ENERGY DISTRIBUTION OF ELECTRONS AND THEIR CONCENTRATION IN A LOW-TEMPERATURE PLASMA

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We investigate the influence of the resonant-radiation yield on the electron energy distribution and on the electron concentration in a low-temperature plasma. We obtain a solution of the kinetic equation with allowance for inelastic collisions of the first and second kind in the case of a large electron concentration, and also the corresponding deviation of the system from the equilibrium value at relative low and relatively large yields of resonant radiation. The problem is also solved for the opposite case of low electron concentration and a strong electric field.

INTRODUCTION

To describe any processes occurring in a nonequilibrium plasma, it is necessary to find the electron velocity distribution function, the electron concentration, and the distribution of the atoms over the excited states. It is known[1] that the continuous electron energy spectrum is in equilibrium with the upper levels of the discrete spectrum. In some papers[2,3] it is noted that in a low-temperature plasma such an equilibrium in first approximation down to the lowest excited level. Thus, the last two problems combine into one in this approximation. On the other hand, the first problem reduces to solving the kinetic equation with allowance for inelastic collision[4-7]. Allowance for the collisions between electrons requires knowledge of their concentration, and allowance for inelastic collisions of the second kind, which eliminate the excitation, calls for knowledge of the concentration of the excited atoms. On the other hand, to calculate these concentrations it is necessary to know the electron energy distribution function. Usually all these problems are solved separately. When solving the kinetic equation for the electrons, their concentration and the population of the excited levels are assumed known[5,6], or else conditions are chosen under which these concentrations and the corresponding terms in the kinetic equation are small[7]. On the other hand, when calculating, say, the electron concentration, the distribution function is assumed known, usually Maxwellian[8]. Biberman et al.[9] attempted to solve these problems simultaneously, However, as will be shown later, in a consistent simultaneous solution, one equation (for the number of the excited atoms) is used in a natural manner to determine the arbitrary constants that appear in the integration of the other equation (for the electron distribution function). On the other hand, when the solutions are obtained separately, all the constants are determined only from the continuity conditions.

Shaw et al.[10] present the results of a numerical simultaneous solution of these problems for the case of an argon-cesium plasma. In calculating the electron energy distribution function, the authors solve a second-order equation subject to the condition $f(\epsilon) \to 0$ as $\epsilon \to \infty$, and stipulate a Maxwellian distribution at $\epsilon = 0$. However, in such a formulation of the problem one cannot expect a unique determination of the form of the distribution function in the region of the excitation threshold, where it can depend strongly on the intensity of the outgoing radiation. This dependence is not limited only to the indirect influence via the concentration of the excited atoms.

In the present paper we solve the kinetic equation for the symmetrical part of the distribution function simultaneously with the kinetic equation for the number of excited atoms. It is assumed that all the excited levels are in equilibrium with one another and with the continuous electron spectrum. This is possible when the atoms are excited by electron impact, and all forms of radiation, except the resonant radiation, do not lead to a noticeable violation of the equilibrium. An explicit dependence on the spectroscopic characteristics of the plasma is obtained for the electron energy distribution function and also for the degree of deviation of the system from equilibrium.

1. YIELD OF RESONANT RADIATION FROM A HOMOGENEOUS PLASMA

The number of quanta radiated from a unit surface of a plasma in a unit time is[11]

$$\frac{S}{h\nu} = \int\int \frac{\gamma n_1(r) p_\nu}{4\pi r^2} e^{-k_\nu r \cos\vartheta}\, dr\, d\nu, \tag{1.1}$$

where s is the energy radiated from a unit surface in a unit time, $h\nu$ is the energy of the quantum, γ is the probability of the spontaneous emission, k_ν is the coefficient of absorption of a quantum of frequency ν, p_ν is the emission line shape (with $\int p_\nu d\nu = 1$), ϑ is the angle between the direction of the beam and the normal to the surface, and n_1 is the concentration of the atoms in the first excited state. We assume that $p_\nu \sim k_\nu$[12]:

$$\frac{p_\nu}{k_\nu} = \frac{8\pi}{\lambda_0^2}\frac{g_0}{g_1}\frac{1}{\gamma n_0}, \tag{1.2}$$

where λ_0 is the wavelength at the center of the line, n_0 is the concentration of the atoms in the ground state, and g_0 and g_1 are the statistical weights of the atom in the ground and first excited states. For a homogene-

