# *INTENSITY FLUCTUATIONS OF CHARGED PARTICLES MOVING IN A RANDOM MAGNETIC FIELD*

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We consider the propagation of charged particles in a random magnetic field under the assumption that the characteristic dimension of the inhomogeneities of the magnetic field is much smaller than the characteristic value of the Larmor radius. An equation for the second moment of the exact distribution function is derived from the collisionless kinetic equation for the exact distribution function, using a diagram technique; this moment characterizes the fluctuations of the intensity of the charged particles. A solution of this equation is obtained for weak intensity fluctuations. It is shown that a study of the intensity fluctuations yields information concerning the structure of the random magnetic field.

## 1. FORMULATION OF THE PROBLEM

DOLGINOV and Toptygin[1] obtained a kinetic equation for the average distribution function of charged particles moving in a random magnetic field. In a number of cases (in particular, for the interpretation of the intensity fluctuations of the relativistic solar protons[2]) it is necessary to know also the characteristics of the fluctuations of the exact distribution function, due to fluctuations of the magnetic field.

In the present paper we derive, using a procedure proposed in[1], an equation for the second moment of the exact distribution function of charged particles moving in a random magnetic field. A solution of this equation is obtained for weak intensity fluctuations. The conditions for the applicability of the solution are indicated.

## 2. DERIVATION OF THE MAIN EQUATION

Assume a stationary random magnetic field with smoothly varying average characteristics and having a normal distribution. We assume that the regular component of the magnetic field vanishes:

$$\langle \mathbf{H} \rangle = 0. \tag{1}$$

The second moment of the field, as shown in[1], should be specified in the form

$$\langle H_\alpha(\mathbf{r}_1) H_\beta(\mathbf{r}_2) \rangle = B_{\alpha\beta}(\mathbf{r}, \mathbf{x}) = \frac{1}{12} H_1^2 \left( \frac{\partial^2}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2}{\partial x_\gamma^2} \delta_{\alpha\beta} \right) \varphi(\mathbf{r}, \mathbf{x}), \tag{2}$$

where $H_1^2$ is the dimensional factor, $\varphi$ is a scalar function of the variables $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, and $\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2$. In the concrete calculations we shall assume

$$\varphi = a^2 \exp(-x^2/a^2). \tag{3}$$

The higher-order even moments of the field are expressed in terms of the sum of products of the second moments (see[1]). All the odd moments vanish.

We assume that in the elementary act of interaction with one inhomogeneity, the particle is scattered through a small angle, so that the inequality

$$a^2/\rho_1^2 = a^2 e^2 H_1^2 / c^2 p^2 \ll 1 \tag{4}$$

is satisfied, where c is the velocity of light, p the momentum, e the electron charge, and a the characteristic dimension of the magnetic-field inhomogeneities.

In the derivation of the equation for the second moment of the exact distribution function, we shall use the method employed by Dolginov and Toptygin[1] to derive an equation for the average distribution function. The exact distribution function satisfies the collision-free kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + (\mathbf{H}\mathbf{D}) f = 0, \tag{5}$$

$$\mathbf{D} = \frac{e}{c} \left[ \mathbf{v} \frac{\partial}{\partial \mathbf{p}} \right], \tag{6*}$$

where $\mathbf{v}$ is the particle velocity, $\mathbf{r}$ the coordinate of the configuration space, and t the time. We carry out in (5) a Fourier transformation with respect to $\mathbf{r}$ and a Laplace transformation with respect to t. Then, solving this equation by iteration, we obtain

$$f(\mathbf{k}, \mathbf{p}, s) = R_0 f_0(\mathbf{k}, \mathbf{p}) + R_0 \sum_{n=1}^{\infty} \int \prod_{m=1}^{n} [(\mathbf{H}\mathbf{D})_{\mathbf{k}_m} R_m \, d\mathbf{k}_m] \cdot \times f_0(\mathbf{k} - \mathbf{k}_1 - \ldots - \mathbf{k}_n, \mathbf{p}), \tag{7}$$

$$R_m = [s + i\mathbf{v}(\mathbf{k} - \mathbf{k}_1 - \ldots - \mathbf{k}_m)]^{-1}, \tag{8}$$

where $\mathbf{k}$ corresponds to $\mathbf{r}$ in Fourier space and s corresponds to t; $f_0(\mathbf{k}, \mathbf{p})$ is the Fourier transformation of the initial distribution function; $(\mathbf{H} \cdot \mathbf{D})_{\mathbf{k}_m}$ is the Fourier transformation of the product $(\mathbf{H} \cdot \mathbf{D})$, which depends on $\mathbf{k}_m$.

