\delta(E_{n0}-E),\quad E > E_F,$$
$$(\Delta_A + \Delta_B R)_{\alpha'\alpha}^{M'M} = \frac{1}{2}\sum_{nM_1}\langle M'|j_\alpha^+|nM_1\rangle\langle M_1 n|j_{\alpha'}|M\rangle\delta(E_{0n}-E),\quad E < E_F, \tag{3}$$

where now there are not less than three particles in the intermediate state n.

Nontrivial behavior of the scattering amplitudes near the Fermi surface (the Kondo effect) is due to the fact that the jump of the analytic function[1)] B at $E > E_F$ and $E < E_F$ are expressed in different manners in terms of the square of the modulus of the same function. We deal here essentially with the so-called threshold singularity

[1)]We recall that the jump of the function Φ is defined as the quantity $(2i)^{-1}[\Phi(E + i\delta) - \Phi(E - i\delta)]$, which in our case is simply equal to ImΦ.

of the amplitude. Namely, the scattering amplitude is an analytic function of E with a cut along the real axis from $E = 0$ to $E = \infty$, and it describes the scattering of a hole or an electron by the impurity when $E < E_F$ and $E > E_F$, respectively: the unitarity conditions for these processes are different. Therefore a decisive factor in what follows is the extent to which the many-particle states change the form of the unitarity conditions when $E > E_F$ and when $E < E_F$. We shall see below that this change is insignificant.

3. DERIVATION OF THE FUNDAMENTAL EQUATIONS

The matrix elements of the operators j and j^+ which enter in (3) are S-matrix elements corresponding to inelastic scattering processes.

Thus, the matrix element $\langle M'|j_{\alpha'}|nM\rangle$ at $E < E_F$ corresponds to the process of coalescence of particles that are in the state n into a single particle with energy E and spin projection α', while the matrix element $\langle M'n|j_{\alpha'}|M\rangle$ at $E < E_F$ corresponds to the decay of a hole with energy E and spin projection α' into the particles present in the state n. The corresponding processes for the case of three-particle states are shown in Fig. 1. This figure explains readily the notation used below. It is easiest to verify the correctness of such an interpretation by considering, for example, the S-matrix element corresponding to the coalescence of n particles into one particle:

$$\langle M'\mathbf{k}'\alpha'|S|nM\rangle = \langle M'|a_{\mathbf{k}'\alpha'}(t=+\infty)S|nM\rangle$$
$$= \frac{1}{\sqrt{V}}\int d\mathbf{x}\exp\{-i\mathbf{k}'\mathbf{x} + iE_{\mathbf{k}'}t\}\langle M'|\Psi_{\alpha'}(\mathbf{x}, t)S|nM\rangle|_{t=+\infty}$$
$$= -\frac{i}{\sqrt{V}}\int d^4x\exp\{-i\mathbf{k}'\mathbf{x} + iE_{\mathbf{k}'}t\}\left(i\frac{\partial}{\partial t} - H_0\right)\langle M'|T(\Psi_{\alpha'}(x)S)|nM\rangle$$
$$= -\frac{i}{\sqrt{V}}\int dt\exp\{iEt\}\langle M'|j_{\alpha'}(t)|nM\rangle$$
$$= -2\pi i\frac{1}{\sqrt{V}}\delta(E_{n0} - E)\langle M'|j_{\alpha'}|nM\rangle. \qquad (4)$$

The derivation of this formula is analogous to the corresponding derivation for the scattering amplitude contained in I and II. The operator $j_{\alpha'}$ is defined in I and II, and is a Heisenberg operator. Before we proceed to a detailed analysis of the matrix elements contained in (3), we shall show that it is impossible to use perturbation theory for their calculation. In fact, let us consider, for example, the coalescence of three particles into one. In the first nonvanishing order of perturbation theory we have for this process the diagram shown in Fig. 2, where the internal dashed line corresponds to the propagation function of the impurity $g(\omega) = (\omega + i\delta)^{-1}$, and each vertex corresponds to a factor $-4\pi f$, where $f = a + bR$ is the Born amplitude for the scattering by the impurity. In addition, the entire expression must be multiplied by $V^{-3/2}$. As a result we obtain

$$\langle M'|j_{\alpha'}|\bar{k}_2k_1k_3\rangle \approx -\frac{(4\pi)^2}{V^{3/2}}\left(\frac{[f_{\alpha'\mu}, f_{\lambda\nu}]}{E_1 - E - i\delta} - \frac{[f_{\alpha'\nu}, f_{\lambda\mu}]}{E_3 - E - i\delta}\right)$$
$$= -\frac{(4\pi)^2b^2}{V^{3/2}}\left(\frac{[R_{\alpha'\mu}, R_{\lambda\nu}]}{E_1 - E - i\delta} - \frac{[R_{\alpha'\nu}, R_{\lambda\mu}]}{E_3 - E - i\delta}\right). \qquad (5)$$

Here and below the bar over the momentum $\bar{k}_2$ denotes that we are dealing with a hole. The main feature of this expression is the presence of poles at $E = E_1$ and $E = E_3$.

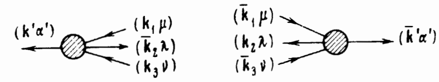

Fig. 1

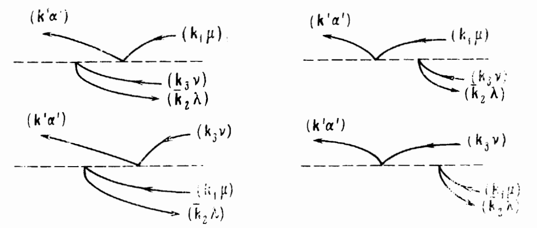

Fig. 2

Because of this, substitution of (5) into (3) leads to the diverging integral $\int_0^{\zeta} d\zeta_1(\zeta - \zeta_1)^{-1}$.