ous plasma, expression (1.1) can be rewritten in the form

$$\frac{S}{h\nu} = \frac{n_1}{n_0}\frac{g_0}{g_1}\frac{\lambda\pi}{\lambda_0^2}\frac{1}{2\pi}\int_{2\pi}\cos\vartheta\, d\Omega\int d\nu\,(1-e^{-k_\nu\rho}) = \frac{n_1}{n_0}\frac{g_0}{g_1}\frac{2\pi}{\lambda_0^2}\cdot 2\overline{\Delta\nu}, \quad (1.3)$$

where Ω is the unit vector in the direction of the beam, $\overline{\Delta\nu}$ is the equivalent line width averaged over a hemisphere with weight $\cos\vartheta$, and $\rho = \rho(\Omega)$ is the thickness of the plasma in the beam direction. For the simplest hemispherical geometry with radius ρ we have in the case of a dispersion line contour[13]

$$2\overline{\Delta\nu} = \Delta\nu = \left(\frac{2\pi e^2}{mc} f n_0 \Delta\nu^c \rho\right)^{1/2}, \quad (1.4)$$

where e is the electron charge, m its mass, c the speed of light, f the oscillator strength, $\Delta\nu^c$ the collision half-width of the line. As shown by appropriate corresponding calculations, in a cylindrical geometry with diameter ρ, the radiation from a unit surface is practically the same as for a hemispherical geometry with radius ρ. If the resonant radiation corresponds not to one line but say to two (doublet), then the result must be summed over both components.

The conversion from the energy S radiated from a unit surface to the energy W radiated from a unit volume entails no difficulty. Thus, for cylindrical geometry $W = 4S/\rho$.

We shall assume henceforth that the equivalent line width does not depend on the electron concentration. This will occur in the case of Doppler or collision broadening. The generalization of the theory to the case of the Stark broadening mechanism entails no great difficulty.

2. FUNDAMENTAL EQUATIONS

The problem of finding the energy distribution of the free electrons reduces to solving the kinetic equation for the symmetrical part of the electron velocity distribution function. In this equation, the terms describing the influence of the elastic collisions of the electrons with the atoms, ions, and other electrons, and also of the field can, as is well known, be written in the form of the divergence of the certain flux j in velocity space[14]. Collisions which transfer the atom from the ground state to the first excited state and back should be written separately.

The difference in the number of collision of the first and second kind, which change the number of particles in the unit volume of phase space in a unit time, is

$$\Phi(v) = vn_0 nq(v)f(v) - v'nn_1 q^*(v')f(v'),$$

where f(v) is the symmetrical part of the electron velocity distribution function, n is the electron concentration, q is the cross section for the excitation of the atom by electron collision, q* is the cross section of the collision of the second kind, $v' = (v^2 - 2h\nu/m)^{1/2}$, and $h\nu$ is the excitation energy of the atom.

Each collision of the first kind leads not only to a vanishing of a fast electron, but also to a production of a slow one. Analogously, a collision of the second kind, as a result of which a fast electron appears, leads to the vanishing of a slow electron. In the kinetic equation, the terms describing the vanishing and creation of slow electrons can be combined into a single term designated $\Phi(v'')$:

$$\frac{\partial f}{\partial t} + \Phi(v) - \Phi(v'') - \frac{1}{v^2}\frac{\partial}{\partial v}v^2 j = 0, \quad (2.1)$$

where $v'' = (v^2 + 2h\nu/m)^{1/2}$.

The cross section of the collisions of the second kind can be expressed in terms of the excitation cross section with the aid of a relation that follows from the detailed balancing principle:

$$(\varepsilon - h\nu)q^*(\varepsilon - h\nu) = \frac{g_0}{g_1}\varepsilon q(\varepsilon),$$

where $\epsilon = mv^2/2$. We change over in the kinetic equation (2.1) from a unit element of phase space to a unit energy interval $(\epsilon, \epsilon + d\epsilon)$ and from the velocity distribution function to the energy distribution function $f(\epsilon)$, normalized to unity with weight $\sqrt{\epsilon}$:

$$\frac{\partial}{\partial t}n\sqrt{\varepsilon}f + \Phi(\varepsilon) - \Phi(\varepsilon + h\nu) - \sqrt{2m}\frac{\partial}{\partial\varepsilon}\varepsilon j(\varepsilon) = 0, \quad (2.2)$$

where

$$\Phi(\varepsilon) = nn_0\varepsilon q(\varepsilon)\sqrt{\frac{2}{m}}\left[f(\varepsilon) - \frac{n_1}{n_0}\frac{g_0}{g_1}f(\varepsilon - h\nu)\right].$$