Let us represent the series (7) graphically. To this end, we introduce certain symbols (they correspond essentially to the symbols employed in[1]):

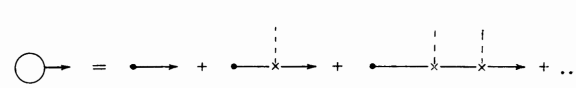


FIG. 1. Series for f (k, p, s).

*$[\mathbf{v}\frac{\partial}{\partial \mathbf{p}}] \equiv \mathbf{v} \times \frac{\partial}{\partial \mathbf{p}}$

1. A black circle on a diagram with n vertices corresponds to the quantity $f_0(\mathbf{k}_1 - \ldots - \mathbf{k}_n, \mathbf{p})$;

2) a shaded-line segment emerging from the m-th vertex corresponds to the factor $(\mathbf{H}\cdot\mathbf{D})_{\mathbf{k}m}$;

3) a straight-line segment joining the vertices m and m + 1 corresponds to a factor $R_m$;

4) an empty circle with a line emerging from it denotes the function $f(\mathbf{k}, \mathbf{p}, s)$;

5) integration with respect to $\mathbf{k}_m$ is carried out over all the internal vertices.

The series (7) is shown graphically in Fig. 1. Out of this series, we make up a combination for the second moment:

$$\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{p}_1, \mathbf{p}_2; s_1, s_2) = \langle f(\mathbf{k}_1, \mathbf{p}_1, s_1)\cdot f(\mathbf{k}_2, \mathbf{p}_2, s_2)\rangle. \tag{9}$$

The angle brackets in (9) denote averaging over the realizations of the random magnetic field.

Graphically, the procedure of averaging consists in the fact that the free ends of the dashed lines are joined pairwise. The segment of the dashed line joining the vertices i and j corresponds to a correlation function $\langle(\mathbf{H}\cdot\mathbf{D})_{\mathbf{k}_i}\cdot(\mathbf{H}\cdot\mathbf{D})_{\mathbf{k}_j}\rangle$. Graphically, the series for $\Phi$ is shown in Fig. 2. In the figure, the rectangle with two emerging lines denotes the function $\Phi$. The vertices of the diagrams that enter in the series for $\Phi$ can be strongly and weakly coupled. A strongly-coupled diagram cannot be divided into parts without cutting at the same time a dashed line. A weakly-coupled diagram can be separated into parts without cutting the dashed line. A weakly-coupled diagram can be combined out of corresponding strongly-coupled diagrams with the aid of weak couplings. The strongly-coupled diagrams, of which the diagrams that enter in the series for $\Phi$ are made up, can be divided into two types. The diagrams of the first type join the vertices located at only one level (Fig. 3a). The diagrams of the second type join vertices located at two levels (Fig. 3b).

It can be shown that the strongly-coupled diagram of order 2n + 2 (containing 2n + 2 vertices) has an extra factor of $a^2/\rho_\mathrm{I}^2$ compared with the strongly-coupled diagram of order 2n. By assumption, this factor is small, so that we can disregard diagrams which contain as components strongly coupled diagrams of order higher than two. This denotes that in the expansion in the parameter $a^2/\rho_1^2$ only the zeroth term is taken into account. The remaining diagrams from the series represented in Fig. 4.

Diagrams containing an equal number of diagrams of the second type can be summed. As a result we obtain the series represented in Fig. 5. In this figure, the heavy straight-line segment emerging from a black circle denotes the average distribution function $F(\mathbf{k}, \mathbf{p}, s) = \langle f(\mathbf{k}, \mathbf{p}, s)\rangle$. The heavy line without the circle denotes the average Green's function $G(\mathbf{k}, \mathbf{p}, s)$. It is easy to see that the series represented in Fig. 5 is a solution of the equation represented in Fig. 6. Let us rewrite this equation analytically in the coordinate representation

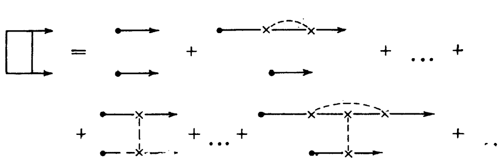


FIG. 2. Series for $\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{p}_1, \mathbf{p}_2; s_1, s_2)$.

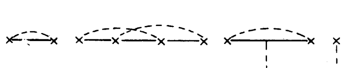



FIG. 3. Examples of strongly connected diagrams of first (a) and second (b) type.

