KINETIC EQUATION IN THE THEORY OF A NORMAL CHARGED FERMI FLUID

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Generalized kinetic equations are obtained with the help of the method of quantum Green's functions. These equations are used for the microscopic foundation of the phenomenological Landau-Silin theory of a degenerate electron Fermi liquid. It is shown that in the case of a normal Fermi system and for slowly changing external perturbations, the phenomenological theory takes exact account of the correlation between electrons.

As is known, the electron theory of metals successfully uses, in the description of many properties of metals, a representation of electrons as independent particles. However, owing to the fact that electrons are in the field of the metal lattice and that they interact with one another, their properties differ appreciably from the properties of free electrons. The mean energy of the Coulomb interaction of electrons of real metals is of the same order of magnitude as their mean kinetic energy. Therefore, an important correlation exists in the electron motions. The fact that the kinetic and potential energies of the electrons are of the same order does not allow us to use, in the description of metallic properties, microscopic theories based on the expansion of a parameter representing the ratio of the mean values of the kinetic and potential energies. Therefore, the study of the properties of metals is carried out on the basis of the Landau phenomenological theory of a normal Fermi liquid;[1] this theory was extended by Silin[2] to the case of Coulomb systems. The phenomenological Landau-Silin theory takes into account adequately the correlation of the electrons and is not limited to the framework of any model representation.

The present work is devoted to the microscopic foundation of the Landau-Silin theory of a normal electron Fermi liquid. The work is based on the use of the method of quantum Green's functions, which makes it possible to go beyond the limits of applicability of the ordinary perturbation theory and allows us to obtain, in principle, the basis of a separately conceived phenomenological theory, making its physical meaning clearer or more precise.[3]

In the first section, the generalized Boltzmann equations are deduced for non-ideal Fermi systems situated in a magnetic field.[4] The resultant equations represent a generalization of the results of Kadanoff and Baym[5] to the case of multi-component systems and for an external magnetic field. In the second section there is an extension of the derived equations to the case of Coulomb systems. In the third and fourth sections, the kinetic equations of Landau-Silin are introduced and a comparison is made of the results with the phenomenological theories.

1. GENERALIZED BOLTZMANN EQUATIONS

We consider an n-component system, consisting of Fermi particles with charges e_j and masses m_j with instantaneous two-particle interaction potential $v_{ij}(|\mathbf{r}_1 - \mathbf{r}_2|)$. We assume that the system as a whole is electrically neutral. Let the system be placed in a slowly changing (in space and time) magnetic field, and let a variable longitudinal electric field, which is described classically, act on the system. The energy operator of the system can be written in the representation of second quantization in the form

$$H = \sum_{j=1}^{n} \frac{1}{2m_j} \int dq \left[-\frac{1}{i}\nabla - \mathbf{A}_j(\mathbf{r}t) \right] \psi_j^+(qt) \left[\frac{1}{i}\nabla - \mathbf{A}_j(\mathbf{r}t) \right] \psi_j(qt)$$
$$+ \frac{1}{2} \sum_{j,k=1}^{n} \int dq\, dq_1 \psi_j^+(qt) \psi_k^+(q_1t) v_{jk}(|\mathbf{r}_1 - \mathbf{r}|) \psi_k(q_1t) \psi_j(qt)$$
$$+ \sum_{j=1}^{n} \int dq U_j(\mathbf{r}t) \psi_j^+(qt) \psi_j(qt). \qquad (1)$$

In this expression, q represents the set of space coordinates $\mathbf{r}$ and spin coordinates σ; integration over q means, as usual, integration over the space variables and summation over the spin variables; ψ_j^+ and ψ_j are the creation and annihilation operators of particles of type j in the Heisenberg representation; $U_j(\mathbf{r}t)$ and $\mathbf{A}_j(\mathbf{r}t)$ are the potentials, multiplied by the charge of particles of type j, of external fields with gauge div $\mathbf{A}_j = 0$, which guarantees the commutability of the momentum and $\mathbf{A}_j$.[6] We use the system of units in which $\hbar = c = 1$.