Each inelastic collision changes not only the state of the electron, but also the number of excited atoms. We write down the kinetic equation for the number of atoms in the excited state under the assumption that all the excited levels are in equilibrium with one another:

$$\frac{\partial n_1}{\partial t} - \int_{h\nu}^{\infty}\Phi(\varepsilon)\,d\varepsilon + \frac{W}{h\nu} = 0. \quad (2.3)$$

In the stationary case, the total difference of the number of electron collisions leading to excitation and de-excitation of the atoms is equal to the number of the resonant radiation quanta leaving the plasma.

We note that $\Phi(\epsilon) \neq 0$ only beyond the excitation threshold, when $\epsilon > h\nu$. The elastic collisions may play an important role only on a segment of order kT beyond the excitation threshold, where the rapidly decreasing distribution function can still make a noticeable contribution to the inelastic-collision integral. Accordingly, $\Phi(\epsilon + h\nu) \neq 0$ in the region $\epsilon \approx kT$. However, it is known that the role of the term $\Phi(\epsilon + h\nu)$ reduces mainly to a description of the source of electrons in the low-energy region[14]. It is essential when writing down the particle or energy balance, but has practically no influence on the form of the distribution function in this region.

For the dependence of the excitation cross section on the energy in a segment of order kT beyond the threshold $h\nu$) we assume the linear approximation

$$\varepsilon q(\varepsilon) = \beta\frac{\pi e^4}{(h\nu)^2}(\varepsilon - h\nu), \quad (2.4)$$

where $\beta = 1$ corresponds to the Thomson cross section.

The interelectron interaction in the "tail" of the distribution function can be approximately taken into account with the aid of the simple expression[14]

$$\varepsilon j_{ee} = \frac{4\pi e^4}{m}\Lambda n^2\left(kT\frac{\partial f}{\partial\varepsilon} + f\right). \quad (2.5)$$

This approximation is perfectly sufficient for energies much higher than kT.

Inasmuch as the elastic collisions of the electrons with atoms, ions, and other electrons are expressed in

the form of the divergence of a certain flux, we can write

$$\sqrt{2m}\,\varepsilon j(\varepsilon) = \varphi_1 \frac{\partial f}{\partial \varepsilon} + \varphi_2 f, \tag{2.6}$$

where

$$\varphi_1 = n\sqrt{2m}\left[2\pi e^4 \Lambda n\left(\frac{2kT}{m} + \frac{kT_a}{M_i}\right) + 2n_a q_a \frac{kT_a}{M}\varepsilon^2 + \frac{e^2E^2}{3m}\frac{\varepsilon}{n_a q_a + n q_{ee}}\right],$$

$$\varphi_2 = n\sqrt{2m}\left[2\pi e^4 \Lambda n\left(\frac{2}{m} + \frac{1}{M_i}\right) + 2n_a q_a \frac{\varepsilon^2}{M}\right],$$

$$q_{ee} = \frac{\pi e^4 \Lambda}{\varepsilon^2} = \pi a_0^2 \Lambda \left(\frac{2I_H}{\varepsilon}\right)^2,$$

a_0 is the Bohr radius, I_H is the ionization energy of the hydrogen atom, M_i is the ion mass, q_a is the cross section for elastic collision of the electron with atoms having a mass M and a concentration n_a, T_a is the temperature of the heavy particles, and E is the intensity of the electric field. At low energies (2.5) no longer holds, and it is necessary to use a more complicated nonlinear expression, and φ_1 and φ_2 become accordingly more complicated. For simplicity, we shall henceforth neglect the dependence of the electron-atom elastic collision cross section on the electron energy (the elastic-sphere model). We confine ourselves only to the stationary case.

