THE PHASE EQUILIBRIUM AND DYNAMICS OF A GAS VOLUME THAT IS HEATED AND COOLED

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The stationary state of a gas heated by cosmic rays and cooled by radiation is considered. For a given density of the cosmic rays the temperature is a function of the gas density. It is known that the pressure is not a monotonic function of the density, therefore the gas decomposes into two phases having different temperatures and densities for the same pressure, the latter being the parameter of the problem. In the present paper it is shown that taking into account heat conduction in the boundary layer allows one to find such a value of the parameter for which the equilibrium is stable. For other pressures the boundary of separation of the two phases moves relatively to the gas. The velocity of this motion has been computed. The theory is applied to the interstellar gas. The conditions applying to a stable state are computed. For deviations from this state, the velocity of boundary motion turns out to be small.

In a paper of one of the authors [1] the interstellar gas was considered subject to heating by cosmic rays and radiative cooling. In its general form the problem may be formulated in such a manner that in a spatially homogeneous region it is governed by the equation

$$c_v \frac{dT}{dt} = qf(\rho, T) - \varphi(\rho, T) \equiv F(q, \rho, T),$$

where q is proportional to the cosmic ray flux, f characterizes the dependence of the ionization losses on the density ρ and temperature T, and the function φ describes the radiative losses. For given q and ρ there exists a solution of the stationarity equation F = 0, of the form $T = T_s(q, \rho)$. Correspondingly one can find the stationary pressure $p_s(q, \rho) = RT_s\rho/\mu_s$, where μ_s is the molecular weight taking into account the composition and the degree of ionization. Here T_s is a decreasing function of ρ and therefore it becomes possible that p_s is a nonmonotone function of ρ for a given field of cosmic rays, characterized by the parameter q. The shape of the curve $p_s(\rho, q = \text{const})$ for the conditions of interstellar space[1] is illustrated in Fig. 1 and reminds one of a van der Waals isotherm (the horizontal axis is in logarithmic scale, k is the Boltzmann constant).

The region BFC, where $\partial p/\partial\rho < 0$ is unstable. The concept of "thermal instability" for a strong dependence of the heat losses on the density was long ago formulated and investigated in detail by Field[2]. For a given average gas density the whole region occupied by the gas splits up into dense cool clouds (the points on the portion AB) and hot gas (the stretch CD). An obvious condition is the equality of the pressures in the cloud and in the hot gas. However this condition is not sufficient. It is obvious that any pair of points situated on the same horizontal line, e.g. AC, EG, or BD satisfy the conditions that the pressures in the two phases are equal, and also the stationarity condition (thermal balance) as well as stability of each phase separately. Thus, the question arises as to which pair of points corresponds to a genuine stationary state, and which processes occur when an arbitrary pair of points is selected.

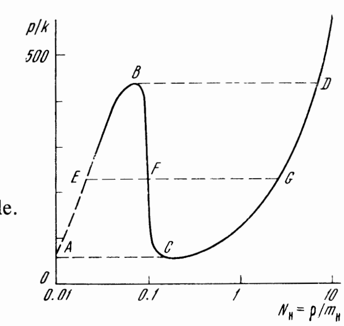

FIG. 1. The dependence of the stationary pressure on the density. The region BC is unstable.

For the van der Waals equation the answer is given by thermodynamics: the stationarity condition consists in equality of the areas spanned by the portions EBF and FCG of the phase equilibrium curve in the coordinates p, $V = \rho^{-1}$. If a different pair of points is selected, there occurs a condensation or evaporation, i.e., a phase transition at the interface. In our case the system has an external energy source and therefore thermodynamic arguments are not applicable. However the considerations regarding the phase transformation at the interface may be carried over to the case under consideration.

We consider the boundary between two phases and take into account the fact that owing to thermal conductivity this boundary is smeared out. For simplicity, in order to clarify the principle of the problem, we shall not take into account diffusion, radiative heat transfer and the finite velocity of establishing the ionization equilibrium. In the transition layer between the two homogeneous phases corresponding to two points, e.g., E and G, all intermediate values of the density (the segment EG) and the corresponding intermediate temperatures are realized. The pressure is constant in the intermediate layer. The states in the intermediate layer do not satisfy the conditions of thermal balance which hold for a homogeneous phase. The difference between heating and cooling is compensated by heat conduction.

