BOSE CONDENSATION AND SHOCK WAVES IN PHOTON SPECTRA

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The process of establishment of equilibrium in a system consisting of radiation and totally ionized plasma is investigated. By solving the kinetic equation it is shown that in the absence of absorption the photons undergo Bose condensation. The process depends essentially on the form of the initial distribution. For a certain form of the initial spectrum a shock wave occurs in the spectrum in the course of its temporal evolution. The process is substantially affected by absorption, in the presence of which Bose condensation is replaced by an accumulation with time of the photons in the region of low frequencies.

INTRODUCTION

V. L. GINZBURG and L. V. Keldysh have posed the problem whether it is possible for Bose condensation to occur for noninteracting bosons in the case when the processes of change of energy and momentum in scattering dominates over processes involving change of particle numbers in their emission and absorption. Below we consider this problem for the concrete example of a system of photons.

In the present paper we shall consider the process of establishment of equilibrium in a system consisting of radiation and a fully ionized plasma. We shall assume that at the initial instant of time the whole system is in a state of incomplete equilibrium. The photon gas is characterized by an arbitrary initial distribution with some average energy E_{ph}. This may, in particular, be even a Planck distribution with a characteristic temperature T_{ph}. The electron gas is considered not to be degenerate and is described by a Maxwell distribution with the temperature T_e, which is maintained at a constant value. We assume that as a whole the system is spatially homogeneous and closed, which also means that no particles leave the system or are added to it from the outside.

The statistical equilibrium between the photons and the plasma will establish itself as a result of both scattering processes, which do not involve a change of the number of photons, and processes involving the emission and absorption of photons. We shall assume that the fundamental process leading to the absorption of photons are free-free transitions in the field of the nuclei. We shall not take into account electron-electron scattering with photon emission or absorption, since these processes cannot introduce radical modifications of the results, and in no case can such processes exceed the contribution from Bremsstrahlung processes on nuclei. It should be noted that scattering is proportional to the density L of the electrons and does not depend on the frequency, whereas absorption is proportional to $L^2\nu^{-3}\ln\nu$. Therefore, the lower the density L, the lower will be the frequency ν_{lim} below which it is no longer possible to neglect absorption.

Usually, in discussions of the establishment of equilibrium between radiation and matter, scattering processes have not been taken into account at all. The first to call attention to the importance of scattering processes in the interaction of photons with the electron gas was Kompaneets[1], who considered the behavior of a system of photons during collisions between the electrons. As a result of free-free transitions quanta are produced with average energies close to $0.5\ kT_e$. After that the photon energy increased to $3kT_e$ owing to collisions with electrons.

Recently the scattering of photons with change in energy was considered by Weyman[2] in connection with cosmological problems[1)].

We shall consider the opposite situation, when the initial photon temperature is substantially higher than the electron temperature. The fundamental result of this paper consists of the following: the kinetics of the relaxation process in the indicated system differs substantially from the kinetics in a system without scattering. The transition to the final equilibrium distribution (a Planck distribution with temperature T_e) is extremely nonuniform across the frequency spectrum. This nonuniformity is an expression of the tendency of a boson system to effect a transition into the state with the lowest energy (Bose condensation). The character of the kinetics of the process of establishment of equilibrium turns out to depend strongly on the form of the initial distribution of the bosons. In particular, for a given initial distribution of the photons the occurrence of shock waves in the energy spectrum becomes possible. Although the results were obtained for a concrete system of photons, some of the results are not related specifically to photons and may be true for other boson systems.

We consider first of all the time variation of the distribution function in a model system without absorption.

1. THE ESTABLISHMENT OF EQUILIBRIUM IN A SYSTEM WITHOUT ABSORPTION IN THE CASE $T_e > T_{ph}$

We first discuss briefly the physical results derived in[1]. The analysis carried out by Kompaneets was based on considering the kinetic equation for the photon

[1)]In particular some considerations were expressed in [2] on the possibility of accumulation of photons in the low-energy region under similar conditions.

distribution function:

$$\frac{\partial n}{\partial t} = -\int d\tau_i \int dW[n(1+n')N(\varepsilon) - n'(1+n)N(\varepsilon + h\nu - h\nu')], \quad (1)$$

where $N(\epsilon)$ is the Maxwellian distribution of the free electrons, dW is the differential transition probability of photons from one state into another during a collision with an electron and $d\tau$ is the phase space volume element of the electrons. In this equation emission and absorption processes have not been taken into account, so that the transitions are produced exclusively by Compton scattering processes. This means that the distribution function in the kinetic equation satisfies the condition of constancy of the particle number.

