INSTABILITY OF A PLASMA IN A STRONGLY INHOMOGENEOUS MAGNETIC FIELD

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We investigate nonpotential electronic plasma oscillations with frequency $eH/m_ic \ll \omega \ll eH/m_ec$ in a strongly inhomogeneous magnetic field. We consider the instabilities of a nonlinear magnetosonic wave in a rarefied plasma. It is shown that a front of width δ smaller than c/ω_{pi} may turn out to be unstable against perturbations of the "whistler" type with scale δ. These instabilities develop within a time shorter than the time of steepening of the nonlinear wave, and should prevent formation of a front with a width smaller than c/ω_{pi}. It is shown in addition that the front of a nonlinear magnetosonic wave with width smaller than $(m_i/m_e)^{1/4}c/\omega_{pe}$ can be unstable against high-frequency perturbations ($\omega > eH/(m_im_e)^{1/2}c$) with $\mathbf{k}\cdot\mathbf{H} = 0$.

1. In the present article we investigate the stability of plasma states realized in experiment on collisionless shock waves in a rarefied plasma situated in a magnetic field.[1-3] In this case the most interesting physical question is that of the mechanism of dissipation of the front of a nonlinear magnetosonic wave under conditions when the pair collisions in a plasma do not play an essential role. Credit for formulating this problem belongs to R. Z. Sagdeev. Theoretical investigations in this direction have shown that the collisionless dissipation necessary to form the profile of the shock wave can be connected with the small-scale instability of the current flowing on the front of the wave.[4-6]

However, besides the small-scale instabilities that lead to a decrease of the effective range of the electrons in the plasma, there can occur on the front of a nonlinear magnetosonic wave also instabilities with wavelengths comparable with the width of the front. Such instabilities are the subject of the present study. We shall show that under certain conditions large-scale instabilities can determine to a considerable degree the structure of the wave front. It should be noted that the instability of a nonlinear magnetosonic wave and, in particular, of a steady-state shock wave, against large-scale perturbations was investigated also earlier, but only within the framework of the equations of magnetogas dynamics[7,8]. However, theoretical considerations and experimental data pertaining to collisionless shock waves show that the width of the front of such a wave lies in the interval of values from c/ω_{pe} to c/ω_{pi} ($\omega_{p\alpha}^2 = 4\pi me^2/m_\alpha$, $\omega_{H\alpha} = eH/m_\alpha c$). At such scales it is necessary to use the equations of two-component gasdynamics for electrons and ions.

The most significant circumstance that makes it possible to simplify the stability investigation within the framework of two-component gas dynamics in a magnetic field is the existence of high-frequency oscillations of the "whistler" type, which can propagate in the plasma transversely to the magnetic field at a velocity much larger than the velocity of the shock wave. This makes it possible, in investigations of stability against such oscillations, to neglect the change of the form of the profile of the magnetic field in the unsteady wave and the flow of the plasma through the wave front. Therefore, in most of the results presented below, the plasma flow velocity through the front and the rate of deformation of the profile of the magnetic field are not contained in explicit form. But the values of the magnetic-field gradient and of the pressure on the front of the nonlinear wave can themselves be determined essentially by the rate of flow of the plasma through the wave front:

$$m_in_0\left\{\frac{\partial \mathbf{v}_0}{\partial t}+(\mathbf{v}_0\nabla)\mathbf{v}_0\right\}=-\nabla p_0+\frac{1}{4\pi}[\operatorname{rot}\mathbf{H}_0,\mathbf{H}_0]. \qquad (1)^*$$

This is precisely why some of our results can be applied directly to a plasma at rest.

We note also the following. As is well known, stationary nonlinear solutions of the equations of two-component magnetogasdynamics in a collisionless plasma,[4] or solutions describing nonlinear flows ahead of a magnetic piston[8] in a dense plasma ($\omega_{pe} \gg \omega_{He}$) have characteristic magnetic-field and density oscillations with a scale c/ω_{pe}. We shall not consider henceforth the stability of smooth and broader profiles, where there are no oscillations on such a scale. We have made this choice, first, because in experiments with a dense plasma the profile of the wave front is relatively smooth and its width is $\delta \gtrsim (7-10)c/\omega_{pe}$.[1-3] The absence of oscillations with scale c/ω_{pe} can be explained by taking into consideration the possible effect of the scattering of electrons by the small-field turbulent electric fields resulting from the instability of the current on the wave front.[5,6]

Second, as shown by calculation[9] and by experiment,[1] if oscillations with scale c/ω_{pe} do appear at all, it is in the final stage of the steepening of the nonlinear magnetosonic wave. The instability observed by us, on the other hand, can develop much earlier.