Let us consider in greater detail the matrix element

$$\langle M'|j_\alpha|\bar{k}_2k_1k_3M\rangle = \langle M'|j_\alpha a_{\bar{k}_2}(t = -\infty)|k_1k_3M\rangle$$
$$= \frac{i}{\sqrt{V}}\int dt e^{iE_2t}\vartheta(-t)\langle M'|\{j_\alpha(0), j_\lambda(t)\}|k_1k_3M\rangle$$
$$= \frac{1}{\sqrt{V}}\sum_n\left\{\frac{\langle M'|j_\alpha|n\rangle\langle n|j_\lambda|k_1k_3M_1\rangle}{E_2 - E_1 - E_3 + E_n - i\delta} + \frac{\langle M'|j_\lambda|n\rangle\langle n|j_\alpha|k_1k_3M_1\rangle}{E_2 - E_n - i\delta}\right\}. \qquad (6)$$

In the derivation of this formula we have used, first, a procedure analogous to that used in the derivation of (4), the only difference being that in lieu of the T-product we introduce the retarded anticommutator, in analogy with the procedure used, for example, in Appendix I of II, and, second, we expanded with respect to the intermediate states of the free particles (the validity of this approach is also demonstrated in Appendix I of II). We assume here that the states n pertain to the instant of time $t = -\infty$.

We confine ourselves here below only to single-particle intermediate states. We have here

$$\langle M_2q|j_\lambda|k_1k_3M\rangle = \frac{4\pi}{\sqrt{V}}\{F_{\lambda\nu}^{M_2M_1}(E_3)\delta_{qk_1} - F_{\lambda\mu}^{M_2M_1}(E_1)\delta_{qk_3}\}$$
$$+ \tau_{\mu\nu M_2M_1}^{\lambda\sigma}(E_{2q}, E_1, E_3, E_q), \qquad (7)$$

where σ is the component of the particle spin with momentum q and

$$\tau_{\mu\nu M_2M_1}^{\lambda\sigma}(E_{2q}, E_1, E_3, E_q) = \langle M_2|j_\lambda|\bar{q}, k_1, k_3\rangle.$$

with $\bar{q} \neq k_1$, k_3 and $E_q + E_{2q} = E_1 + E_3$. The equality of τ and the corresponding matrix element j can be demonstrated with the aid of a procedure analogous to that used in the derivation of (4). Terms with δ-symbols are obtained with the aid of a similar procedure; their appearance can be readily understood by taking into account the possibility that q in the left side of (7) may be equal to k_2 or k_2. A single-particle contribution to the right side of (6) can also be made by the three-particle intermediate states, if the momenta of two particles in these states coincide with k_1 and k_3. As a result we obtain for τ the equation

$$\tau_{\mu\nu}^{\alpha\lambda}(E, E_1, E_3, E_2) = -\frac{(4\pi)^2}{V^{3/2}}\left\{\frac{[F_{\alpha\mu}(E_1), F_{\lambda\nu}(E_3)]}{E_2 - E_3 - i\delta} - \right.$$

$$-\frac{[F_{\alpha\nu}(E_1),F_{\lambda\mu}(E_3)]}{E_2-E_1-i\delta}\Big\}+\frac{4\pi}{V}\sum_{q>k_F}\Big\{\frac{F_{\alpha\sigma}(E')\tau^{\lambda\sigma}_{\mu\nu}(E_2',E_1,E_3,E')}{E'-E-i\delta}$$

$$-\frac{F_{\lambda\sigma}(E')\tau^{\alpha\sigma}_{\mu\nu}(E_2',E_1,E_3,E')}{E'-E_2+i\delta}\Big\}-\frac{4\pi}{V}\sum_{q<k_F}\Big\{\frac{\tau^{\alpha\sigma}_{\mu\nu}(E_2',E_1,E_3,E')F_{\lambda\sigma}(E')}{E'-E_2+i\delta}$$

$$-\frac{\tau^{\lambda\sigma}_{\mu\nu}(E_2',E_1,E_3,E')F_{\alpha\sigma}(E')}{E'-E-i\delta}\Big\}, \tag{8}$$

where $E + E_2 = E_1 + E_3$, $E' = E_q$, and we have omitted the projections of the spin of the impurity, since they follow in natural order. From this formula it follows, in particular, that

$$\tau^{\alpha\lambda}_{\mu\nu}(E+i\delta,E_1,E_3,E_2-i\delta)=-\tau^{\lambda\alpha}_{\mu\nu}(E_2-i\delta,E_1,E_3,E+i\delta). \tag{9}$$

Taking this equation into account, and also the fact that $F = A + BR$, we can readily show that τ can be represented in the form

$$\tau^{\alpha\lambda}_{\mu\nu}(E,E_1,E_3,E_2)=-\frac{(4\pi)^2}{V^{3/2}}B(E_1)B(E_3)f^{\alpha\lambda}_{\mu\nu}(E+i\delta). \tag{10}$$

We then have for the function f the equation

$$f^{\alpha\lambda}_{\mu\nu}(E+i\delta)=\frac{[R_{\alpha\mu},R_{\lambda\nu}]}{E_1-E-i\delta}-\frac{[R_{\alpha\nu},R_{\lambda\mu}]}{E_3-E-i\delta}+\frac{1}{\pi}\int_0^\infty\frac{dE'\,k'}{E'-E-i\delta}$$

$$\times\{F_{\alpha\sigma}(E')f^{\sigma\lambda}_{\mu\nu}(E'-i\delta)\vartheta_+(E')+f^{\sigma\lambda}_{\mu\nu}(E'-i\delta)F_{\alpha\sigma}(E')\vartheta_-(E')\}$$

$$-\frac{1}{\pi}\int_0^\infty\frac{dE'\,k'}{E'-E_2+i\delta}\{F_{\lambda\sigma}(E')f^{\sigma\alpha}_{\mu\nu}(E'-i\delta)\vartheta_+(E')$$

$$+f^{\sigma\alpha}_{\mu\nu}(E'-i\delta)F_{\lambda\sigma}(E')\vartheta_-(E')\}, \tag{11}$$

where $\vartheta_+(E) = 1$ when $E > E_F$, $\vartheta_+(E) = 0$ when $E < E_F$, and $\vartheta_+(E) + \vartheta_-(E) = 1$.