In the sequel it will be convenient to use dimensionless variables related to the constant electron temperature T_e:

$$t' = t\frac{mc^2}{kT_e}\left(\frac{l}{c}\right), \quad x = \frac{h\nu}{kT_e};$$

here t′ is the real time, l is the Compton scattering mean free path, determined by the total Thomson scattering cross section. In terms of these variables the condition that the particle number be constant can be written in the form

$$N = \int_0^\infty n(x, t)x^2dx = \mathrm{const}, \quad (2)$$

where N is the total number of photons, and n(x, t) is the number of photons in a phase space cell with energy x at time t.

If the average electron energy or photon energy is smaller than the rest energy mc^2 of the electrons, one may assume that the energy exchange between electrons and photons (frequency variation of the photons) will occur in small portions:

$$h\Delta = h(\nu' - \nu) \ll h\nu. \quad (3)$$

Under these conditions it was shown in[1] that the equation (1) can be reduced to a differential equation

$$\frac{\partial n}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n^2 + n\right)\right]. \quad (4)$$

It is not difficult to see that the right-hand side of the equation (4) vanishes not only if one inserts the Planck distribution belonging to temperature T_e, i.e.,

$$n(x) = [e^x - 1]^{-1}, \quad (5)$$

but also for a Bose distribution with dimensionless chemical potential $\mu = \mu'/kT_e$, i.e., the function

$$n(x, \mu) = [e^{x-\mu} - 1]^{-1} \quad (6)$$

for $\mu < 0$.

The distribution (6) corresponds to a total number of quanta, $N(\mu)$, smaller than the one corresponding to a total Planck equilibrium distribution with $\mu = 0$, i.e.,

$$N(\mu) < N(0).$$

In the absence of emission and absorption N = const. Consequently the stationary distribution (6) appears in the case in which for the initial state such an n(0, x) is given that

$$N_0 = \int n(0, x)x^2dx < N(0).$$

Thus, if one defines the initial distribution in the form of a Planck distribution with the temperature $T_{ph} = T_e/\alpha$, $\alpha > 1$, we find

$$N_0 = \alpha^{-3}N(0) < N(0).$$

Here the chemical potential is $\mu \sim -\ln\alpha$. A similar situation with μ slowly approaching zero from the left owing to emission of quanta (but with a distorted spectrum for small values of x) has been considered in[1],

2. THE ESTABLISHMENT OF EQUILIBRIUM IN A SYSTEM WITHOUT ABSORPTION IN THE CASE $T_{ph} > T_e$

It is completely obvious that the preceding reasoning cannot be applied to the case $T_{ph} > T_e$ (or $\alpha < 1$). Indeed, if at the initial instant of time we have $\mu = 0$ and the establishment of equilibrium is accompanied by a lowering of the photon temperature, this would imply in the Bose distribution a transition to values $\mu > 0$, which is impossible. Therefore the transition to equilibrium in the system must have the characteristic of a Bose condensation: in an equilibrium state with fixed number of photons their cooling down is possible only by means of a transition of the excess into states of zero energy. We consider the kinetics of the process of Bose condensation in a quantitative way and then take into account the modifications required by considering absorption-emission processes.

Thus, we shall consider that an initial distribution is given for the photons with respect to energy: n(0, x), such that the total number of particles $N_0 > N(0)$ and E_{ph} is larger than the average electron energy E_e.

Starting at time t = 0 the photons are subject to Compton scattering on free electrons. We shall only be interested in the form of the distribution function in the region of large occupation numbers. We assume that in the region of low frequencies the occupation numbers are extremely large, so that one may consider that $n \gg 1$. This leads to an essential simplification of the kinetic equation. It follows from (4) that

$$\frac{\partial n}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left\{x^4\left(\frac{\partial n}{\partial x} + n^2\right)\right\}. \quad (7)$$

We shall assume that in Eq. (7) the following inequality is satisfied:

$$n^2 \gg |\partial n/\partial x|. \quad (8)$$

Below we shall discuss the meaning of this inequality and the limits of its applicability.

In the special case of an initial Planck distribution the terms $|\partial n/\partial x|$ and n^2 are of the same order in x at time t = 0, but their ratio is of the order of $\alpha < 1$.

Thus the equation for the distribution function n(x, t) takes the form

$$\frac{\partial n}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}(x^4n^2). \quad (9)$$

Introducing the new unknown function $f = x^2n$, $N = \int f dx$, we have

$$\partial f/\partial t = \partial f^2/\partial x, \quad (10)$$

which can be written in terms of characteristics as df/dt = 0 along the characteristic dx/dt = f. The general solution of the characteristic equation (10) is

$$x = F(f) - 2tf, \quad (11)$$

where the form of the function F(f) is determined by the initial condition. This solution satisfies, of course, the requirement that the area under the curve f(x, t) be constant, which corresponds to the condition (2) that the particle number be constant.