Thus, let us consider the stability of the profile of a nonlinear magnetosonic wave against small perturbations within the framework of the equations of two-component gas dynamics. We confine ourselves here to an investigation of the stability of the front of a plane nonlinear wave moving along the x axis perpendicular to the main magnetic field $\mathbf{H}_0$, which in turn is directed along the z axis. The plasma mass velocity v_{0x} is directed along the x axis, and the electron current veloc-

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*$[\operatorname{rot}\mathbf{H}_0, \mathbf{H}_0] \equiv \operatorname{curl}\mathbf{H}_0 \times \mathbf{H}_0$.

ity v_{0y} is directed along the y axis. All the foregoing quantities, and also the density n_0 and the pressure p_0, vary only in the direction of the x axis. We denote the width of the wave front by δ, and in some cases the profiles of the different quantities may have different widths. In the investigation of the stability, we use the hydrodynamic approximation for the electrons:

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = 0, \tag{2}$$

$$m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}[\mathbf{vH}] - \frac{\nabla p_e}{n} - m_e \nu \mathbf{v}, \tag{3}$$

$$\frac{d}{dt}(p_e n^{-\gamma}) = 0 \tag{4}$$

and Maxwell's equations.

The motion of the ions in oscillations of the "whistler" type can be neglected if $\omega \gg \omega_{Hi}$ and $k_z \gg \omega_{pi}/c$. Thus, we shall consider the instabilities connected only with the motion of the electronic component of the plasma, and not affecting the ionic component. Owing to the quasineutrality we have here $n_{1e} = n_{1i} = 0$.

The hydrodynamic equations (3) and (4) are valid only if the collision frequency ν is high enough. The collisions can be either of the pair type or scattering by small-scale fluctuations of the electric field. The choice of the equation for p_e is determined by the character of the oscillations. Thus, if the thermal conductivity of the electron gas along the magnetic field is high, $\omega \ll k_z^2 v_{Te}^2/\nu$, then $\gamma = 1$. In this case the electron temperature on the force line is constant, i.e.,

$$\frac{\partial T_e}{\partial t} + c\frac{[\mathbf{EH}]}{H^2}\nabla T_e = 0. \tag{5}$$

2. In this section we consider perturbations for which $k_z/k \gg \omega/\omega_{He}$. Then the motion of the electronic component of the plasma in small oscillations can be regarded as non-inertial. The effect of the ohmic dissipation in such oscillations can be neglected if $\nu/\omega \ll \omega_{pe}^2/k^2c^2$. If we use the assumptions made and take the curl of Eq. (3), expressing the electron velocity $\mathbf{v}$ in this equation in terms of curl $\mathbf{H}$, then the equations for the investigation of the stability of the nonlinear wave take the form

$$\frac{4\pi e}{c}\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}\left(\frac{1}{n}[\mathbf{H}\operatorname{rot}\mathbf{H}] + \frac{4\pi}{n}\nabla p_e\right), \tag{6}$$

$$\frac{d}{dt}(p_e n^{-\gamma}) = 0. \tag{7}$$

The simplest way of reducing the linearized system of Eqs. (6) and (7) to a single equation is as follows. It is necessary to take the x components of the first equation and of its curl. Then we obtain, together with the second equation, a system of equations with respect to the variables H_{1x}, $(\text{curl } H)_{1x}$, and p_{1e}. Eliminating from them $p_{1e} = iH_{1x}p'_{0e}/k_z H_0$ and $(\text{curl } H)_{1x}$, we obtain one differential equation for H_{1x}

$$\frac{d^2H_x}{dx^2} + H_x\left\{-k^2 - \frac{n}{H}\left(\frac{H'}{n}\right)' + \frac{k^2}{k_z^2}\frac{4\pi n'\bar{p}'_e}{nH^2} + \frac{\omega_{pi}^4}{k_z^2c^4}\frac{(\omega - k_y v_H)(\omega - k_y v_n)}{\omega_{Hi}^2}\right\} = 0,$$

где $\bar{p}_e' = n^\gamma (p_e n^{-\gamma})'$, $k^2 = k_z^2 + k_y^2$,

$$v_H = \frac{c^2}{\omega_{pi}^2}\omega_{Hi}\frac{H'}{H}, \quad v_n = \frac{c^2}{\omega_{pi}^2}\omega_{Hi}\frac{n'}{n}, \tag{8}$$

and the prime denotes differentiation with respect to x. We have omitted the indices designating the unperturbed and perturbed quantities.

If the plasma is homogeneous, the result is a dispersion equation for oscillations of the "whistler" type:

$$-k^2 - k_x^2 + \frac{\omega^2}{\omega_{Hi}^2}\frac{\omega_{pi}^4}{k_z^2c^4} = 0.$$

To take into account the ion motion, it is necessary to add to the right side of this equation the term $(k^2 + k_x^2)\,\omega_{pi}^2/k_z^2c^2$. Then the oscillations in question go over into magnetic sound when $k_z \ll \omega_{pi}/c$.