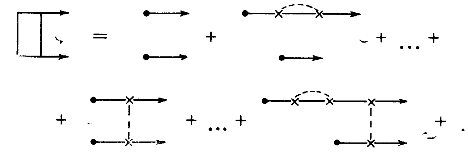


FIG. 4. Series for $\Phi$, in which the strongly-coupled diagrams of order higher than two are discarded.

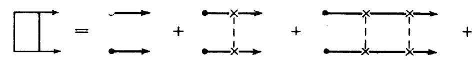


FIG. 5. Partly summed series for $\Phi$.

$$\begin{aligned}\Phi(\mathbf{r}_1, \mathbf{r}_2; \mathbf{p}_1, \mathbf{p}_2; t_1, t_2) &= F(\mathbf{r}_1, \mathbf{p}_1, t_1)F(\mathbf{r}_2, \mathbf{p}_2, t_2)\\ &+ \int G(\mathbf{r}_1-\mathbf{r}_1', \mathbf{p}_1, t_1-t_1')G(\mathbf{r}_2-\mathbf{r}_2', \mathbf{p}_2, t_2-t_2')\\ &\times D_\alpha D_\beta B_{\alpha\beta}(\mathbf{r}_1'+\mathbf{r}_2', \mathbf{r}_1'-\mathbf{r}_2')\Phi(\mathbf{r}_1', \mathbf{r}_2'; \mathbf{p}_1, \mathbf{p}_2; t_1', t_2')\\ &\cdot d\mathbf{r}_1'\, d\mathbf{r}_2'\, dt_1'\, dt_2'.\end{aligned} \tag{10}$$

## 3. WEAK INTENSITY FLUCTUATIONS IN THE STATIONARY CASE

Let us consider first a bounded volume with a random field, in which the integral scattering is through a small angle, and that the particles not be scattered outside this volume. The average distribution function will be assumed stationary. The stationary average Green's function, assuming small scattering, has in accord with[1] the form

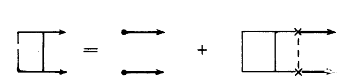
$$G(|\mathbf{r}-\mathbf{r}'|, \psi) = \frac{3}{4\pi q|\mathbf{r}-\mathbf{r}'|^3}\exp\left(-\frac{3\psi^2 v}{4q|\mathbf{r}-\mathbf{r}'|}\right), \tag{11}$$

$$q = \frac{\sqrt{\pi}\, e^2 a H_1^2}{12 m^2 c^2 v}, \tag{12}$$

where $\psi$ is the polar angle characterizing the direction of the momentum $\mathbf{p}$, reckoned from the direction of the vector $\mathbf{r} - \mathbf{r}'$.

If we choose as F the solution for a pointlike source contained in the volume under consideration, and represent the solution of Eq. (10) in the form of an iteration series, then it is easy to see that all the integrals that enter in the series diverge. The mean square of the intensity fluctuations becomes infinite in the entire volume under consideration. Physically this is explained by the fact that the inhomogeneities of the magnetic field can focus the radiation of a pointlike source into a point. To eliminate the singularities it is necessary to assume that the average distribution function has a sufficiently smooth dependence on the angles

FIG. 6. Equation for $\Phi$.



in velocity space in the entire volume under consideration.

Let us consider the case of a weakly anisotropic average distribution function. Let the dependence of F on $\mathbf{p}$ be so weak, that in the solution of (10) by an iteration method it is possible to confine oneself to iteration of first order. We assume also that F depends little on the coordinates, so that it can be taken outside the integral sign. Quantitatively these limitations will be considered below. Thus, under these limitations, we have in the stationary case

$$\Phi(\mathbf{r}_1, \mathbf{r}_2; \mathbf{p}_1, \mathbf{p}_2) = F(\mathbf{r}_1, \mathbf{p}_1) F(\mathbf{r}_2, \mathbf{p}_2)$$
$$+ D_{\alpha_1} D_{\alpha_2} F(\mathbf{r}_1, \mathbf{p}_1) F(\mathbf{r}_2, \mathbf{p}_2) \int_V G(|\mathbf{r}_1 - \mathbf{r}_1'|, \psi_1) G(|\mathbf{r}_2 - \mathbf{r}_2'|, \psi_2) B_{\alpha_1\alpha_2}\, d\mathbf{r}_1'\, d\mathbf{r}_2', \tag{13}$$

where $\psi_i$ is the angle between $\mathbf{r}_i - \mathbf{r}_i'$ and $\mathbf{p}_i$.