We define the Green's function of real time argument

$$g_j(qt; q't') = \frac{1}{i}\langle T[\psi_j(qt)\psi_j^+(q't')]\rangle,$$
$$g_j^>(qt; q't') = \frac{1}{i}\langle \psi_j(qt)\psi_j^+(q't')\rangle,$$
$$g_j^<(qt; q't') = -\frac{1}{i}\langle \psi_j^+(q't')\psi_j(qt)\rangle,$$
$$g_{jk}(qt, q_1t_1; q't', q_1't_1') = \frac{1}{i^2}\langle T[\psi_j(qt)\psi_k(q_1t_1)\psi_k^+(q_1't_1')\psi_j^+(q't')]\rangle. \qquad (2)$$

The symbol T is the usual Wick operator which makes possible time ordering; ⟨ ⟩ denotes averaging over the grand canonical ensemble of eigenstates of the system at the instant of time proceeding the moment at which the external fields are introduced. With the help of (1) and (2), we obtain the following equation for the Green's function:

$$\left\{ i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m_j} - U_j(\mathbf{r}_1t_1) - \frac{i}{m_j}\mathbf{A}_j(\mathbf{r}_1t_1)\nabla_1 - \frac{1}{2m_j}A_j^2(\mathbf{r}_1t_1) \right\} g_j(q_1t_1; q_2t_2)$$

$$= \delta(q_1 - q_2)\,\delta(t_1 - t_2) - i \sum_{k=1}^{n} \int dq_3 dt_3 V_{jk}(\mathbf{r}_1 t_1; \mathbf{r}_3 t_3)$$

$$\times g_{jk}(q_1 t_1, q_3 t_3; q_2 t_2, q_3 t_3 + 0) \tag{3}$$

and the conjugate equation, whose structure is obvious; here,

$$V_{jk}(\mathbf{r}_1 t_1; \mathbf{r}_2 t_2) = v_{jk}(|\mathbf{r}_1 - \mathbf{r}_2|)\,\delta(t_1 - t_2).$$

In what follows, we shall not speak of the conjugate equation, it being understood that it is always included.

To solve Eq. (3), it is necessary, somehow, to approximate the two-particle Green's function g_{jk}. As is known,[5] it is not possible to write down the approximation higher than the Hartree-Fock approximation directly for the Green's function of real time arguments. Therefore, we must use the method of analytic continuation suggested in[5]. We define the Green's functions G_j an G_{jk} in the following interval of complex time arguments:

$$0 < i(t - t_0) < \beta,$$

t_0 is real, β the parameter of the Gibbs distribution. We have

$$G_j(q_1 t_1; q_2 t_2; t_0) = \frac{1}{i} \frac{\langle T[S\varphi_j(q_1 t_1)\varphi_j^+(q_2 t_2)]\rangle}{\langle T[S]\rangle},$$

$$G_{jk}(q_1 t_1, q_2 t_2; q_1' t_1', q_2' t_2'; t_0)$$

$$= \frac{1}{i^2} \frac{\langle T[S\varphi_j(q_1 t_1)\varphi_k(q_2 t_2)\varphi_k^+(q_2' t_2')\varphi_j^+(q_1' t_1')]\rangle}{\langle T[S]\rangle}. \tag{4}$$

The operator T carries out the ordering in the following way: field operators with large argument $i(t - t_0)$ are moved to the left:

$$S = \exp\Big\{ - i \sum_{j=1}^{n} \int_{t_0}^{t_0 - i\beta} dt \int dq \Big[U_j(\mathbf{r}t)\varphi_j^+(qt)\varphi_j(qt)$$

$$+ \frac{i}{m_j}\varphi_j^+(qt)\mathbf{A}_j(\mathbf{r}t)\nabla\varphi_j(qt) + \frac{A_j^2(\mathbf{r}t)}{2m_j}\varphi_j^+(qt)\varphi_j(qt)\Big]\Big\}, \tag{5}$$

φ_j and φ_j^+ are the Fermi field operators in the interaction representation.

The functions G_j and G_{jk} introduced above satisfy the same boundary conditions as the equilibrium Green's function. For them we can use the ordinary techniques of obtaining an approximation for G_{jk}. In particular, the energy eigenfunctions σ_j of particles of type j satisfy the following functional-integral equation:

$$\sigma_j(q_1 t_1; q_2 t_2; t_0) = -\delta(q_1 - q_2)\,\delta(t_1 - t_2) \sum_{k=1}^{n} \int_{t_0}^{t_0 - i\beta} dt_3 dq_3 V_{jk}(\mathbf{r}_1 t_1; \mathbf{r}_3 t_3)$$

$$\times C_k(q_3 t_3; q_3 t_3 + 0; t_0) + iV_{jj}(\mathbf{r}_1 t_1; \mathbf{r}_2 t_2) G_j(q_1 t_1; q_2 t_2; t_0)$$

$$+ i \sum_{k=1}^{n} \int_{t_0}^{t_0 - i\beta} dt_3 d\mathbf{r}_3 dt_4 dq_4 V_{jk}(\mathbf{r}_1 t_1; \mathbf{r}_3 t_3) G_j(q_1 t_1; q_4 t_4; t_0) \frac{\delta\sigma_j(q_4 t_4; q_2 t_2; t_0)}{\delta U_k(\mathbf{r}_3 t_3)}. \tag{6}$$