We shall now show with the aid of Eq. (8) that a weak nonlinear wave ($\Delta H \ll H$) is stable against perturbations of the surface-wave type, $k\delta \ll 1$. We choose the origin in the center of the profile of the nonlinear wave. Then outside the front of the nonlinear wave, where the magnetic field and the plasma are assumed homogeneous, the solution of Eq. (8) is

$$H_x = \begin{cases} A\exp(\varkappa x), & x < 0, \\ A\exp(-\varkappa x), & x > 0, \end{cases} \quad \varkappa^2 = k^2 - \left(\frac{\omega}{\omega_{Hi}}\frac{\omega_{pi}^2}{k_z c^2}\right)^2, \quad \operatorname{Re}\varkappa > 0.$$

We integrate (8) with respect to x over a region whose dimensions δ_1 satisfy the condition $\delta \ll \delta_1 \ll \kappa^{-1}$. Neglecting under the integral sign the change of H_x and the terms containing no derivatives, we obtain the following dispersion equation for the surface oscillations:

$$2\varkappa = \int dx\left\{-\frac{n}{H}\left(\frac{H'}{n}\right)' + \frac{k^2}{k_z^2}\frac{4\pi n'\bar{p}'_e}{nH^2} + \frac{k_y^2}{k_z^2}\frac{H'n'}{Hn} - \frac{\omega_{pi}^2}{k_z^2c^2}\frac{\omega k_y}{\omega_{Hi}}\frac{(Hn)'}{Hn}\right\}. \tag{9}$$

In order to be able to neglect the change of H_x on the wave front, it is necessary to assume that

$$\frac{k_y^2}{k_z^2}\frac{\Delta H}{H}\frac{\Delta n}{n} \ll 1.$$

It is easy to verify that the dispersion equation written out above has unstable solutions. Thus, neglecting in this equation the two first and last terms under the integral sign, something possible in the case when $\beta_e \ll 1$ and $k_y^2/k_z^2 \gg H/\Delta H$, we obtain

$$\left[k_y^2 - \frac{\omega_{pi}^4}{k_z^2c^4}\frac{\omega^2}{\omega_{Hi}^2}\right]^{1/2} = \frac{k_y^2}{2k_z^2}\int dx\frac{H'n'}{Hn}. \tag{9'}$$

It is seen from the equation that when

$$\frac{|k_y|}{2k_z^2}\int dx\frac{H'n'}{Hn} > 1 \tag{10}$$

the weak shock wave is unstable. If we take as an estimate $|H'|/H \sim |n'|/n \sim 1/\delta$ and use the following limitations on k_z and k_y:

$$k_z \gg \frac{\omega_{pi}}{c}, \quad \frac{H}{\Delta H} \ll \frac{k_y^2}{k_z^2} \ll \left(\frac{H}{\Delta H}\right)^2,$$

Then the instability condition obtained above can be written in the form of the following condition on the front width δ of the nonlinear magnetosonic wave:

$$\frac{1}{\delta} \sim \int dx(\ln\Delta H)'(\ln\Delta n)' \gg \frac{\omega_{pi}}{c}\frac{\Delta H}{H}.$$

The increment of the oscillations under consideration is of the order of

$$\operatorname{Im}\omega \gtrsim \frac{v_{Ai}}{\delta}\frac{k_z c}{\omega_{pi}}\frac{\Delta H}{H} \tag{11}$$

As already noted, the developed theory can be regarded as valid only if the increment of the oscillations greatly exceeds the reciprocal of the characteristic time of deformation of the front of the nonlinear wave. This is true if

$$\frac{H}{\Delta H} \ll \frac{k_z c}{\omega_{pi}} \ll \frac{c}{\delta \omega_{pi}}.$$

Thus, we have shown that a weak straight magnetosonic wave, in which the density and the magnetic field are not very strongly "unfrozen," is stable against perturbations of the "whistler" type. As a result of the development of such an instability the magnetic field components H_x and H_y appear on the front of the nonlinear wave in addition to the fundamental component H_z of the magnetic field.

Of course, strong waves may also turn out to be unstable. Unfortunately, however, it is impossible to write a general dispersion relation for a strong wave. An exact solution can be obtained only for certain particular cases. For perturbations where $k_y = 0$, however, the situation is more definite.

Let us consider Eq. (8) against the background of a strong nonlinear wave at $k_y = 0$ and $\gamma = 1$:

$$\frac{d^2 H_x}{dx^2} + H_x\left\{\left(\frac{\omega}{\omega_{Hi}}\frac{\omega_{pi}^2}{k_z c^2}\right)^2 - k_z^2 - \frac{n}{H}\left(\frac{H'}{n}\right)' + \frac{4\pi n' T_e'}{H^2}\right\} = 0. \quad (12)$$

In the case of a weak wave ($\Delta H \ll H$), this equation is unstable against surface perturbations. The increment of such a solution is determined by the dispersion relation (9), in which it is necessary to put $k = 0$ and $\tilde{p}_e' = nT_e'$.