In a layer at rest with respect to the gas the following equation holds:

$$-\frac{d}{dx}\left(\lambda\frac{dT}{dx}\right)=\rho F(\rho, T, q)=\Phi(p, T, q),$$

where the transition to the variables was carried out because in the limits of the layer under consideration p and q are constant parameters. The thermal conductivity λ can also be expressed as a function of p and T. For given p and q is is necessary to find a solution which gives the temperature distribution T(x) in the transition layer. This solution has a form which is schematically illustrated in Fig. 2. The curve T(x) asymptotically approaches T_E for $x \to -\infty$, and approaches T_G for $x \to +\infty$. The equilibrium curve in Fig. 1 corresponds to the condition of thermal balance, consequently at each point of that curve, including the points E and G, the following equality holds

$$y=\lambda\frac{dT}{dx},\quad \frac{d}{dx}\lambda\frac{dT}{dx}=\frac{y}{\lambda}\frac{dy}{dT}=\frac{1}{2\lambda}\frac{dy^2}{dT}=-\Phi,$$

It is easy to see that $\Phi > 0$ above the curve ABCD and $\Phi < 0$ underneath this curve.

A stationary solution is obtained by means of the elementary transformation:

$$y^2|_{T=T_G}-y^2|_{T=T_E}=0=-2\int_{T_E}^{T_G}\lambda(p, T)\Phi(p, T, q)\,dT. \qquad (1)$$

The integral condition determines the selection of the pair E, G of points corresponding to dynamic equilibrium, i.e., selects such a value of the pressure p, that under the given conditions of heating and given parameter q guarantees a stationary equilibrium of the two phases. The condition (1) replaces (Maxwell's) area rule of the thermodynamic theory of a nonideal gas. It is obvious that the solution lies somewhere between the extreme lines AC and BD, that the integral vanishes owing to compensation of the contributions of the portions EF and FG, along which the sign of Φ is different.

What happens in the case of "incorrect" selection of the pressure, when the integral does not vanish? Obviously, one phase will transform into the other with a definite rate, corresponding to a flow of mass m, in g/cm^2 sec. In a coordinate system tied to the separation surface the temperature distribution does not depend on time, but at each point there is flow of matter (to the left, Fig. 2) at a rate $|u| = m/\rho$. The heat equation has the form

$$\frac{c_p m}{\rho}\frac{dT}{dx}-\frac{d}{dx}\lambda\frac{dT}{dx}=\Phi. \qquad (2)$$

The parameters p and q are given, Φ, λ, and c_p depend on T, and Φ vanishes at the endpoints of the region of integration, as well as at some interior point. The equation has a solution of the form represented in Fig. 2; $T \to$ const for $x \to -\infty$ and $x \to +\infty$ for a single value of the parameter m, value which has to be determined. The problem is posed in a manner similar to the problem of flame propagation[3,4]. The difference consists in the fact that Φ may be both positive or negative, and thus also the sign of m is not given a priori (i.e., whether the colder phase turns into the hotter one or vice versa). At the middle of the interval under investigation there is a value of p for which m = 0. The exact solution of the problem requires a numerical integration of the equation, or, more precisely, of that first-order equation for y(T) which is obtained from (2). The integration has to be done by trial and error, selecting a value of m for which the boundary conditions are satisfied. According to[4] one can then check the uniqueness of the solution.

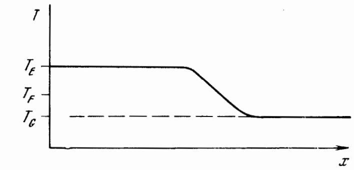

FIG. 2. Schematic temperature variation at the boundary between the two phases.

Following the ideas of[3] one can propose a semi-empirical method, yielding a result close to reality over the whole range of p. We start from the integral relation

$$m\int_{T_1}^{T_2}c_p\,dT=m[H(T_2)-H(T_1)]=\int_{-\infty}^{+\infty}\Phi\,dx,$$

where $T_1 = T(x = -\infty)$, $T_2 = T(x = +\infty)$, H(T) is the enthalpy of the gas, $H = \int c_p dT$. It would seem that the improper integral is hardly suitable for numerical estimates. It can, however, be converted into a "good" integral with respect to dT, if one derives from the equation m = 0 an approximate relation between T and x. Thus a first step of an iterative procedure is proposed: given $m_0 = 0$, find m_1 from the integral relationship. Unfortunately the next step of iteration cannot be done in an elementary way. It is practical to introduce the variable z:

$$z=\int^{T}\lambda\Phi\,dT,\qquad dz=\lambda\Phi\,dT=\Phi y\,dx.$$

For m = 0, $y = (2\int^{T}\lambda\Phi dT)^{1/2} = (2z)^{1/2}$, so that

$$\int\Phi\,dx=\int\frac{\Phi y\,dx}{y}=\int\frac{dz}{\sqrt{2z}}=\sqrt{2z}$$

Since in the general case Φ changes sign in the middle of the region, one must integrate separately over each of the subregions. The boundary is T_3 at the point F, and we denote the temperatures at the points E and G by T_1 and T_2, respectively ($T_1 > T_3 > T_2$). In each subregion we take the integral from the infinitely remote (in x) point, i.e.,

$$y=-\left(2\int_{T}^{T_1}\lambda\Phi\,dT\right)^{1/2},\qquad y=-\left(-2\int_{T_2}^{T}\lambda\Phi\,dT\right)^{1/2}$$

corresponding to the first and second zones. Finally, we obtain:

$$m=\left\{-\left(2\int_{T_2}^{T_3}\lambda|\Phi|\,dT\right)^{1/2}+\left(2\int_{T_3}^{T_1}\lambda\Phi\,dT\right)^{1/2}\right\}\Big/[H(T_1)-H(T_2)], \qquad (3)$$

where we have selected as positive that sign of m which corresponds to the transformation of the cold phase into the hot one. It is easy to see that m = 0 is an exact result, since for this it is required that the integrals over the two subregions be equal.

We verify the degree of approximation on a model. We choose

$$T=\alpha e^{-(1-\alpha)x}\quad(x<0),\qquad T=1-(1-\alpha)e^{-\alpha x}\quad(x>0),$$

which satisfies the requirements of continuity of T and dT/dx everywhere, including x = 0, and the boundary conditions $T = 0$, $x = -\infty$, $T = 1$, $x = +\infty$. We choose m_0

arbitrary in the equation

$$m_0 \frac{dT}{dx} + \frac{d^2T}{dx^2} = \Phi.$$

The exact solution Φ corresponding to the chosen T(x) and m is

$$\Phi(T) = [m_0(1-\alpha) + (1-\alpha)^2]T, \quad 0 < T < \alpha,$$
$$\Phi(T) = -[-m_0\alpha + \alpha^2](1-T), \quad \alpha < T < 1,$$

and for this Φ we determine an approximate value of m_1 according to the proposed formula

$$m_1 = \{[m_0(1-\alpha) + (1-\alpha)^2]\alpha^2\}^{1/2} - \{[-m_0\alpha + \alpha^2](1-\alpha)^2\}^{1/2}.$$

For $m_0 = 0$, $m_1 = 0$. For $\alpha \approx 0.5$ and $m_0 \ll 1$ we find

$$m_1 = (1-\alpha)\cdot\frac{1}{2}\frac{m_0}{1-\alpha} + (1-\alpha)\alpha\cdot\frac{1}{2}\frac{m_0}{\alpha} = \frac{m_0}{2}.$$

The maximal limits of variation of m_0, compatible with $\Phi > 0$ in the first region and $\Phi < 0$ in the second are:

$$-(1-\alpha) < m_0 < \alpha.$$

At the ends of the region it is easy to see that $m_1 = m_0$. Thus it is likely that the proposed approximate formula does not give errors larger than two times and yields the condition m = 0 exactly.

We now apply the relations obtained here to the interstellar gas heated by cosmic rays. The quantity Φ can be written for a weakly ionized gas in the form

$$\Phi = N_{\mathrm{H}}\left[q\frac{13}{35}\left(1 + 13\frac{N_e}{N_{\mathrm{H}}}\right) - N_e L(T)\right],$$

where the ionization is determined by the condition

$$\frac{N_e^2}{N_{\mathrm{H}}} = \frac{1{,}38}{35\cdot 1{,}6\cdot 10^{-12}}\frac{q}{\alpha'(T)} = bq\,T^{1/2}.$$

Here q are the ionization losses of the cosmic rays per neutral hydrogen atom per second, $N_H N_e L(T)$ is the rate of radiative cooling per cm^3, $\alpha'(T) = 2.55 \times 10^{-11} T^{-1/2}$ is the recombination coefficient for hydrogen at a level higher than the first. Replacing N_e and N_H in terms of the pressure we finally bring the expression of Φ to the form

$$\Phi(p, T) = \frac{13}{35}\frac{q^2}{kT}\frac{p}{q}\left[1 + 13T^{3/4}\left(kb\frac{q}{p}\right)^{1/2} - \frac{35}{13}T^{-1/4}\left(\frac{b}{k}\frac{p}{q}\right)^{1/2}L(T)\right].$$