We integrate both parts of (2.2) with respect to ϵ from a certain value of ϵ to infinity, and introduce the function

$$\psi(\varepsilon) = \frac{h\nu}{W}\int_{\varepsilon}^{\infty} \Phi(\varepsilon)\,d\varepsilon:$$

$$\varphi_1 \frac{\partial f}{\partial \varepsilon} + \varphi_2 f = -\frac{W}{h\nu}[\psi(\varepsilon) - \psi(\varepsilon + h\nu)], \tag{2.7}$$

$$\frac{W}{h\nu}\frac{\partial \psi}{\partial \varepsilon} = nn_0 \varepsilon q(\varepsilon)\sqrt{\frac{2}{m}}\left[f(\varepsilon) - \frac{n_1}{n_0}\frac{g_0}{g_1} f(\varepsilon - h\nu)\right]. \tag{2.8}$$

These two first-order equations with two unknown functions $f(\epsilon)$ and $\psi(\epsilon)$ and with the condition $\psi(\epsilon) = 1$ when $\epsilon \leq h\nu$ are equivalent to the single second-order equation (2.2). It may be convenient to solve the second-order equation beyond the excitation threshold, after using the substitution

$$f = z\sqrt{\frac{\varphi_1(h\nu)}{\varphi_1(\varepsilon)}}\exp\left(-\frac{1}{2}\int_{h\nu}^{\varepsilon}\frac{\varphi_2}{\varphi_1}d\varepsilon\right)$$

to eliminate the terms with the first derivative:

$$\varphi_1 \frac{\partial^2 z}{\partial \varepsilon^2} - \varphi_3 z = nn_0 \varepsilon q(\varepsilon)\sqrt{\frac{2}{m}}\left[z - \frac{n_1}{n_0}\frac{g_0}{g_1}\exp\left(\frac{1}{2}\int_{h\nu}^{\varepsilon}\frac{\varphi_2}{\varphi_1}d\varepsilon\right) f(\varepsilon - h\nu)\right], \tag{2.9}$$

where

$$\varphi_3 = \frac{1}{2}\left\{\frac{1}{2\varphi_1}\left[\varphi_2^2 - \left(\frac{\partial \varphi_1}{\partial \varepsilon}\right)^2\right] - \frac{\partial \varphi_2}{\partial \varepsilon} + \frac{\partial^2 \varphi_1}{\partial \varepsilon^2}\right\}.$$

In the absence of inelastic collisions, $\epsilon j = 0$. The solution for this case is known[14]:

$$f(\varepsilon) = Ce^{-\Phi(\varepsilon, 0)}. \tag{2.10}$$

where

$$\varphi(\varepsilon, a) = \int_a^{\varepsilon}\frac{\varphi_2}{\varphi_1}d\varepsilon.$$

C is the normalization constant.

The inelastic collisions influence principally the "tail" of the distribution function, and we can therefore expect that in the presence of inelastic collisions in the region of sufficiently low energies ($\epsilon \leq \epsilon_1 \ll h\nu$) the distribution function is of the form (2.10). We take into account in this region only the indirect influence of the radiation yield via the energy balance.

With increasing electron energy, the inelastic-collision cross section beyond the excitation threshold increases linearly from a zero value, and is quite small at sufficiently low energies. Thus, there always exists a small region beyond the excitation energy, where the right-hand side can still be neglected in (2.9), and consequently also in (2.7):

$$0 \leqslant \varepsilon - h\nu \ll \frac{kT}{a}, \quad a = kT\frac{nn_0\beta\pi e^4}{(h\nu)^2\varphi_3(h\nu)}\sqrt{\frac{2}{m}}. \tag{2.11}$$

The main contribution to the inelastic collision integral is made by a region of the order of kT beyond the threshold $h\nu$, so that we can expect that when $a \ll 1$ the inelastic collisions practically do not change the form of the distribution function even in the "tail."

Let us find the distribution function in the pre-threshold region. Equation (2.7) simplifies to

$$\varphi_1 \frac{\partial f}{\partial \varepsilon} + \varphi_2 f = -\frac{W}{h\nu}, \quad kT \ll \varepsilon_1 \leqslant \varepsilon \leqslant h\nu, \tag{2.12}$$

and has a solution

$$f(\varepsilon) = \left[f(\varepsilon_1) - \frac{W}{h\nu}\int_{\varepsilon_1}^{\varepsilon}\frac{1}{\varphi_1(t)}e^{\varphi(t,\,\varepsilon_1)}\,dt\right]e^{-\varphi(\varepsilon,\,\varepsilon_1)}. \tag{2.13}$$

The presence of the lower limit of the region is connected with the requirement that the quantity $\psi(\epsilon + h\nu)$ be small, Later we shall verify, using concrete examples, that the distribution function is practically independent of the choice of the lower limit ϵ_1. The form of the distribution function beyond the excitation threshold will be obtained later for individual particular cases.