This conclusion can be formulated also as follows: Eq. (12) has at least one solution with $\kappa^2 > 0$ provided the following integral condition is satisfied:

$$\int dx \left\{\frac{n}{H}\left(\frac{H'}{n}\right)' - \frac{4\pi n' T_e'}{H^2}\right\} < 0.$$

This solution corresponds to stable oscillations when $k_z^2 > \kappa^2$ and unstable oscillations when $k_z^2 < \kappa^2$.

It follows from the foregoing obvious reasoning that the front of a strong wave is also unstable against perturbations with $k_y = 0$, if Eq. (12) for such a front has at least one proper solution.

Let us consider two examples.

In the presence of ionic viscosity, a nonlinear wave may be established, in which the density changes over a length δ_n, whereas the magnetic field in the wave changes over a length $\delta_j = c^2/4\pi\sigma v_x$, with $\delta_j \gg \delta_n$ (isomagnetic jump[10]). Then when δ = const we have in the region in front of the wave ($x > 0$)

$$H_z = H_{z1} + H_{z2}\frac{n_2 - n_1}{n_2}\exp\left(-\frac{x}{\delta_j}\frac{n_2}{n_1}\right), \quad \frac{H_{z1}}{H_{z2}} = \frac{n_1}{n_2}.$$

Here H_{z1}, n_1 and H_{z2}, n_2 denote the values of the magnetic field and the density ahead and behind the wave front, respectively. Let us consider the stability of such a front relative to perturbations of the surface-wave type with $k_y = 0$, described by Eq. (12). In the region behind the wave ($x < 0$), a solution of this equation is

$$H_x = A\exp(\varkappa x), \quad \varkappa^2 = k_z^2 - \left(\frac{\omega}{\omega_{Hi}}\frac{\omega_{pi}^2}{k_z c^2}\right)^2, \quad \operatorname{Re}\varkappa > 0.$$

In the region ahead of the wave ($\delta_n \ll x \ll H/|H'|$), a solution that attenuates as $x \to +\infty$ is

$$H_x = A\left\{1 - x\left[\varkappa_1^2 + \frac{1}{\delta_j^2}\frac{n_2(n_2 - n_1)}{n_1^2}\right]^{1/2}\right\},$$

$$\varkappa_1^2 = k_z^2 - \left(\frac{\omega}{\omega_{Hi}}\frac{\omega_{pi}^2}{k_z c^2}\right)^2\frac{n_1^2}{n_2^2}$$

Joining these solutions together, we obtain the following dispersion equation:

$$\varkappa + \left[\varkappa_1^2 + \frac{1}{\delta_j^2}\frac{n_2(n_2 - n_1)}{n_1^2}\right]^{1/2} = \ln\frac{n_2}{n_1}\left\langle -\frac{H'}{H} - \frac{4\pi n T_e'}{H^2}\right\rangle, \quad (13)$$

where $\langle\ldots\rangle$ denotes the mean value of the quantity at the point of the density jump. When $\beta_e \ll 1$ there are no unstable oscillations, since the left side of (13) is always larger than the right side. Instability is possible if the electron temperature T_e and the magnetic field H fall off to one side and

$$\frac{1+\bar{\beta}_e}{2}\ln\frac{n_2}{n_1} > \sqrt{\frac{n_2}{n_2 - n_1}}, \quad \bar{\beta}_e = \delta_j\frac{n_2 + n_1}{n_2 - n_1}\left\langle\frac{4\pi n_1 T_e'}{H^2}\right\rangle.$$

We now consider the stability of a strong shock wave, behind the front of which we have $\beta_e \gg 1$, and in which the magnetic field, density, and pressure p_e vary over the same scale δ. Let $k_y = 0$, and let the perturbations of the magnetic field be of small scale, $k_x\delta \gg 1$.

In this case there exist local growing perturbations described by the dispersion equation

$$\int dx\left[\left(\frac{\omega}{\omega_{Hi}}\frac{\omega_{pi}^2}{k_z c^2}\right)^2 - k_z^2 + \frac{4\pi n' T_e'}{H^2}\right]^{1/2} = \pi\left(l + \frac{1}{2}\right). \quad (14)$$

As a result of the development of such an instability with an in current $\operatorname{Im}\omega \sim \beta_e\omega_{Hi}c^2/\delta^2\omega_{pi}^2$, the entire front of the nonlinear wave breaks up into individual cells with scale $\lambda \sim \delta/\sqrt{\beta_e} \approx \delta/M$, where M is the Mach number.

In the derivation of Eq. (8) we have neglected the motion of the ions in the oscillations, a procedure valid when $k_z \gg \omega_{pi}/c$, and the motion of the plasma through the wave front with variable velocity $v_x = Mv_{Ai}$. It is obvious if $\operatorname{Im}\omega \gg Mv_{Ai}/\delta$, then the perturbation will have time to grow before it is carried away by the stream from the region of the wave front. It follows therefore that the front of a strong shock wave may turn out to be unstable if $\delta < c\sqrt{\beta_e}/\omega_{pi}$ ($\beta_e \approx M^2$).