Let us examine in greater detail expression (13) for the case $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$, $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$, i.e., for the mean square of the exact distribution function. We write the integration variables in a rectangular coordinate system: $\mathbf{r}_i - \mathbf{r}_i' = x_i', y_i', z_i'$. We direct the $z'$ axis along the vector $\mathbf{p}$. Inasmuch as in the region significant for the integration we have $x_i' \ll z_i'$ and $y_i' \ll z_i$, we can write approximately

$$G(|\mathbf{r}_i - \mathbf{r}_i'|, \psi_i) = \frac{3}{4\pi q z_i'^3} \exp\left[-\frac{3v(x_i'^2 + y_i'^2)}{4q z_i'^3}\right]. \tag{14}$$

The off-diagonal components of the tensor $B_{\alpha_1\alpha_2}$ make no contribution to the integral, and furthermore $D_z = 0$, since $v_x = v_y = 0$; we can therefore put in (13) $\alpha_1 = \alpha_2 = \alpha$, whereby $\alpha$ assumes the values x and y.

The functions $B_{\alpha\alpha}(|\mathbf{r}_1' - \mathbf{r}_2'|)$ depend on the difference $z_1' - z_2'$ much more strongly than the functions $G(|\mathbf{r}_i - \mathbf{r}_i'|, \psi)$, and therefore, in integrating with respect to $z_1' - z_2'$, we can disregard the dependence of G on $z_i'$:

$$\Phi = F^2 + \gamma_{\alpha\alpha}(D_\alpha F)^2 \int_0^{z_0} dz' \int G(x_1', y_1', z')$$
$$\times G(x_2', y_2', z') R_{\alpha\alpha}(x_1' - x_2', y_1' - y_2')\, dx_1'\, dy_1'\, dx_2'\, dy_2', \tag{15}$$

where $z_0$ is determined by the shape of the volume and

$$\gamma_{\alpha\alpha} = \int_{-\infty}^{+\infty} B_{\alpha\alpha}(z)|_{x=y=0}\, dz,$$
$$R_{\alpha\alpha} = \int_{-\infty}^{+\infty} B_{\alpha\alpha}(x, y)\, dz / \gamma_{\alpha\alpha}. \tag{16}$$

If the concrete form of $B_{\alpha\alpha}$ is determined by expressions (2) and (3), then (15) can be integrated with respect to $x_i'$ and $y_i'$:

$$\Phi = F^2 + (D_\alpha F)^2 \frac{\sqrt{\pi} H_1^2}{6} \int_0^{z_0} \frac{a^3}{a^2 + 8qz'^3/3v}\left(1 - \frac{8qz'^3/3v}{a^2 + 8qz'^3/3v}\right) dz'. \tag{17}$$

An important feature of the integrand in (17) is the fact that it decreases rapidly (like $z'^{-6}$) when $z' > \zeta$, where $\zeta$ is determined by the expression

$$\zeta = (3a^2 v / 8q)^{1/3}, \tag{18}$$

and when $z' < \zeta$ the integrand is not dependent on $z'$. Thus, only a small region adjacent to the point of observation in the direction of the line of sight takes part in the formation of the fluctuations. This region is determined quantitatively with the following relations:

$$x'^2 \lesssim {}^4/_3 q z'^3, \qquad y'^2 \lesssim {}^4/_3 q z'^3, \qquad z' \lesssim \zeta. \tag{19}$$

The first two inequalities in (19) are determined by the form of the function G.

If F has circular symmetry in the angular spectrum, then, denoting by $\theta$ the polar angle measured from the symmetry axis, and approximately integrating with respect to $z'$, we obtain the simple relation

$$\Phi = F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2 \left(\frac{aq}{v}\right)^{2/3} 3^{1/3}. \tag{20}$$

Expression (20) has a simple meaning: scattering by the inhomogeneities located near the point of observation shifts the average angular spectrum by a random small angle $\Delta\theta \approx \sqrt{q\zeta/v}$, and this is the cause of the intensity fluctuations.

The correlation function can be calculated quite similarly. We present the final expressions for a number of cases, assuming that F depends only on $\theta$. Let $p_1 = p_2 = p$, $\mathbf{r}_1 \neq \mathbf{r}_2$, and let the vector $\mathbf{p}$ be perpendicular to the vector $\mathbf{r}_1 - \mathbf{r}_2$. We direct the x axis along the vector $\mathbf{r}_1 - \mathbf{r}_2$. In this case we have

$$\Phi_\perp(x_1 - x_2) = F^2 + 3^{1/3}\left(\frac{\partial F}{\partial \theta}\right)^2 \left(\frac{aq}{v}\right)^{2/3}$$
$$\times \left[1 - \frac{2(x_1 - x_2)^2}{a^2}\cos^2\lambda\right] \exp\left[-\frac{(x_1 - x_2)^2}{a^2}\right], \tag{21}$$