Let us write down the electron energy balance equation. Multiplying both sides of (2.2) by ϵ and integrating over all the energies, we have, after simple transformations,

$$n\sqrt{2m}\left\{2\pi e^4 \Lambda n\frac{1}{M_i}\left[\int f d\varepsilon - f(0)kT_a\right] + 2n_a q_a \frac{1}{M}\int(\varepsilon - 2kT_a) f\varepsilon\,d\varepsilon + \frac{e^2E^2}{3m}\int\frac{f(\varepsilon)\,d\varepsilon}{n_a q_a + n q_{ee}}\right\} = W. \tag{2.14}$$

In the energy balance equation it is usually possible to neglect the singularities of the behavior of the function in the "tail," and we can confine ourselves to the solution (2.10).

We assume that the concentration of all the excited atoms and the concentration of the electrons have identical[2,3] deviations from the corresponding equilibrium values, defined by the Boltzmann and Saha formulas:

$$\frac{n_1}{n_0} = y\frac{g_1}{g_0}e^{-h\nu/kT}, \quad \frac{n^2}{n_0} = y\frac{2g_i}{g_0}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}e^{-\varepsilon_i/kT}, \tag{2.15}$$

where g_i is the statistical weight of the ion, and ϵ_i is the ionization energy. Here $y \leq 1$. Knowing the distribution function, we can determine from relations (2.14) and (2.3), taking (2.15) into account, the average energy of the electrons (or a suitably defined temperature) and the degree of deviation of the system from the equilibrium state at a given electron temperature.

3. CASE OF LARGE ELECTRON CONCENTRATION

As seen from (2.6), the relative role of the terms describing the interelectron interaction decreases with increasing electron energy. Let us consider a case when these terms remain principal in a sufficiently large region, up to energies $\epsilon \approx h\nu$:

$$\frac{m}{M}\left(\frac{h\nu}{2I_H}\right)^2 \frac{1}{2\Lambda\pi a_0^2 nl} \ll 1, \tag{3.1}$$

where $l = (n_a q_a + n q_{ei})^{-1}$ is the electron mean free path, and M is the mass of the atoms to which the electrons give up energy in the elastic collisions. It is easy to verify that in this case the distribution function ahead of the excitation threshold (2.13) is given by

$$f(\varepsilon) = Ce^{-\varepsilon/kT} - ACe^{-h\nu/kT}(1 - e^{-(\varepsilon-\varepsilon_1)/kT}) \approx C(e^{-\varepsilon/kT} - Ae^{-h\nu/kT}), \quad \varepsilon_1 \leqslant \varepsilon \leqslant h\nu, \tag{3.2}$$

where

$$A = \frac{\pi}{4}\frac{g_0}{g_1}\frac{\Delta\nu}{\mathrm{Ry}}\frac{1}{\Lambda}\frac{1}{\rho\lambda^2 n_0}e^{\varepsilon_1/kT},$$

Ry is the Rydberg constant. It is assumed here that the normalization factor remains practically the same as for a Maxwellian distribution: $C = 2\pi^{-1/2}(kT)^{-3/2}$.

Beyond the excitation threshold, Eq. (2.9) reduces to

$$\frac{\partial^2 z}{\partial t^2} - tz = 0, \tag{3.3}$$

where

$$z = e^{x/2}\left(f - C\frac{n_1}{n_0}\frac{g_0}{g_1}e^{-x}\right), \quad t = (2a)^{-2/3}(1 + ax),$$
$$a = \frac{n_0}{n}\left(\frac{kT}{h\nu}\right)^2\frac{\beta}{\Lambda}, \quad x = \frac{\varepsilon - h\nu}{kT}.$$

A solution decreasing at infinity is expressed in terms of a modified Bessel function of the second kind (MacDonald function) or in terms of an Airy function[15]:

$$z = \mathrm{const}\cdot\sqrt{t}K_{1/3}(^2/_3 t^{3/2}) = \mathrm{const}\cdot v(t). \tag{3.4}$$

Let us transform from the function z back to the function f, and determine the integration constant from the requirement that the distribution function be continuous at the point $\epsilon = h\nu$:

$$f = Ce^{-h\nu/kT} + \left\{ye^{-x} + (1 - A - y)e^{-x/2}(1 + ax)^{1/2}K_{1/3}\left[\frac{(1+ax)^{3/2}}{3a}\right]K_{1/3}^{-1}\left(\frac{1}{3a}\right)\right\} \tag{3.5}$$