Using a procedure perfectly analogous to that described above, we can obtain an equation for the matrix element $\langle M'\bar{k}_1\bar{k}_3k_2|j_\alpha|M\rangle$, describing the decay of a hole. It turns out here that

$$\langle M'\bar{k}_1\bar{k}_3k_2|j_\alpha|M\rangle=\tau^{\alpha\lambda}_{\mu\nu}(E,E_1,E_3,E_2),$$

where now $E_{1,3} < E_F$ and $E_2 > E_F$. But since E_1 and E_3 enter in (8) as parameters, while E and E_2 can be arbitrary, these matrix elements need not be investigated separately.

As already mentioned in the Introduction, the paramagnetic impurities lead to the appearance of a certain additional interaction between the electrons. It is perfectly clear that the amplitude of the electron-electron scattering, which is shown graphically in Fig. 3, should be directly connected with the amplitude for the coalescence of three particles into one, which we are considering.

Inasmuch as in this case $E_2 > E_F$, it is natural to expect the scattering amplitudes to be obtained from the coalescence amplitude, by replacing in the latter $E_2 - i\delta$ by $E_2 + i\delta$. That this is actually the case can be readily verified by confirming the matrix element of the S-matrix corresponding to scattering, and by using a procedure similar to that employed above. As a result we obtain

$$\langle M'kk_2|S|k_1k_3M\rangle=\frac{2\pi i}{\sqrt{V}}\delta(E+E_2-E_1-E_3)\tau^{\alpha\lambda}_{\mu\nu}(E+i\delta,E_1,E_3,E_2+i\delta)$$

$$\tau^{\alpha\lambda}_{\mu\nu}(E+i\delta,E_1,E_3,E_2+i\delta)=\tau^{\alpha\lambda}_{\mu\nu}(E+i\delta,E_1,E_3,E_2-i\delta)$$

$$+2ik_2F_{\lambda\sigma}(E_2)\tau^{\alpha\sigma}_{\mu\nu}(E+i\delta,E_1,E_3,E_2-i\delta)$$

$$-2\pi i\frac{(4\pi)^2}{V^{3/2}}\{\delta(E_1-E)F_{\lambda\nu}(E_3)F_{\alpha\mu}(E_1)-\delta(E_3-E)F_{\lambda\mu}(E_1)F_{\alpha\nu}(E_3)\} \tag{12}$$

for the scattering of two electrons and

$$\langle\bar{k}_3\bar{k}_1|S|\bar{k}\bar{k}_2\rangle=\frac{2\pi i}{\sqrt{V}}\delta(E_1+E_3-E-E_2)\tau^{\alpha\lambda}_{\mu\nu}(E-i\delta,E_1,E_3,E_2-i\delta),$$

$$\tau^{\alpha\lambda}_{\mu\nu}(E-i\delta,E_1,E_3,E_2-i\delta)=\tau^{\alpha\lambda}_{\mu\nu}(E+i\delta,E_1,E_3,E_2-i\delta)$$

$$+2ik\tau^{\lambda\sigma}_{\mu\nu}(E_2+i\delta,E_1,E_3,E-i\delta)F_{\alpha\sigma}(E)$$

$$-2\pi i\frac{(4\pi)^2}{V^{3/2}}[\delta(E_1-E)F_{\lambda\nu}(E_3)F_{\alpha\mu}(E_1)-F_{\lambda\mu}(E_1)F_{\alpha\nu}(E_3)\delta(E_3-E)] \tag{13}$$

for the scattering of two holes. In these expressions, the terms containing δ-functions describe the scattering of particles by impurities, not accompanied by a redistribution of the energy along the particles. Such terms are present also in pure potential scattering, when there are no inelastic processes.

In conclusion, let us consider briefly the question of a five-particle amplitude. As will be shown later, the character of the contribution of the three-particle states to the unitarity condition is determined by the pole term in τ. The same should take place also for five-particle terms. Leaving out the corresponding cumbersome although straightforward calculations, we present directly an expression for the pole term of the amplitude of coalescence of five particles into one, shown graphically in Fig. 4:

$$\langle M'|j_\alpha|\bar{k}_2\bar{k}_4k_1k_3k_5M\rangle=\frac{(4\pi)^3}{V^{5/2}}B(E_1)B(E_3)B(E_5)$$

$$\times\Big\{\frac{[[R_{\alpha\mu_1},R_{\lambda_2\mu_3}],R_{\lambda_4\mu_5}]}{(E_1-E-i\delta)(E_5-E_4)}+\ldots\Big\}, \tag{14}$$

where the dots denote terms obtained for the first term by all possible permutations of the "incoming" (k_1, k_3, k_5) and "outgoing" (k, k_2, k_4) particles, and the sign of each term is determined as the product of the parities of these two permutations.

4. ANALYSIS OF THREE-PARTICLE AMPLITUDE

We shall consider below only the case of "antiferromagnetic" interactions ($b < 0$), since in "ferromagnetic" interactions ($b > 0$) the problem was solved by Abrikosov[10] with the aid of a graphic technique, and needs no proof. All the results obtained below can be readily rewritten for $b > 0$; it turns out here that the three-particle state makes a negligibly small contribution to the unitarity condition. Before we investigate the three-particle amplitude τ, we recall several properties of the scattering amplitude $F = A + BR$, obtained earlier in the single-particle approximation (see I, II, and[11]); we shall henceforth denote the amplitude in this approximation by F_0). As $\zeta \to 0$ we have

$$A_0\approx\frac{i}{k_F}\Big(1-\frac{\pi^2S(S+1)}{\ln^2(\varepsilon_0/|\zeta|)}\Big),\quad B_0\approx-\frac{\pi}{2k_F\ln(\varepsilon_0/|\zeta|)}. \tag{15}$$

Here $\epsilon_0 = E_F\exp(-1/|g|)$ is the Kondo energy, and $g \approx 2k_Fb/\pi$.

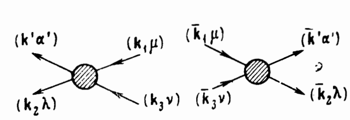

Fig. 3

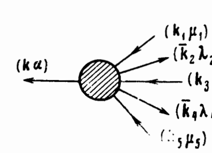

Fig. 4

These formulas are valid when $\ln(\epsilon_0/|\zeta|) \gg 1$. In addition, in the expression for A_0 we have neglected the small term $-a$, which is due to the nonexchange interaction. We shall need later an expression for $|B_0|^2$. It can be readily obtained from the formulas of I and is of the form

$$|B_0|^2 \approx \left(\frac{\pi}{2k_F}\right)^2\left[\left(\ln\frac{\varepsilon_0}{|\zeta|}\right)^2+\frac{\pi^2}{4}(2S+1)^2\right]^{-1}. \tag{16}$$

This formula is valid when $|\zeta| \ll E_F$. It follows from it that $|B_0| \leq [k_F(2S+1)]^{-1}$.