The investigated instability leads to a local pinching of the current in the plane of the wave. Let us consider first the configurations of the perturbed force lines to which the solutions of Eq. (8) correspond when $\omega \ll \omega_{Hi}k_z^2c^2/\omega_{pi}^2$ and $k_y = 0$. In such perturbations, the electric field can be regarded as potential, $E = -\nabla\varphi$. Then the quasiequilibrium states of the electron gas are determined by the equations

$$en_0\frac{\partial\varphi}{\partial x} - \frac{\partial}{\partial x}(n_0 T_{1e}) - \frac{1}{c}j_y H_z = 0,$$

$$en_0\varphi - n_0 T_{1e} + \frac{1}{c}j_{0y}\xi_x H_{0z} = 0,$$

$$T_{1e} = -\xi_x T_{0e}', \quad \xi_x = -\frac{iH_x}{k_z H_{0z}}, \quad (15)$$

ξ is the displacement of the electrons in the perturbations. The magnetic field is "frozen in" the electrons.

From the system (15) there follows Eq. (8) for H_x in the case when $\omega = 0$. If this equation has localized solutions for a certain set of values of k_{zl}^2, then this signifies that there exist a number of quasiequilibrium states.

We now consider a perturbation with a smaller value of k_z^2. In this case the restoring force $j_y H_z/c$, at the same displacement amplitude ξ_x, is smaller and cannot balance completely the force

$$en_0\frac{\partial\varphi}{\partial x}-\frac{\partial}{\partial x}(n_0T_{1e})=n_0'T_{0e}'\xi_x-n_0\frac{\partial}{\partial x}\left(\xi_x\frac{H_0H_0'}{4\pi n_0}\right)$$

The difference amounting to $(k_{zl}^2 - k_z^2)H_0^2\xi_x/4\pi$. This leads to displacement of the electrons along the x axis. The force $n_0 e v_x H_{0z}/c$ displaces the electrons in the y direction and causes a field H_y to appear. The instability should cease when the perturbation of the magnetic field becomes comparable with H_{0z}.

The formulas of this section do not contain the electron scattering frequency ν. Neglect of the effects of ohmic dissipation in the oscillations, as already noted, is justified if $\omega \gg \nu k^2c^2/\omega_{pe}^2$. But ν determines also the minimum dimension of δ_H over which the magnetic field changes: $\delta_H \sim c\nu/\omega_{pi}\omega_{He}$. Therefore, for unstable perturbations with $\mathrm{Im}\,\omega \sim \omega_{Hi}k_z^2c^2/\omega_{pi}^2$ and $k_z < 1/\delta$, this condition is satisfied if $\nu \ll \omega_{He}$ or $\delta_H \ll c/\omega_{pi}$. The assumption that $\omega \ll k_z^2 v_{Te}^2/\nu$, on the basis of which we put $\gamma = 1$ in Eq. (4), is justified if $\delta_H < c\beta_e/\omega_{pi}$.

3. We now consider purely electronic oscillations on the front of a nonlinear magnetosonic wave when $k_z/k \ll \omega/\omega_{He}$, $\omega \gg (\omega_{He}\omega_{Hi})^{1/2}$, and $\beta_e \ll 1$. In this case it is impossible to neglect in (3) the inertia of the electrons. We are interested, as before, in instabilities that develop more rapidly than the deformation of the wave front. Under the conditions formulated above, the equation for H_z has in the quasiclassical limit $k_x\delta \gg 1$ the following form:

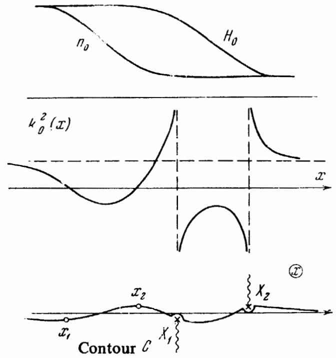
$$\frac{d^2H_z}{dx^2}-\left(k_y^2+\frac{\omega_{pe}^2}{c^2}\frac{\omega-k_yv_n}{\omega-k_yv_H+i\nu}\right)H_z=0, \qquad (16)$$

where

$$v_H=\frac{c^2}{\omega_{pe}^2}\,\omega_{He}\frac{H_0'}{H_0},\qquad v_n=\frac{c^2}{\omega_{pe}^2}\,\omega_{He}\frac{n_0'}{n_0}.$$

We shall henceforth neglect the collision frequency ν compared with ω. This equation is similar to the equation obtained in hydrodynamics in the investigation of the stability of plane-parallel flow of an ideal liquid.[11] It is known that such flows are unstable in the presence of an inflection point on the velocity profile. As applied to our problem, this means that Eq. (16) has exponentially growing solutions only in the case when at least at one point we have $v_n(x_0) = v_H(x_0) = \omega/k_y$. However, there is a difference between these problems. In the hydrodynamic formulation the existence of a localized solution is guaranteed by the presence of walls. In our problem it is still necessary to show that a solution that is localized, i.e., that decreases away from the front, exists.