We substitute the obtained distribution function in (2.3):

$$(1 - A - y)\int_0^\infty e^{-x/2}(1 + ax)^{1/2}K_{1/3}\left[\frac{(1+ax)^{3/2}}{3a}\right]K_{1/3}^{-1}\left(\frac{1}{3a}\right)x\,dx = \frac{4A}{a}. \tag{3.6}$$

Since $a = a^* y^{-1/2}$, where a^* is the value of a at the equilibrium electron concentration corresponding to the given temperature, Eq. (3.6) determines a complicated implicit dependence of the degree of nonequilibrium of the plasma on its gas-kinetic and spectroscopic characteristics. We shall simplify this in particular cases when the role of the inelastic collisions is very small or, conversely very large compared with the elastic collisions.

As is well known, the main contribution to the inelastic collision integral is made by the distribution function in a region of the order of kT beyond the excitation threshold, at $x \approx 1$. When the inelastic collisions make a small contribution in this region compared with the elastic collisions, i.e., when $a \ll 1$, the argument of the function $K_{1/3}(x)$ assumes large values, and it is possible to use its asymptotic expression[16]

$$K_{1/3}\left[\frac{(1+ax)^{3/2}}{3a}\right]K_{1/3}^{-1}\left(\frac{1}{3a}\right) \approx e^{-x/2}.$$

The distribution function retains a Maxwellian energy dependence:

$$f(\varepsilon) = C(1 - A)e^{-\varepsilon/kT}, \quad 0 \leqslant \varepsilon - h\nu \ll kT/a, \tag{3.7}$$

and Eq. (3.6) is transformed into an equation quadratic in $\sqrt{y}$:

$$1 - A - y = 4A\sqrt{y}/a^*. \tag{3.8}$$

In (3.8), the right-hand side is smaller than unity, so that it is compatible with the condition $a \ll 1$ only when $A < a/4 \ll 1$, when the emission of the radiation disturbs the equilibrium little. Hence $y \approx 1 - 4A/a^*$.

Let us consider the opposite case, when the inelastic collisions prevail over the elastic collisions almost immediately beyond the excitation threshold, $a \gg 1$. In the interval $0 < x < 1$ the argument of the function $K_{1/3}$ varies in a wide range, $^1/_3 a < t < \sqrt{a/3}$, and the function $K_{1/3}(t)$ decreases much more rapidly than the exponential function under the integral sign in the left side of (3.6). Let us replace the exponential by unity and change over to the integration variable $t = (1 + ax)^{3/2}/3a$. Replacing further the lower integration limit $^1/_3 a$ by zero and using the equation[16]

$$\int_0^\infty t^{m-1}K_n(t)\,dt = 2^{m-2}\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right),$$

where $\Gamma(x)$ is the Γ function, we obtain for the integral in the left side of (3.6)

$$I \approx a^{-2/3}\frac{2\cdot 3^{1/3}}{\sqrt{\pi}}\Gamma\left(\frac{5}{6}\right)\left[1 - (6a)^{-2/3}\Gamma\left(\frac{1}{3}\right)\right] \approx 1{,}84a^{-2/3};$$

whence

$$y \approx 1 - A\left[1 + 2.17(a^*)^{1/3}\right]. \tag{3.9}$$

In the case of a small radiation yield, $A \ll 1$ and the equilibrium is disturbed little; $y \approx 1 - A[1 + 2.17(a^*)^{1/3}]$. In the opposite extreme case we have $A \approx 1$ and $\sqrt{y} \approx a^*[(1 - A)/2.17A]^3$. Here, however, it is already necessary to exercise caution. If we increase the role of the radiation, by decreasing, for example, the geometrical dimensions of the plasma, then we have $y \to 0$ when $A \to 1$ and accordingly $n \to 0$. An analysis of (3.6) shows that when inequality (3.1) is satisfied we always have $A < 1$.

At not too low an electron concentration, when the inequality (3.1) is satisfied and the source of the equilibrium is upset by the emission of the resonant radiation, we have a simple criterion for the applicability of the Saha equation: $A \ll 1$.

When the inelastic collisions play a minor role compared with the elastic collisions, the radiation yield, as expected, can be only small and the distribution function remains Maxwellian. On the other hand, when the frequency of the inelastic collisions beyond the excitation threshold is very large, the radiation yield can be both small and relatively large, greatly disturbing the equilibrium. In the former case the distribution function also remains Maxwellian (the first term in the curly brackets is the principal term in (3.5)). In the latter case, the concentration of the electrons and of the excited atoms decreases, and accordingly the role

of collisions of the second kind decreases, and the principal term in the distribution function beyond the excitation threshold will be the rapidly decreasing term connected with the excess of the number of collisions of the first kind over the number of collisions of the second kind as a result of the radiation yield.