We now discuss the meaning of the solution obtained in this manner on the example of an initial distribution of the form illustrated in Fig. 1. According to Eqs. (10) and (11), all points on the initial curve f(x, t = 0) (curve 1) move along characteristic lines—straight lines parallel to the x axis—in the direction of decreasing x (Fig. 1, curve 2) with a velocity proportional to f. The time in which a given point reaches the axis x = 0 is obviously determined by the expression

$$\tau = F(f)\,/\,2f. \tag{12}$$

The solution (11) is formally valid both for positive and for negative values of x. Thus, as time increases f(x, t) must take on the form represented by curve 3 in Fig. 1. It is clear that since a transition to negative x is impossible, the particles reaching the f-axis accumulate in the state x = 0, i.e., undergo Bose condensation. Formally the shaded area in Fig. 1 determines the number of particles in the Bose condensate at each instant of time.

In order to follow in more detail how the process of Bose condensation occurs and to explain the meaning of the original inequality (8), we consider the special case of an initial Planck distribution with $T_{ph} > T_e$ in the frequency region which corresponds to large occupation numbers, i.e., we set

$$n(0,\,x) = [e^{\alpha x}-1]^{-1} \approx [\alpha x + \alpha^2 x^2/2]^{-1}, \quad \alpha x < 1. \tag{13}$$

It will be explained below why second order terms have been retained in the expansion (13) of the Planck formula. The initial condition yields

$$F(f) = \alpha f[1-\alpha^2 f/2]^{-1}. \tag{14}$$

Whence, substituting (14) into (11) we obtain

$$f = \frac{1}{\alpha^2 t}\left\{\left(2t\alpha - \alpha - \frac{\alpha^2 x}{2}\right) \pm \left[\left(\alpha - 2t + \frac{\alpha^2 x^2}{2}\right)^2 + 2\alpha^2 t x\right]^{1/2}\right\}. \tag{15}$$

For x > 0 one has to choose the plus sign in front of the square root in (15). We obtain for the duration of the Bose condensation

$$\tau = \frac{\alpha}{2(1-\alpha^2 f/2)}. \tag{16}$$

The time $\tau = \alpha/2$ is the minimal critical time in which the state x = 0 starts filling up with particles. To each value of f corresponds a specific, always well-defined finite time of Bose condensation. We note that if the initial distribution would, in the region $x < 1/\alpha$, have the character of a straight line through the origin: $f = x/\alpha$ we would be dealing with a degenerate case for which the whole straight line would approach the f-axis in the (f, x) plane during the same amount of time $\tau = \alpha/2$. This is the reason why we have retained in the expansion (15) small quantities of second order: even a small curvature of the initial distribution has an essential influence on the kinetics of the process. A direct calculation shows that the inequality (8) is valid for all values of $x \geq 0$. In particular, for $t = \tau = \alpha/2$ and $x \to 0$ (from the side of positive values of x) $\partial n/\partial x \backsim x^{-5/2}$, whereas $n^2 \backsim x^{-3}$.

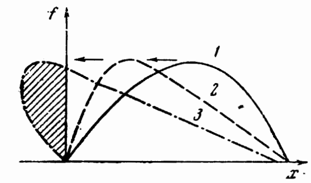

FIG. 1.

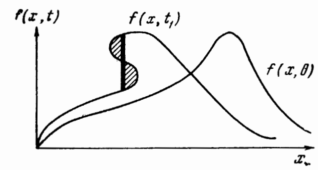

FIG. 2.

It is quite clear that the solution of the kinetic equation is meaningful only in the region $x \geq 0$, $f > 0$.

However, the extension of the solution of the simplified equation (9) to the region x < 0 allows one to determine the number of particles which have left the region x > 0 and have effected transitions into the Bose condensate.

We have already stressed the fact that the character of the kinetics depends in many respects on the form of the initial distribution n(0, x). Let us consider, in particular, the initial distribution illustrated in Fig. 2. In distinction from the distribution in Fig. 1 it has an inflection point. According to the solution (11), the distribution function will be subject to such a deformation with time, that it will take on the form shown in Fig. 2. The velocity of approach to the f-axis increases as f increases, so that the upper parts of the graph of f(x, t) will advance compared to the lower parts. As a result the distribution function may bend over so much that the curve f(x, t) stops being single-valued, as illustrated in Fig. 2.

There appears a situation completely analogous to the formation of shock waves in the one-dimensional flow of a nonviscous fluid. In reality the function f(x, t) does not become many-valued. As the shock front (represented by the heavy line in Fig. 2) is approached, the derivative $\partial n/\partial x$ tends to infinity. This leads to a violation of the condition (8). Consequently, in the same manner as for the appearance of shock waves in a viscous fluid, the presence of higher derivatives (cf. Eq. (7)) insures that the front is not strictly vertical and exhibits a certain structure and width.