Equation (16) coincides in form with a Schrödinger equation whose effective "potential" depends on the profiles of the magnetic field H_0 and the density n_0 (see the figure). Arguments in favor of precisely such profiles of the magnetic field H_0 and the density n_0 are given in Sec. 1. It is clear here from physical considerations that the profile of the magnetic field should



"lead" the density profile.

It is seen from the figure that at a fixed phase velocity of the oscillations ω/k_y the effective "potential" of (16) has four turning points, two of which are zeroes and the other two are poles. At different values of the parameters $k_y^2c^2/\omega_{pe}^2$ and $\omega/k_y v_0$, where v_0 is the velocity at the intersection point $v_H = v_n$, there can be different sequences of the turning points. We shall show subsequently that the existence of quasiclassical localized solutions is possible only for the singular-point sequence shown in the figure. This corresponds to parameter values $k_y^2c^2/\omega_{pe}^2 < |v_n'/v_H'|$ and $\omega/k_y v_0 > 1$. If we take into account the rule for going around the poles in accordance with the Landau rule[12] ($\omega \to \omega + i\epsilon$, $\epsilon > 0$), then Eq. (16) can be represented in the form

$$\frac{d^2H_z}{dx^2}-k_0^2(x)H_z=-i\pi\frac{\omega_{pe}^2}{c^2}(\omega-k_yv_n)\,\delta(\omega-k_yv_H)H_z, \quad (17)$$

where

$$k_0^2(x)=k_y^2+\frac{\omega_{pe}^2}{c^2}\,\frac{\omega-k_yv_n}{\omega-k_yv_H}.$$

In the complex x plane it is possible to introduce a contour C, along which the phase $\int^x k_0(x')\,dx'$ is real or imaginary, depending on our position relative to the turning point. The usual rules of quasiclassical quantization are applicable along such a contour. The figure shows the arrangement of the turning points and of the contour C for the profiles of H_z and n_0 given above when $k_y > 0$.

We now find the localized solution for the chosen sequence of singular points. We construct first in the zero-zero well between the turning points x_1 and x_2 a quasiclassical solution that falls off to the left of the point x_1. We then construct an analogous solution in the pole-pole well between the turning points X_1 and X_2; this solution falls off to the right of the point X_2. Then we join the obtained solutions in the region of the "potential" barrier (x_2X_1) separating the zero-zero and pole-pole wells. The general method for continuing the quasiclassical solution through the turning point is as follows. In the direct vicinity of the turning point we construct an exact solution, which goes over asymptotically far from the turning point into the quasiclassical solution. Joining the exact and quasiclassical solution at a certain intermediate region, where both solutions are valid with sufficient degree of accuracy, and using the boundary conditions at the turning point for the exact solution, we can obtain the connection between the coefficients of the quasiclassical solutions. The only difference between a zero and a pole is that the boundary conditions for H_Z' at the turning point are different. Using this method we find that the solution falling off to the left of the zero x_1

$$H_z = \frac{A}{(k_0^2)^{1/4}} \exp\left(\int_{x_1}^{x} \sqrt{k_0^2} dx'\right), \quad (18)$$

corresponds on the right side to an oscillating solution

$$H_z = \frac{2A}{(-k_0^2)^{1/4}} \cos\left(\int_{x_1}^{x} \sqrt{-k_0^2}\, dx - \frac{\pi}{4}\right). \quad (19)$$

In turn, corresponding to this solution on the right of the zero x_2 is

$$H_z = \frac{A}{(k_0^2)^{1/4}} \exp\left(-\int_{x_2}^{x} \sqrt{k_0^2}\, dx'\right) \cos\left(\int_{x_1}^{x_2} \sqrt{-k_0^2}\, dx'' - \frac{\pi}{2}\right)$$
$$- \frac{2A}{(k_0^2)^{1/4}} \exp\left(\int_{x_2}^{x} \sqrt{k_0^2}\, dx'\right) \sin\left(\int_{x_1}^{x_2} \sqrt{-k_0^2}\, dx'' - \frac{\pi}{2}\right). \quad (20)$$

This solution is valid provided the second term is smaller than or of the order of the first. When finding the quasiclassical solutions in the region of the pole-pole well it is necessary to take into account the fact that H_0' experiences a discontinuity at the pole:

$$\{H_z'\} = H_z i\pi \frac{\omega_{pe}^2}{c^2} \frac{v_n - v_H}{|v_H'|}. \quad (21)$$