We now proceed to analyze Eq. (11). We seek its solution in the form

$$f(\zeta+i\delta)=\frac{P(\zeta+i\delta)}{\zeta_1-\zeta-i\delta}+\frac{Q(\zeta+i\delta)}{\zeta_3-\zeta-i\delta}. \tag{17}$$

Substituting this expression in (16), we obtain

$$P(\zeta+i\delta)=[R_1,R_2]+\frac{1}{\pi}\int_{-E_F}^{\infty}d\zeta'k'(F(\zeta'),P(\zeta'-i\delta))_1$$
$$\times\left(\frac{1}{\zeta'-\zeta-i\delta}-\frac{1}{\zeta'-\zeta_1-i\delta}\right)+\frac{1}{\pi}\int_{-E_F}^{\infty}d\zeta'k'(F(\zeta'),Q(\zeta'-i\delta))_2$$
$$\times\left(\frac{1}{\zeta'-\zeta_2-i\delta}-\frac{1}{\zeta'-\zeta_3-i\delta}\right)-2ik_2(F(\zeta_2),Q(\zeta_2-i\delta))_2,$$
$$Q(\zeta+i\delta)=-\overline{[R_1,R_2]}+\frac{1}{\pi}\int_{-E_F}^{\infty}d\zeta'k'(F(\zeta'),Q(\zeta'-i\delta))_1$$
$$\times\left(\frac{1}{\zeta'-\zeta-i\delta}-\frac{1}{\zeta'-\zeta_3-i\delta}\right)+\frac{1}{\pi}\int_{-E_F}^{\infty}d\zeta'k'(F(\zeta'),P(\zeta'-i\delta))_2$$
$$\times\left(\frac{1}{\zeta'-\zeta_2-i\delta}-\frac{1}{\zeta'-\zeta_1-i\delta}\right)-2ik_2(F(\zeta_2),P(\zeta_2-i\delta))_2. \tag{18}$$

where we have introduced the notation $R_1 = R_{\alpha\mu}$, $R_2 = R_{\lambda\nu}$, the bar over the product R_1R_2 denotes permutation of the indices μ and ν, and

$$(F,Z)_1=F_{\alpha\sigma}(\zeta)Z_{\mu\nu}^{\sigma\lambda}(\zeta)\vartheta_+(\zeta)+Z_{\mu\nu}^{\sigma\lambda}(\zeta)F_{\alpha\sigma}(\zeta)\vartheta_-(\zeta),$$
$$(F,Z)_2=F_{\lambda\sigma}(\zeta)Z_{\mu\nu}^{\sigma\alpha}(\zeta)\vartheta_+(\zeta)+Z_{\mu\nu}^{\sigma\alpha}(\zeta)F_{\lambda\sigma}(\zeta)\vartheta_-(\zeta). \tag{19}$$

It is easy to verify with the aid of (18) and (19) that the quantities P and Q have the following properties:

$$P_{\nu\mu}^{\alpha\lambda}(\zeta,\zeta_3,\zeta_1,\zeta_2)=-Q_{\mu\nu}^{\alpha\lambda}(\zeta,\zeta_1,\zeta_3,\zeta_2). \tag{20}$$

In addition, it follows from (9) and (18) that

$$\frac{P_{\mu\nu}^{\alpha\lambda}(\zeta+i\delta,\zeta_1,\zeta_3,\zeta_2-i\delta)}{\zeta_1-\zeta-i\delta}+\frac{Q_{\mu\nu}^{\alpha\lambda}(\zeta+i\delta,\zeta_1,\zeta_3,\zeta_2-i\delta)}{\zeta_3-\zeta-i\delta}$$
$$=\frac{P_{\mu\nu}^{\lambda\alpha}(\zeta_2-i\delta,\zeta_1,\zeta_3,\zeta+i\delta)}{\zeta_3-\zeta-i\delta}+\frac{Q_{\mu\nu}^{\lambda\alpha}(\zeta_2-i\delta,\zeta_1,\zeta_3,\zeta+i\delta)}{\zeta_1-\zeta-i\delta} \tag{21}$$

We now consider the question of the behavior of the quantities P and Q at the pole, i.e., when ζ_1 or ζ_3 tends to ζ. We assume here that as $\zeta \to 0$ the amplitudes A and B behave in the same manner as the amplitudes A_0 and B_0.

We shall show below that this is indeed the case. Let, for example, $\zeta_1 \to \zeta$; then by virtue of the energy conservation law $\zeta \to 0$ and $\zeta_3 \to 0$, and we obtain from (18)

$$P_{\mu\nu}^{\alpha\lambda}(\zeta+i\delta,\zeta,0,0)=[R_{\alpha\mu},R_{\lambda\nu}]+2Q_{\mu\nu}^{\lambda\alpha}(0,\zeta,0,\zeta+i\delta). \tag{22}$$

On the other hand, it follows from (21) that $P_{\mu\nu}^{\alpha\lambda}(\zeta,\zeta,0,0) = Q_{\mu\nu}^{\lambda\alpha}(0,\zeta,0,\zeta)$, and therefore $P_{\mu\nu}^{\alpha\lambda}(\zeta,\zeta,0,0) = -[R_{\alpha\mu},R_{\lambda\nu}]$. We similarly obtain $Q_{\mu\nu}^{\alpha\lambda}(\zeta,0,\zeta,0) = [R_{\alpha\nu},R_{\lambda\mu}]$, and consequently the residues of the function f at its poles differ only in sign from the residues of the pole term in the equation that determines f.