3. CONSIDERATION OF PHOTON ABSORPTION

Above it was assumed that the existence time of the photons is large compared to the time of Bose condensation. In a real system one must take into account the absorption of photons which increases sharply as the frequency decreases. Taking into account absorption the kinetic equation takes on the form (for x < 1)

$$\frac{\partial f}{\partial t} = 2f\frac{\partial f}{\partial x} - \frac{Af}{x^2}\ln\frac{2{,}35}{x^2}, \tag{17}$$

where A is a constant.

We see from (17) that the absorption may be neglec-

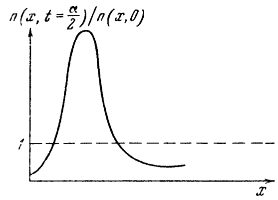

FIG. 3.

ted if the following inequality holds:

$$\frac{\partial f}{\partial x} \gg \frac{A}{x^2} \ln \frac{2{,}35}{x^2}. \tag{18}$$

It is clear that since $\partial f/\partial x$ is everywhere finite, the inequality determines the limiting frequency x_{lim} below which absorption predominates over scattering from the very beginning of the process. In this region, over a time smaller than the time of Bose condensation a Planck distribution with temperature T_e establishes itself. Thus, the absorption does not allow the Bose condensation to be realized as such. However in the region where the inequality (18) holds the character of the temporal development of the distribution function corresponds to the picture described in the previous section.

Figure 3 illustrates schematically the dependence of the ratio $n(x, t)/n(x, 0)$ on x at the time $t = \alpha/2$ for the special case of a Planck initial distribution. We see that the processes considered above lead to a significant increase in the number of particles in the region of sufficiently small x, but such that the condition $x > x_{\text{lim}}$ is still satisfied. It is completely clear that the concrete value of the limiting frequency as well as the time during which scattering processes are essential without an essential influence of absorption, depends on the form of the initial distribution function. In addition, the quantity x_{lim} is mainly determined by the constant $A \sim L/T_e^{9/2}$. Therefore the described phenomenon is extremely clearly expressed in a rarefied hot plasma.

It is easy to note however that for photon scattering in a rarefied plasma the electron temperature will, in general, increase.

One can, of course, imagine a situation for which the temperature of the rarefied plasma remains constant, so that the situation described above is realized in full. Namely, let the initial distribution $n(x, 0)$ be such that in the region of small x the number of photons is larger than that of an equilibrium Planck distribution, and that in the region of large x it is smaller than that of the Planck distribution with temperature T_e. At the same time the total number of photons N_0 is larger than $N(0)$, and $E_{ph} = E_e$. Then in the region $x < 1$ there will occur an accumulation of photons as described above. In the region of large photon frequencies, photons which are scattered by "hot" electrons a Wien distribution with temperature T_e will establish itself. The system of electrons as a whole remains at constant temperature T_e.

Until now we have considered the kinetics in the system consisting of photons and free electrons. It is important, however to keep in mind that the results formulated above have a general character. The method of using the kinetic equation can be extended to other boson systems, e.g., excitons in solids, or transverse plasma oscillations. The principal difference will be a different dependence of the cross sections for scattering and absorption on the energy.

It is not excluded that similar phenomena occur in astrophysical conditions. In a series of cases hot radiation from interior regions passes through "cold" layers of plasma. In distinction from the problem considered above, the phenomenon bears a stationary character. But the density distribution of matter is not isotropic and homogeneous. Therefore it is impossible to carry over directly the preceding results to this case. However, it is not excluded that an excess (in comparison with the Planck equilibrium) number of soft photons may appear as a result of processes which are analogous to those described above. This question requires further study.

CONCLUSIONS

1. From the solutions of the kinetic equations it is shown that in a system of photons undergoing scattering without absorption there occurs Bose condensation.
2. The time (duration) of Bose condensation has been determined. This time is always finite and differs for particles with different energies.
3. Owing to the nonlinearity of the process its evolution depends significantly on the form of the initial photon distribution function. In particular, for certain forms of the initial distribution the formation of shock waves in the spectrum becomes possible.
4. Taking into account absorption shows that the tendency to undergo Bose condensation leads to an accumulation of photons in the region in the low-energy region. However the absorption process in the very low-frequency region occurs faster than the process of Bose condensation.

[1] A. S. Kompaneets, Zh. Eksp. Teor. Fiz. 31, 876 (1956) [Sov. Phys.-JETP 4, 730 (1957)].
[2] R. Weyman, Phys. of Fluids 8, 12 (1965).

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