Here and throughout we shall assume that $k_y > 0$. It is then easy to find that the solution

$$H_z = \frac{B}{(k_0^2)^{1/4}} \exp\left(-\int_{X_2}^{x} \sqrt{k_0^2}\, dx'\right), \quad (22)$$

which decreases to the right of the pole X_2, corresponds on the left of this pole, inside the pole-pole well, to the quasiclassical solution

$$H_z = \frac{B}{(-k_0^2)^{1/4}} \exp\left\{-i\left(\int_{X_2}^{x} \sqrt{-k_0^2}\, dx' + \frac{\pi}{4}\right)\right\}. \quad (23)$$

It is significant that, unlike (19), we obtain a traveling wave [13]. In the region to the left of the pole X_1, the traveling wave corresponds, with exponential accuracy, to a solution that increases away from the pole, namely

$$H_z = \frac{B}{(k_0^2)^{1/4}} \exp\left(i \int_{X_1}^{X_2} \sqrt{-k_0^2}\, dx'\right) \exp\left(-\int_{X_1}^{x} \sqrt{k_0^2}\, dx''\right). \quad (24)$$

In the subsequent calculations of the increment of the instability, we shall need the values of H_Z at the poles X_1 and X_2:

$$H_z(X_1) = \frac{iB}{\sqrt{\pi}} \left(\frac{\omega_{pe}^2}{c^2} \frac{v_H - v_n}{v_H'}\right)_{X_1}^{-1/2} \exp\left(i \int_{X_1}^{X_2} \sqrt{-k_0^2}\, dx'\right),$$
$$H_z(X_2) = \frac{B}{\sqrt{\pi}} \left(\frac{\omega_{pe}^2}{c^2} \frac{v_n - v_H}{v_H'}\right)_{X_2}^{-1/2}. \quad (25)$$

In order to join the solutions in the region (x_2X_1), we make use of the fact that the wells (x_1x_2) and (X_1X_2) are separated by a broad barrier and consequently the coupling between the oscillations is exponentially weak in the wells. Therefore the equation

$$\int_{x_1}^{x_2} \sqrt{-k_0^2}\, dx = \pi\left(l + \frac{1}{2}\right) \quad (26)$$

is, with exponential accuracy, the quantization condition for the entire system. We then obtain the following condition for joining together the amplitudes:

$$B \exp\left(i \int_{X_1}^{X_2} \sqrt{-k_0^2}\, dx\right) \cong (-1)^l A \exp\left(-\int_{x_2}^{X_1} \sqrt{k_0^2}\, dx\right), \quad (27)$$

where l is the number of the level in the zero-zero well.

We now find the small imaginary correction to ω. To this end we multiply Eq. (12) by H_Z^*, subtract from the resultant expression the complex conjugate, and after integrating with respect to real x we obtain

$$\int_{-\infty}^{+\infty} dx |H_z|^2 \frac{\omega_{pe}^2}{c^2} \frac{\gamma(v_H - v_n)}{(\omega_0 - k_y v_H)^2 + \gamma^2} = 0,$$
$$\mathrm{Re}\,\omega = \omega_0, \quad \mathrm{Im}\,\omega = \gamma > 0. \quad (28)$$

Using the joining condition (27) and the expression for H_Z at the poles (25), and assuming that $\gamma \ll \omega_0$, we obtain the following expression for the increment:

$$\gamma \mathrm{P} \int_{x_1}^{x_2} \frac{dx}{\sqrt{-k_0^2}} \frac{\omega_{pe}^2}{c^2} \frac{k_y(v_H - v_n)}{(\omega_0 - k_y v_H)^2} = \exp\left(-2 \int_{x_2}^{X_1} \sqrt{k_0^2}\, dx\right). \quad (29)$$

Here the integration on the left side is over the region inside the zero-zero well, where $v_H - v_n > 0$. The expression for the increment can be represented also in the form

$$\gamma \frac{\partial}{\partial \omega} \int_{x_1}^{x_2} \sqrt{-k_0^2}\, dx = \frac{1}{2} \exp\left(-2 \int_{x_2}^{X_1} \sqrt{k_0^2}\, dx\right). \quad (30)$$

Let us dwell briefly on the validity of the neglects made in the derivation of the equation for H_Z. From the very procedure of continuing the solution through the pole it follows that we go around the pole $[\omega - k_y v_H(x)]^{-1}$ in the complex x domain along a certain semicircle of finite radius R_ϵ above or below the pole. On the other hand, when rigorous account is taken of the quasiclassical small terms, which we have neglected, poles of second and higher order appear in the equation for H_Z. We now choose a circle with a radius R_ϵ such that it is possible to neglect on it the contribution due to these additional poles. In particular, a second-order pole appears when the term with the first derivative is excluded from the equation for H_Z. To be able to neglect this pole, it is necessary to go around the singularity along a semicircle with radius

$$\delta \gg R_\varepsilon \gg \delta (c/\delta\omega_{pe})^4,$$

where δ is the width of the wave front.