where $\lambda$ is the angle between $\nabla_p F$ and $\mathbf{r}_1 - \mathbf{r}_2$. We see from (21) that the characteristic transverse spatial scale of the intensity fluctuations is approximately equal to the dimension of the inhomogeneities of the magnetic field. If the vector $\mathbf{p}$ is directed at any angle to the vector $\mathbf{r}_1 - \mathbf{r}_2$, then, directing the x axis along the projection of the vector $\mathbf{r}_1 - \mathbf{r}_2$ on the x, y plane, we obtain

$$\Phi(x_1 - x_2, z_1 - z_2) = \Phi(x_1 - x_2)|_{z_1 = z_2} = \Phi_\perp(x_1 - x_2). \tag{22}$$

Thus, the correlation function depends only on the projection of the base on the plane perpendicular to the line of sight. This indicates that the longitudinal correlation scale is much larger than the transverse correlation scale.

Let now $\mathbf{p}_1 \neq \mathbf{p}_2$ and $\mathbf{r}_1 = \mathbf{r}_2$. When $\Delta\mathbf{k} = \mathbf{p}_1/|\mathbf{p}_2| - \mathbf{p}_2/|\mathbf{p}_2| \ll 1$, we have

$$\Phi(\Delta\mathbf{k}) = F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2 \frac{q}{3v} \frac{a}{|\Delta\mathbf{k}|} \operatorname{erf}\left(-\frac{|\Delta\mathbf{k}|\zeta}{a}\right),$$
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{23}$$

From (23) we see that the characteristic scale of the angular correlation equals $a/\zeta$.

Let us refine now the limitations imposed above on F. Since the fluctuations are determined only by the region (19), we can lift the limitations on the magnitude of the volume V: integral scattering in the entire volume can be large; it merely suffices that the scattering occur through small angles in the region (10):

$$q\zeta / 3v = (aq / 3v)^{2/3} \ll 1. \tag{24}$$

It is sufficient to impose on the dependence of F on $\mathbf{r}$ the condition that F changes weakly within the limits of the region (19). An estimate of the second-order iteration terms shows that, on top of the first-order iteration terms, there appear terms with order of magnitude $(\partial F/\partial\theta)^4(q\zeta)^2$. It is therefore necessary to impose on the dependence of F on $\theta$ the condition

$$3^{1/3}\left(\frac{\partial F}{\partial\theta}\right)^2\left(\frac{aq}{v}\right)^{2/3}\Big/F^2 \ll 1, \qquad (25)$$

i.e., that the density fluctuations be small compared with the average intensity.

The obtained results can be applied also to the case of a nonstationary average distribution function, if the characteristic scale of variation of F with time greatly exceeds $\zeta/v$.

It should be noted that the intensity fluctuations yield new information compared with the average intensity. Indeed, F depends only on $q \sim H_1^2 a$. Observations of the intensity fluctuation scale yield directly the characteristic dimension of the inhomogeneities of the magnetic field a. A comparison of the mean square of the intensity fluctuations with the average intensity yields the parameter $qa \sim H_1^2 a^2$. On the other hand, if fluctuations are not observed, then it is possible to impose limitations on the maximum dimension of the inhomogeneities of the magnetic field and the minimum field intensity in them.

In conclusion, let us compare the theoretical results with the observed data on the flare of solar cosmic rays of 15 November 1960[2]. The motion of the inhomogeneities of the magnetic field with the velocity of the solar wind ($V \approx 300$ km/sec) has changed the spatial fluctuation picture into a temporal one. Therefore, the theory developed above can be used for the temporal picture of the intensity fluctuations of solar cosmic rays. According to data by McKracken[2], for particles with energy $\approx 2 \times 10^9$ eV, the characteristic time dimension of the fluctuations amounted to approximately 10 minutes, giving $a \approx 3 \times 10^{10}$. For the same station $(\Phi - F^2)/(\partial F/\partial\theta)^2 \approx 0.04$, giving $H_1 \approx 10^{-4}$ G. These figures agree well with Coleman's data[3].


[1] A. Z. Dolginov and I. N. Toptygin, Zh. Eksp. Teor. Fiz. 51, 1771 (1966) [Soviet Phys. JETP 24, 1195 (1967)].

[2] K. G. McKracken, Geophys. Res. 67, 435 (1962).

[3] P. J. Coleman, ibid. 71, 5509 (1967).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