The thermal conductivity is determined by the neutral atoms. Substituting into (1) Φ and $\lambda \approx 0.7 \times 10^3\, T^{1/2}$ we calculate the function in the integrand for different values of the parameter p/q. In Fig. 3 the curves are illustrated for p/q equal to 7×10^{11}, 11×10^{11} and 14×10^{11}. The mean value satisfies the condition of stationarity. The limiting temperatures for the two media are then equal to 90 and 12000°. From the condition $\Phi = 0$ for T = 12000° we find for the hot phase the relations $N_H = 6.5 \times 10^{23}q$, $N_e = 2.7 \times 10^{23}q$, i.e. $N_H = 2.4\, N_e$. The electron concentration is not very small compared to the concentration of atoms, so that the result given above has the character of an estimate, with a relatively small error. The average value of N_e may be estimated from the magnitude of the Faraday rotation for polarized radio emission, and N_H can be estimated from the radio emission in the line of 21 cm wavelength. In correspondence with these data we adopt the pair of values $N_H \approx 0.05\ cm^{-3}$, $N_e \approx 0.02\ cm^{-3}$, then $q \approx 8 \times 10^{-26}$ erg/sec. If the ionization is produced by low energy cosmic rays, 10^{-7} particles per cm^3 are necessary; if the ionization is produced by relativistic electrons, the number must be several times larger. The average proton energy must be smaller than 10 MeV, and that of the electrons smaller than 3–5 MeV, in order that the total pressure of these particles be smaller than the magnetic pressure in the interstellar gas. This condition is necessary for stability of the gas in a magnetic field.

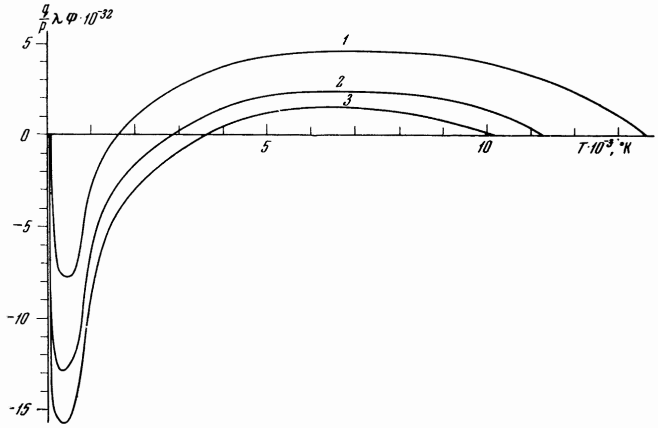

FIG. 3. The curves representing $q\lambda\Phi/p$ as a function of T for various values of the parameter p/q: $1 - 7 \times 10^{11}$, $2 - 11 \times 10^{11}$, $3 - 14 \times 10^{11}$ cgs esu.

In order to estimate the thickness of the transition layer we use the condition (1). The quantity y is maximal at the point T_3 and equals

$$y(T_3) = \left\{2\left|\int_{T_1}^{T_3}\lambda\Phi\,dT\right|\right\}^{1/2}.$$

This yields a thickness of the transition layer of $\Delta x \approx 2\lambda(T_3)T_3/y(T_3) \approx 5 \times 10^{13}$ cm.

The average density of interstellar gas changes with time. Through this gas spiral arms move in the form of waves[5]. When the gas gets into the sleeve, it becomes compressed. Between the sleeves the gas will in general be in the rarefied phase[1]. The compression should reach the point B, after which clouds are formed. When the process of compression of the sleeve stops, the clouds evaporate gradually, and the conditions approach stable ones. The velocity of motion of the boundary between the two phases computed according to Eq. (3) is maximal for maximal deviation from equilibrium and decreases asymptotically as the stable state is approached. If the deviation corresponds to the curve p/q = 14×10^{11}, the mass flow is $m = 4 \times 10^{-22}$ g/cm$^2\cdot$sec. This corresponds to a velocity of the front relative to the cloud of $u \approx 40$ cm/sec (for $\rho \approx 10^{-23}$ g/cm^3). Over 10^6 years the front will move 10^{15} cm, i.e. over a small portion of the cloud. Therefore one may consider that the state which is realized is determined more by the initial conditions, than by the stationarity conditions.

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