4. CASE OF LOW ELECTRON DENSITY IN A STRONG ELECTRIC FIELD

Let us consider the case when the field is so strong that the energy acquired by the electron from the field during the time between two collisions exceeds the energy that the electron can be obtained from the atom as a result of elastic collision,

$$kT_a \ll \frac{2\alpha^2}{h\nu}, \quad \alpha = \frac{1}{2}\sqrt{\frac{M}{3m}}eEl,$$

and the electron concentration is sufficiently small to be able to neglect the interelectron interaction in the "tail" of the distribution function. Nevertheless, the electron concentration should exceed a certain minimum corresponding to the equilibrium concentration at the temperature of the atoms, $n \gg n^*(T_a)$. The presence of equilibrium between the excited levels also presupposes a certain minimum of the electron concentration. In the energy region $\epsilon \approx kT$, the distribution can be Maxwellian

$$\frac{kT}{h\nu} \ll b\frac{kT}{h\nu} \ll 1, \quad b = \frac{\pi e^4}{2\alpha^2}\Lambda nl\frac{M}{m},$$

or of the Druyvesteyn type ($b \ll 1$):

$$f = Ce^{-(\varepsilon/2\alpha)^2}, \qquad 0 \leqslant \varepsilon \leqslant \varepsilon_1. \tag{4.1}$$

In the latter case $\alpha \approx kT$.

When the inelastic collisions are infrequent

$$a_1 = \frac{M}{m}\beta n_0 \pi e^4 l\frac{(2\alpha)^4}{2(h\nu)^6} \ll 1, \tag{4.2}$$

and have little effect on the distribution function, which at high energies has the form (4.1), Eq. (2.3), accurate to small terms, reduces to the form

$$Ce^{-(h\nu/2\alpha)^2} - ye^{-h\nu/hT}\int_0^\infty f\left(\frac{2\alpha^2}{h\nu}x\right)x\,dx = \sqrt{y}\frac{4B^*}{a_1}, \tag{4.3}$$

where

$$x = \frac{h\nu(\varepsilon - h\nu)}{2\alpha^2}, \quad B^*\sqrt{y} = B = \frac{1}{n}\frac{W}{(h\nu)^3}\frac{lM}{2\sqrt{2m}}.$$

Under the integral sign, the function $f(\epsilon - h\nu)$ can be close to Maxwellian or to the Druyvesteyn type. From (4.3) we can obtain the degree of non-equilibrium of the system in the case of trapped or almost trapped radiation. When $a_1 \ll 1$, the radiation yield cannot be large, since $B < (1/4)a_1 C \exp[-(h\nu/2\alpha)^2]$. The condition (4.2) may be satisfied, for example, in a helium plasma with a very small admixture of argon.

Let us consider now another case. We introduce in lieu of the condition (4.2) that the number of inelastic collisions of the first kind be small the condition that the number of collisions of the second kind be small compared with the number of collisions of the first kind:

$$ye^{-h\nu/kT} \ll e^{-(h\nu/2\alpha)^2}. \tag{4.4}$$

Near the excitation threshold, when x changes by unity, the distribution function changes by a factor e. It is easy to verify that when ϵ_1 satisfies the condition

$$1 \ll \frac{h\nu(h\nu - \varepsilon_1)}{2\alpha^2} \ll \frac{(h\nu)^2}{2\alpha^2},$$

the distribution function in the pre-threshold region (2.13) can be written, accurate to small terms, in the approximate form

$$f(\varepsilon) = Ce^{-(\varepsilon/2\alpha)^2} - B, \qquad \varepsilon_1 \leqslant \varepsilon \leqslant h\nu. \tag{4.5}$$

Beyond the excitation threshold, when $2\alpha^2/(h\nu)^2 \ll 1$, Eq. (2.9) can be rewritten in the form (3.3), whereas $v = e^{x/2}f$, $t = (2a_1)^{-2/3}(1 + a_1x)$.