We now examine the question of the contribution of the pole terms of the amplitude τ to the unitarity condition. Taking (8) into account, we obtain after simple calculations

$$\Delta_A^{(p)}(\zeta)=\frac{8S(S+1)}{\pi^2}k_F{}^3\int_0^{\zeta}d\zeta_1\int_0^{\zeta-\zeta_1}d\zeta_3|B_1B_3|^2(\zeta-\zeta_1)^{-2},$$
$$\Delta_B^{(p)}(\zeta)=-\frac{4k_F{}^3}{\pi^2}\int_0^{\zeta}d\zeta_1\int_0^{\zeta-\zeta_1}d\zeta_3|B_1B_3|^2[(\zeta-\zeta_1)^{-2}$$
$$-(\zeta-\zeta_1)^{-1}(\zeta-\zeta_3)^{-1}]\,\varepsilon(\zeta), \tag{23}$$

where $B_i = B(\zeta_i)$ and $|\zeta| \ll E_F$. Let us ascertain now the structure of $\Delta_{A,B}^{(p)}$ if we substitute in them B_0 in place of B (see (17)). Integrating by parts, we readily obtain the formula

$$\Delta_A^{(p,0)}\approx-2S(S+1)\Delta_B^{(p,0)}\varepsilon(\zeta)\approx2S(S+1)k_F|B_0|^2\frac{1}{d}\left(\frac{\pi}{2}-\operatorname{arctg}\frac{L}{d}\right), \tag{24}$$

where $L = \ln(\epsilon_0/|\zeta|)$ and $b = (1/2)\pi(2S+1) > 1$. In the derivation of this equation we have neglected the terms that are of order $|L|^{-4}$ when $|L| \gg 1$, and of order d^{-4} when $|L| \lesssim 1$. When $L \gg 1$ we have $\Delta_{A,B}^{(p,0)} \sim L^{-3}$, and substitution of (24) in (2) leads to unitarity conditions that practically coincide with the single-particle conditions. When $-L \gg 1$ we have

$$\Delta_A^{(p,0)}\approx2\pi S(S+1)\frac{k_F}{d}|B_0|^2,$$

and in this case the unitarity conditions differ strongly from the single-particle conditions. This is connected with the fact that at large values of ζ ($E_F \gg |\zeta| \gg \epsilon_0$) and when $\zeta_1 \sim \zeta$ and $\zeta_3 \sim \epsilon_0$ the amplitude of the coalescence of three particles into one is anomalously large (of the order of $bk_F^{-1}(\zeta-\zeta_1)^{-1}$), and therefore the contribution made to the unitarity condition by the three-particle state turns out to be of the order of b^2, and not b^4 (at such values of ζ we have $|B_0|^2 \approx b^2$). It can be shown, by using the exact formulas for B_0 (cf. I) that the three-particle contribution becomes of the order of b^4 only when $|\zeta| > E_F$; this is the real condition for the applicability of perturbation theory to the Kondo effect. It will be shown below that the allowance for the three-particle states reduces in the main to a renormalization of the Kondo energy, i.e., to replacement of ϵ_0 by $\epsilon_1 > \epsilon_0$.

If this is so, then we can replace, with logarithmic accuracy, $|B_1|^2$ in the expression for $\Delta_A^{(p)}$, by $|B(\zeta)|^2$ and $|B_3|^2$ by $|B(\zeta-\zeta_1)|^2$, since the principal role under the integral sign is played by $\zeta_1 - \zeta$ and $\zeta_3 \sim \zeta - \zeta_1$. With the same accuracy, we can neglect the second term in the expression for $\Delta_B^{(p)}$. As a result, the unitarity conditions assume the following form:

$$\operatorname{Im}A=k\{|A|^2+|B|^2S(S+1)(1+2\gamma)\},$$
$$\operatorname{Im}B=k\{A^*B+AB^*-|B|^2(1+\gamma)\varepsilon(\zeta)\},$$
$$\gamma(\zeta)=\frac{1}{2}\left(\frac{2k_F}{\pi}\right)^2\int_0^{\zeta}\frac{d\zeta_1}{\zeta-\zeta_1}|B(\zeta-\zeta_1)|^2. \tag{25}$$

These formulas are valid in the region $|L_1| = |\ln(\epsilon_1/|\zeta|)| \gg 1$. When $|L_1| \lesssim 1$, the functions $\Delta^{(p)}_{A,B}$ differ from the approximate values used in the derivation of (25) by terms of order d^{-4}. As will be shown later, the concrete form of the functions $\Delta_{A,B}$ when $|\zeta| \sim \epsilon_1$ is immaterial for our purposes.

We shall seek the functions A and B satisfying the conditions (25) by a method analogous to that used in I. We introduce the function $u = B^{-1}(1 + 2ikA)$. By virtue of the unitarity condition for B, we have

$$u(\zeta + i\delta) - u(\zeta - i\delta) = 2ik\varepsilon(\zeta)[1 + \gamma(\zeta)]. \tag{26}$$

Reconstructing $u(\zeta)$ with the aid of the dispersion integral, we obtain

$$u(\zeta) = u_0(\zeta) + \frac{k_F}{\pi}\int_{-E_F}^{E_F} \frac{d\zeta'\varepsilon(\zeta')\gamma(\zeta')}{\zeta' - \zeta - i\delta},$$

$$u_0(\zeta) = \frac{2k_F}{\pi}\left(\frac{1}{g} - \ln\frac{\zeta}{E_F} + i\frac{\pi}{2}\right). \tag{27}$$

Here $u_0(\zeta)$ is a function of u, calculated in the single-particle approximation (see I). The region of integration with respect to ζ' in (27) is bounded from above by the quantity E_F, for when $\zeta' > E_F$ the function $\gamma(\zeta')$ should decrease rapidly (perturbation theory becomes applicable).