Allowance for the motion of matter through the front of the wave with velocity $v_{0x}(x)$ leads also to the appearance of additional pole terms of fourth order. These can be neglected by going around the singularity along a semicircle with radius $\delta \gg R \gg \delta\, (\delta\omega_{pi}/c)^{2/3}$. Therefore, in order for the formulas of the present section to be regarded as valid, it is necessary to assume that the condition $c/\omega_{pe} \ll \delta \ll c/\omega_{pi}$ is fulfilled.

In Eq. (16) we disregarded terms connected with the friction of the electrons against the ions. This is justified if $\nu \ll \gamma$. If the "unfreezing" of n_0 and H_0 on the front of the nonlinear wave is determined by the ohmic dissipation, then the effective collision frequency is not a free parameter in the foregoing inequality. The quantity ν itself determines the "unfreezing" of the profiles of the density n_0 and the magnetic field H_0 on the front of the nonlinear wave, and consequently also the oscillation increment γ. Using formula (30) for the increment on the limit of its applicability, when the distance between the singular points x_2 and X_1 is of the order of c/ω_{pe}, we obtain the following condition for the applicability of all the conclusions of the present section:

$$\frac{c}{\omega_{pe}} < \delta < \left(\frac{m_i}{m_e}\right)^{1/4} \frac{c}{\omega_{pe}}$$

We now clarify the physical nature of the obtained solution. We note first that when the solution is continued through the pole, the decreasing solution goes over into a traveling wave whose phase velocity is directed to the side of this pole, thus greatly differing from the behavior of the solution in the zero-zero well, where it represents a standing wave. Obviously, therefore, neither the zero-pole well nor the pole-pole well has localized solutions. To this end it would be necessary to join in the first well the traveling wave with the standing wave, and in the second well two waves traveling in opposite directions. Neither is possible. We note that if the value of ω/k_y is very close to $v_H(x_0) = v_n(x_0)$, so that $|X_1 - x_2|\,\omega_{pe}/c < 1$, then again there is no localized solution. This circumstance limits the maximal increment γ.

Let us calculate the energy of the oscillations in the local approximation, when $dH_z/dx = ik_xH_z$. In the reference frame where the electrons are at rest, this energy is obviously equal to the sum of the vibrational energy of the electrons as they drift and of magnetic oscillation energy

$$W_k' = \frac{|H_z|^2}{8\pi} \frac{v_n - v_H}{\omega'} k_y, \tag{31}$$

ω' is the frequency in a reference frame moving with velocity v_H. Using the conservation of the adiabatic invariant W_k/ω, we go over to the laboratory frame:

$$W_k = \frac{|H_z|^2}{8\pi} \frac{k_y(v_n - v_H)\omega}{(\omega - k_y v_H)^2},$$

We see therefore that the energy of the oscillations is positive in the region of the zero-zero well and negative in the region of the pole-pole well. Thus, the traveling wave propagating in the region of the pole-pole well from the pole X_1 to the pole X_2 carries energy in the opposite direction, i.e., in the direction towards the zero-zero well. In the poles, owing to the resonant interaction with electrons, which drift with velocity $v_H = \omega/k_y$, the wave becomes intensified. The additional energy connected with the wave passes through the barrier (x_2X_1) into the region of the zero-zero well and causes an increase of the oscillations in this well. The exponential smallness of the increment is obviously connected with the presence of the barrier (x_2X_1) in the path of the energy flux.

We shall now show that the resultant electronic plasma oscillations lead to an irreversible change of the profile of the magnetic field H_0. Using the hydrodynamic equations of motion of the electronic components, the induction equation, and expression (16), and averaging over the random phases of the oscillations, we can obtain the following equation:

$$\frac{\partial}{\partial t}\left(\frac{H_0}{n_0}\right) + v_{0x}\frac{\partial}{\partial x}\left(\frac{H_0}{n_0}\right) = \frac{1}{n_0 L}\sum_l \int dk_y \frac{\partial}{\partial x}\delta(\omega_{lk_y} - k_y v_H)\,|v_{lky}|^2 n_0 \frac{\partial}{\partial x}\left(\frac{H_0}{n_0}\right). \tag{32}$$

Here L is the length of the plasma layer in the y direction, l is the number of zeroes of the solution of Eq. (16) on the x axis, and v_{lky} is the Fourier component of the velocity v_x.

We see therefore that the coefficient of diffusion of the magnetic field H_0 is of the order of $D_H \sim v_x^2\gamma/\omega^2$. In conjunction with the equation

$$\frac{\partial}{\partial t}|v_{lky}|^2 = 2\gamma_{lky}|v_{lky}|^2 \tag{33}$$

the quasilinear equation (32) determines the collisionless diffusion of the magnetic field on the front of the shock wave. It must be emphasized that the quasilinear effects do not lead to a change in the density n_0 of the electrons, since $n_{1e} = 0$ in the oscillations in question.

It is seen from the equations that the buildup of the oscillations is a consequence of the resonant interaction between the wave and the electrons drifting with velocity $v_H = \omega/k_y$. The increase of the energy of the oscillations results in this case from the realignment of the magnetic field H_0.

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