We substitute the solution

$$f = [Ce^{-(h\nu/2\alpha)^2} - B]e^{-x/2}(1 + a_1x)^{1/2}K_{1/3}(t)K_{1/3}^{-1}\left(\frac{1}{3a_1}\right), \quad x \geqslant 0, \tag{4.6}$$

in Eq. (2.3):

$$[Ce^{-(h\nu/2\alpha)^2} - B]K_{1/3}^{-1}\left(\frac{1}{3a_1}\right)\int_0^\infty e^{-x/2}(1 + a_1x)^{1/2}K_{1/3}(t)x\,dx = \frac{4B}{a_1}. \tag{4.7}$$

When the role of the inelastic collisions is very large ($a_1 \gg 1$), we have

$$Ce^{-(h\nu/2\alpha)^2} - \sqrt{y}B^*(1 + 2{,}17a_1^{-1/3}) = 0. \tag{4.8}$$

In the general case

$$y = (B/B^*)^2. \tag{4.9}$$

As follows from (4.7), the quantity B cannot be large: $B < Ce^{-(h\nu/2\alpha)^2}$. On the other hand, B* cannot be an arbitrarily small quantity, for otherwise the inequality (4.4) must be violated as a result of the increase of the electron concentration.

The quantity a_1 does not depend on y, and therefore B, which is proportional to the outgoing radiation, is determined completely by Eq. (4.7). Thus, the distribution function (4.5) and (4.6) does not depend on the spectral characteristics and is determined entirely by the gas kinetic properties of the plasma, and can be determined without calculating the degree of deviation of the system from the equilibrium state. This is connected with the fact that in the case when collisions of the second kind are neglected, each act of excitation of the atom leads to subsequent emission of a quantum out of the plasma.

5. ON THE CONDITIONS FOR JOINING TOGETHER THE DISTRIBUTION FUNCTION

The kinetic equation (2.2) is a differential equation of second order. The solution is determined in three different regions, and the distribution functions are then joined together, In the low energy region $0 \leq \epsilon \leq \epsilon_1$, the flux j increases from zero to a certain value, but has practically no influence on the form of the distribution function, and was therefore neglected. The only arbitrary constant that appears in the solution of the corresponding first-order equation that the majority of the electrons must satisfy ($kT \ll \epsilon_1$) should naturally be the normalization constant.

In the pre-threshold region $\epsilon_1 \leq \epsilon \leq h$, the electrons can still not excite the atoms, but the influence of the radiation yield on the distribution function can already be noticeable. The solution of the second-order Fokker-Planck type equation gives rise to two arbitrary constants. One of them, which determines the magnitude of the flux, is expressed with the aid of the imposed condition (2.3) in terms of the intensity of the

outgoing radiation. The second is chosen in such a way as to ensure continuity of the distribution function at the point $\epsilon = \epsilon_1$.

Finally, beyond the excitation threshold $\epsilon > h\nu$, both the flux and the distribution function should decrease at infinity. One of the two arbitrary constants obtained in the solution of the corresponding equation must be used to satisfy this physical requirement, and the other to satisfy the condition of continuity of the distribution function at the point $\epsilon = h\nu$.

We note that, within the framework of the employed approximation, the continuity of the first derivative would be simultaneously a sufficient condition of the continuity of the second derivative of the distribution function, since it is easy to see from the very form of Eq. (2.2) that the derivative of the flux j is continuous. In the presence of resonant radiation from the plasma, the electron energy distribution and concentration depend not only on the gas-kinetic characteristics but also on the equivalent width of the spectral line and optical thickness of the plasma. Only when the number of elastic collisions of the second kind is negligibly small compared with the number of collisions of the first kind is the electron distribution function (but not their concentration) independent of the spectroscopic characteristics of the plasma.

Note added in proof (7 January 1969). By equating the flux of the electronic states to the flux of the quanta leaving the plasma we have, as it were, left no constants with which to satisfy the requirement of the continuity of the derivative of the distribution function. However, the very manner in which the differentiated terms are written out and their physical meaning require this continuity. Indeed, as noted in a recent paper by Yu. B. Golubovskiĭ, Yu. M. Kagan, and R. I. Lyagushchenko (Zh. Tech. Fiz. 38, 1934 (1968)) [Soviet Phys.-Tech. Phys. 13, 1553 (1969)] the continuity of the derivative of the distribution function is fulfilled automatically. The continuity of the collision flux was essentially used already in the derivation of expression (2.7) or (2.12), and if the distribution function itself is continuous, the continuity of the flux is equivalent to continuity of the derivative of this function.

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