We assume that the influence of the three-particle state should reduce in the main to a renormalization of the Kondo energy. If this is so, then when $L_1 = \ln(\epsilon_1/|\zeta|) \gg 1$ the function γ should have the form $\gamma_0 L_1^{-1}$, and when $-L_1 \gg 1$ we have $\gamma(\zeta) \approx \gamma_1$, where γ_0 and γ_1 are constants and therefore $u(\zeta)$ can be represented in the form

$$u(\zeta) = \frac{2k_F}{\pi}\left\{\ln\frac{\varepsilon_1}{\zeta} + \frac{i\pi}{2} + v(\zeta)\right\},$$

$$v(\zeta) = \begin{cases} \gamma_1 \ln\dfrac{\varepsilon_1}{\zeta}, & |\zeta| \gg \varepsilon_1 \\ \gamma_0 \ln\ln\dfrac{\varepsilon_1}{\zeta}, & \ln\dfrac{\varepsilon_1}{|\zeta|} \gg 1 \end{cases}, \tag{28}$$

where

$$\varepsilon_1 = E_F \exp\{-[|g|(1 + \gamma_1)]^{-1}\},$$

and, in addition, when $\zeta \sim \epsilon_0$ we have $v(\zeta) \sim 1$.

We now introduce the amplitudes $\alpha_\pm = A \pm B(S + 1/2 \pm 1/2)$. We seek them in the form

$$\alpha_\pm = \frac{1}{2ik}(S_\pm - 1), \quad S_\pm = \exp(2i\delta_\pm). \tag{29}$$

Using the definition of u, we obtain

$$\frac{S_+(\zeta)}{S_-(\zeta)} = \frac{u + 2ikS}{u - 2ik(S + 1)}. \tag{30}$$

From (28) and (30) it follows, in particular, that $S_+(0) + S_-(0)$, and therefore $\delta_+(0) - \delta_-(0) = m\pi$, where m is an integer. It is shown in I that in the single particle approximation $\delta_\pm \approx \pm\pi/2$ and $m = -1$. We represent the phases $\delta_\pm$ in the form

$$\delta_\pm = \nu_\pm + \varphi_\pm,$$

$$\varphi_\pm = -\frac{k}{4\pi}\int_{-E_F}^{\infty} \frac{d\zeta' \ln \eta_\pm^2}{k'(\zeta' - \zeta - i\delta)} = -\frac{i}{2}\ln\eta_\pm - \frac{k}{4\pi}\int_{-E_F}^{\infty}\frac{d\zeta' \ln\eta_\pm^2}{k'(\zeta' - \zeta)}, \tag{31}$$

where $k = \sqrt{E_F + \zeta} \approx k_F$, and the phases $\nu_\pm$ are real.

The integral representing $\varphi_\pm$ has in I limits from $-E_F$ to zero. In our case it is necessary to use the limits $-E_F$ and infinity, since the expression for Im A contains terms describing inelastic scattering. With the aid of (25) we get

$$(S + 1)\eta_+^2 + S\eta_-^2 = (2S + 1)[1 - 8k^2|B|^2\gamma S(S + 1)]. \tag{32}$$

From this, taking (30) into account, we can readily determine $\eta_\pm^2$:

$$\eta_\pm^2 = [1 - 8k^2|B|^2\gamma S(S + 1)]\{1 \pm 4(S + 1/2 \mp 1/2)k^2 D^{-1}[\varepsilon(\zeta)(1 + \gamma) - 1]\},$$

$$D = |u|^2 + 4S(S + 1)k^2 = u_1^2 + k^2[4S(S + 1) + (1 + \gamma)^2], \tag{33}$$

where $u_1 = \mathrm{Re}\, u$. These formulas contain the unknown functions $|B|^2$ and γ.

Using (29), (30), (33), and the formula $B = (2S + 1)^{-1}(\alpha_+ - \alpha_-)$, we get an expression relating these functions:

$$|B|^2 = [D + 8k^2\gamma S(S + 1)]^{-1}. \tag{34}$$

Substituting (34) in the expression for γ, taking (28) into account, and introducing a new integration variable $L_1' = \ln(\epsilon_1/|\zeta - \zeta_1|)$, we obtain

$$\gamma(L_1) = \frac{1}{2}\int_{L_1}^{\infty} dL_1'\Big\{(L_1' + v_1(L_1'))^2 + \frac{\pi^2}{4}[\gamma^2(L_1') + 2(2S + 1)^2\gamma(L_1') + (2S + 1)^2]\Big\}^{-1}, \tag{35}$$

where $v_1 = \mathrm{Re}\, v$, from which it follows that

$$\frac{d\gamma}{dL_1} = -\frac{1}{2}\Big\{(L_1 + v_1(L_1))^2 + \frac{\pi^2}{4}[\gamma^2(L_1) + 2(2S + 1)^2\gamma(L_1) + (2S + 1)^2]\Big\}^{-1} \tag{36}$$

and the boundary condition $\gamma(\infty) = 0$. By virtue of (36), γ decreases monotonically with increasing L_1, with

$$\gamma(L_1) = (2L_1)^{-1}, \quad L_1 \to +\infty;$$

$$\gamma(L_1) = \gamma_1 + \frac{1}{2L_1}, \quad L_1 \to -\infty. \tag{37}$$

These equations justify the assumptions made above concerning the properties of the function γ.

We now determine γ_1. From (35) we have

$$\gamma_1 = \frac{1}{2}\int_{-\infty}^{\infty} dL_1\Big\{(L_1 + v_1)^2 + \frac{\pi^2}{4}[(2S + 1)^2 + 2(2S + 1)^2\gamma + \gamma^2]\Big\}^{-1}. \tag{38}$$

The second term in the denominator of the integrand can be assumed large compared with unity, and therefore the main contribution to the integral is made by the region $|L_1| \gg 1$, in which we can use for v_1 and γ the asymptotic formulas (28) and (37). As a result we obtain for γ_1 an equation whose solution can be readily obtained in the form of a series in powers of $(2S + 1)^{-1}$:

$$\gamma_1 = (2S + 1)^{-1}\left(1 - \frac{1}{2(2S + 1)} + \frac{3}{4(2S + 1)^2} - \ldots\right). \tag{39}$$

The obtained formulas make it possible to write the solution of our problem in the regions $|\zeta| \ll \epsilon_1$ and $|\zeta| \gg \epsilon_1$, or more accurately, when $|L_1| \gg 1$.

Taking (33) into account, the phases $\varphi_\pm$ can be represented in the form

$$\varphi_\pm = \varphi_\pm^{(1)} + \varphi_\pm^{(2)},$$

$$\varphi_\pm^{(1)} \approx -\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{d\zeta'}{\zeta' - \zeta - i\delta}\Big\{\ln[1 - 8k_F^2|B|^2\gamma S(S + 1)] + \ln\Big[1 \pm 4\Big(S + \frac{1}{2} \mp \frac{1}{2}\Big)k_F^2\gamma D^{-1}\Big]\Big\},$$

$$\varphi_{\pm}^{(2)} \approx -\frac{1}{4\pi}\int_{-E_F}^{0}\frac{d\zeta'}{\zeta'-\zeta-i\delta}\ln\frac{D\mp 4(S+{}^1/_2\mp{}^1/_2)(2+\gamma)k_F^2}{D\pm 4(S+{}^1/_2\mp{}^1/_2)k_F^2\gamma}. \quad (40)$$

These formulas are valid only when $|\zeta| \ll E_F$, and therefore, just as in I, we have replaced k by k_F throughout. In addition, in the expression for $\varphi_{\pm}^{(1)}$ we replace the lower limit by $-\infty$, since the integral converges well. In the formula for $\varphi_{\pm}^{(2)}$ this cannot be done for the same reason as in the expression for $\varphi_{\pm}$ in I.

Further, $\varphi_{\pm}^{(1)}(0) = 0$, for in this case the corresponding integrand is an odd function of ζ'.

Using the limiting expressions for γ, $|B|^2$, and D, we can readily obtain an asymptotic expression for $\varphi_{\pm}^{(1)}(\zeta)$ in the region $L_1 \gg 1$. To this end it suffices to determine the imaginary part of $\varphi_{\pm}^{(1)}$, and then reconstruct its real part by using the fact that $\varphi_{\pm}^{(1)}$ is an analytic function of ζ and Re $\varphi_{\pm}^{(1)}$ is odd; such a reconstruction is possible accurate to the constant $\varphi_{\pm}^{(1)}(0)$, which, as we know, equals zero. As a result, taking (28), (33), (34), and (37) into account, we obtain

$$\varphi_{\pm}^{(1)}(\zeta) \approx \frac{i\pi^2}{8}(2S+1)\left(S+\frac{1}{2}\mp\frac{1}{2}\right)\left(\ln\frac{\varepsilon_1}{\zeta}+\frac{i\pi}{2}\right)^{-3}$$
$$= \frac{i\pi^2}{8}(2S+1)\left(S+\frac{1}{2}\mp\frac{1}{2}\right)L_1^{-3}\left(1-\frac{3\pi i}{2L_1}\varepsilon(\zeta)\right). \quad (41)$$

The last part of this equation is valid only on the real axis (we define $\ln(\epsilon_1/\zeta)$ in such a way that when $\zeta > 0$ it is real). In the region $-L_1 \gg 1$, the functions $\varphi_{\pm}^{(1)}$ decrease like L_1^{-2}.

In analogous fashion, we obtain expressions for the phases $\varphi_{\pm}^{(2)}$:

$$\varphi_{\pm}^{(2)}(\zeta) = \varphi_{\pm}^{(2)}(0) \pm \frac{\pi(S+{}^1/_2\mp{}^1/_2)}{2\ln(\varepsilon_1/\zeta)}$$
$$\approx \varphi_{\pm}^{(2)}(0) \pm \frac{\pi}{2}\left(S+\frac{1}{2}\mp\frac{1}{2}\right)(L_1^{-1}+i\pi L_1^{-2}\vartheta(-\zeta)),$$

$$\varphi_{\pm}^{(2)}(0) \approx -\frac{1}{4\pi}\int_{-\infty}^{\infty}dL_1\ln\frac{(L_1+v_1(L_1))^2+{}^1/_4\pi^2[2S+1\mp(2+\gamma)]^2}{(L_1+v_1(L_1))^2+{}^1/_4\pi^2(2S+1\pm\gamma)^2} \quad (42)$$

In the derivation of these formulas we replaced the lower limit of integration in (42) by $-\infty$. Just as in I, it is necessary in this case to introduce a correction term, which together with $\nu_{\pm}$ gives a general phase $\nu \approx k_F a$.

The integrals (40) can be calculated in the same manner as the integral in (38). As a result we obtain $\varphi_{\pm}^{(2)}(0) \approx \pm\pi/2$, i.e., when the three-particle states are taken into account at $\zeta = 0$ the phases reach the maximum possible value from the point of view of the unitarity condition. Moreover, the asymptotic form of the phases (40) coincides exactly with the corresponding asymptotic form in I.

Thus, three-particle states essentially lead only to a renormalization of the constant that characterizes the exchange interaction, and a renormalization of the Kondo energy. Such a renormalization was obtained earlier by Appelbaum and Kondo[12]. Further, inasmuch as the asymptotic behavior of the phases is the same as in the single-particle case, it is to be expected that at finite temperature allowance for the three-particle states will lead only to a renormalization of the Kondo temperature, i.e., all the limiting formulas for the conductivity, thermal emf, and specific heat, obtained in[11], remain unchanged.

We now consider the extent to which our results depend on the made approximations which enabled us to represent the single-particle unitarity condition in the form (25). Once this was done we have essentially used only the limiting formulas (37) for the function γ, and the concrete value γ_1 for the final result was practically immaterial, since γ_1 entered only in the definition of the quantity ϵ_1, which should be regarded as the main parameter of the theory.

Allowance for the terms discarded in the derivation of the equations in (25) would lead to their replacement by the formulas

$$\operatorname{Im} A = k\{|A|^2+S(S+1)|B|^2(1+2\gamma_A)\},$$
$$\operatorname{Im} B = k\{AB^*+A^*B-|B|^2(1+\gamma_B)\varepsilon(\zeta)\},$$

$$\gamma_{A,B} = \frac{1}{2}\int_0^{\zeta}\frac{d\zeta_1}{\zeta-\zeta_1}\Phi_{\zeta,\zeta_1}^{(A,B)}[A,B], \quad (43)$$

where $\Phi_{\zeta,\zeta_1}^{(A,B)}[A, B]$ are certain functionals of A and B. In this form, the unitarity conditions can be represented not only by taking into account the three-particle terms which we have discarded, but also in the more general case, for example if account is taken of the pole term of the five-particle amplitude (14), and in all cases we shall have $\gamma_A \approx \gamma_B \approx (2L)^{-1}$ when $L \gg 1$ and constant γ_A and γ_B when $-L \gg 1$. It should be noted, however, that generally speaking when $-L \gg 1$ we have $\gamma_{A,B} = \gamma_{1A,B} + \epsilon(\zeta)\gamma_{2A,B}$, i.e., at large values of ζ the values of $\gamma_{A,B}$ depend on the sign of ζ. This behavior of $\gamma_{A,B}$ has a general character and does not depend on the assumptions made.

With the aid of conditions (41) we can construct the function u. The renormalized Kondo energy will then depend only on the constant γ_{1B}. Furthermore, we can introduce the phases $\varphi_{\pm}(\zeta)$ in the same manner as above. The expressions for $\eta_{\pm}^2$ now take the form

$$\eta_{\pm}^2 = [1-8k^2\gamma_A|B|^2S(S+1)]\{1\pm 4(S+{}^1/_2$$
$$\mp{}^1/_2)k^2D^{-1}[\varepsilon(1+\gamma_B)-1]\},$$
$$D = u_1^2+k^2[4S(S+1)+(1+\gamma_B)^2],$$
$$|B|^2 = [D+8k^2\gamma_A S(S+1)]^{-1} \quad (44)$$

and the function u_1 = Re u is described by the limiting formulas (28) in which γ_1 is replaced by γ_{1B}. Substituting the expression for A and B in the formula for $\gamma_{A,B}$, we obtain an equation for the constants $\gamma_{1,2;A,B}$. The expressions for the phases $\varphi_{\pm}$ are best represented in the form similar to (40), where the combination of the logarithms in the integrand expression for $\varphi_{\pm}^{(1)}$ is an even function of ζ, in any case in the region $|L_1| > 1$, and therefore $\varphi_{\pm}^{(1)} \approx 0$, and the asymptotic formula (41) is valid for $\varphi_{\pm}^{(1)}$. The function $\varphi_{\pm}^{(2)}(\zeta)$ can also be represented in the form (42) when $L_1 \gg 1$, but the expressions for $\varphi_{\pm}^{(2)}(0)$ are now much more complicated:

$$\varphi_{+}^{(2)}(0) = \frac{1}{4\pi}\int_{-\infty}^{\infty}dL\ln\frac{[D_-+8S(S+1)\gamma_A^{(-)}k_F^2][D_++4Sk_F^2\gamma_B^{(+)}]}{[D_++8S(S+1)\gamma_A^{(+)}k_F^2][D_--4Sk_F^2\gamma_B^{(-)}]},$$

$$\varphi_{-}^{(2)}(0) = \frac{1}{4\pi}\int_{-\infty}^{\infty}dL\ln\frac{[D_-+8S(S+1)\gamma_A^{(-)}k_F^2][D_+-4(S+1)k_F^2\gamma_B^{(+)}]}{[D_++8S(S+1)\gamma_A^{(+)}k_F^2][D_-+4(S+1)k_F^2\gamma_B^{(-)}]} \quad (45)$$

where $Z_{\pm} = Z_1 \pm Z_2$, and Z_1 and Z_2 are respectively the even and odd parts of Z.

These integrals can be calculated already by the method described above, and as a result we obtain

$$\varphi_{\pm}^{(2)}(0) = \mp\frac{\pi}{2} + \lambda,$$

$$\lambda = \frac{\pi}{4}\frac{2S+1}{1+\gamma_{1B}}\left(\sqrt{1+c_1+c_2} - \sqrt{1+c_1-c_2}\right),$$

$$c_1 = (2S+1)^{-2}[2\gamma_{1B} + \gamma_{1B}^2 + \gamma_{2B}^2 + 8S(S+1)\gamma_{1A}],$$
$$c_2 = (2S+1)^{-2}[2\gamma_{2B}(1+\gamma_{1B}) + 8S(S+1)\gamma_{2A}]. \qquad (46)$$

Thus, in this case one of the phases will be larger than $\pi/2$ in absolute value, and the other smaller. At the same time we expect that $\lambda \ll 1$, since the terms $\gamma_{2A,B}\epsilon(\zeta)$ in the expression for $\gamma_{A,B}$ appear, as a minimum, only in the fourth power of the terms containing $|B|$, and each $|B|$ introduces into the corresponding term a small factor which roughly speaking is proportional to $(2S+1)^{-1}$ (we recall in this connection that in the simplest case γ_{1B} is equal to $(2S+1)^{-1}$).

In conclusion let us stop to discuss one more question. As already noted in Sec. 3, the inelastic-process amplitudes considered by us are connected with the amplitudes for scattering of two electrons (holes) by an impurity. On the other hand, the scattering amplitude should satisfy the reciprocity theorem (see, for example, the book of Landau and Lifshitz[13]), according to which the interchange of the initial and final particles can change only the sign of the amplitude. At the same time, in the pole approximation employed by us the scattering amplitude does not possess this property. But the pole approximation, obviously, is valid only near the pole, i.e., when $\zeta_{1,3} \approx \zeta$. This suggests that the indicated violation is only illusory. In order to verify this, we solved Eqs. (16) for f with allowance for the non-pole terms in the limiting case $L \gg 1$. The corresponding rather complicated calculations will be published separately in connection with the question of scattering by impurities and the influence of this scattering on superconductivity. We present here only the final result:

$$\tau(\zeta + i\delta, \zeta_1, \zeta_3, \zeta_2 - i\delta)$$
$$\approx -\left(\frac{\pi}{2k_F}\right)^2 (L_1L_3L_2L)^{-1/2}\left(\frac{[R_1, R_2]}{\zeta_1 - \zeta - i\delta} - \frac{\overline{[R_1, R_2]}}{\zeta_3 - \zeta - i\delta}\right). \qquad (47)$$

The corresponding scattering amplitude $\tau(\zeta + i\delta, \zeta_1, \zeta_3, \zeta_2 + i\delta)$ differs from $\tau(\zeta + i\delta, \zeta_1, \zeta_3, \zeta_2 - \delta)$ only in sign, as can be readily verified with the aid of (12) and satisfies the reciprocity theorem. It is also easy to verify that substitution of (47) in the expression for $\Delta_{A,B}$ leads to formulas that coincide with (24) when $L \gg 1$.

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77