The explicit equation for σ_j can be obtained by solving Eq. (6) by the iteration method; σ_j consists of two parts: the energy eigenfunction in the Hartree-Fock approximation σ_j^{HF}, and the correlation function σ_j^{C}, which is in turn composed of two analytic functions of the time arguments (in this sense, that any iteration expression for σ_j^{C} consists of such functions):

$$\sigma_j^{c}(q_1 t_1; q_2 t_2; t_0) = \begin{cases} \sigma_j^{>}(q_1 t_1; q_2 t_2; t_0) & i(t_1 - t_2) > 0, \\ \sigma_j^{<}(q_1 t_1; q_2 t_2; t_0) & i(t_1 - t_2) < 0. \end{cases} \tag{7}$$

When the Coulomb interaction potential is used, the expressions for the separate terms of the expansion σ_j^{C} turn out to be divergent. The problem of the applicability of similar expansions in the case of a Coulomb potential will be discussed below.

The equations for the Green's function of real time argument can be obtained from the equations for the functions G_j by exactly the same method as in[5].

If the functions $U_j(\mathbf{r}t)$ and $\mathbf{A}_j(\mathbf{r}t)$ are analytic functions for $0 > \operatorname{Im} t > -\beta$ and satisfy the condition

$$\lim_{\operatorname{Re} t \to -\infty} U_j(t) = 0, \qquad \lim_{\operatorname{Re} t \to -\infty} \mathbf{A}_j(t) = 0, \tag{8}$$

then

$$\lim_{t_0 \to -\infty} G_j^{<}(q_1 t_1; q_2 t_2; t_0) = g_j^{<}(q_1 t_1; q_2 t_2),$$

while the equation for the function $g_j^{<}(q_1 t_1; q_2 t_2)$ has the form

$$\Big\{ i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m_j} - U_j(\mathbf{r}_1 t_1) - \frac{i}{m_j}\mathbf{A}_j(\mathbf{r}_1 t_1)\nabla_1 - \frac{1}{2m_j}A_j^2(\mathbf{r}_1 t_1)\Big\} g_j^{<}(q_1 t_1; q_2 t_2)$$

$$= \int dq_3 \sigma_j^{HF}(q_1 t_1; q_3 t_1) g_j^{<}(q_3 t_1; q_2 t_2) + \int_{-\infty}^{t_1} dt_3 dq_3 \{\sigma_j^{>}(q_1 t_1; q_3 t_3)$$

$$- \sigma_j^{<}(q_1 t_1; q_3 t_3)\} g_j^{<}(q_3 t_3; q_2 t_2) - \int_{-\infty}^{t_2} dt_3 dq_3 \{g_j^{>}(q_3 t_3; q_2 t_2)$$

$$- g_j^{<}(q_3 t_3; q_2 t_2)\}\sigma_j^{<}(q_1 t_1; q_3 t_3), \tag{9}$$

where

$$\sigma_j^{HF,>,<}(q_1 t_1; q_2 t_2) = \lim_{t_0 \to -\infty} \sigma_j^{HF,>,<}(q_1 t_1; q_2 t_2; t_0).$$

We note that if the external magnetic field is constant and homogeneous, then it can be included in the basic Hamiltonian system; of course, $\mathbf{A}_j(\mathbf{r}t)$ is absent from Eq. (5) for this case. Here the form of Eqs. (9) remains unchanged.

Equations (9) can be used for the description of various transport phenomena. However, in the form in which they are written, these equations are too involved. They can be considerably simplified when the external fields are such that their potentials vary slowly in time and space. The equations obtained in this case are a generalization of the ordinary Boltzmann equations for transport phenomena. We note that in the chosen Hamiltonian system (1) the function g_j and the energy eigenfunctions σ_j depend on the spins in the following way:

$$g_j(q_1 t_1; q_2 t_2) = g_j(\mathbf{r}_1 t_1; \mathbf{r}_2 t_2)\,\delta_{\sigma_1\sigma_2},$$

$$\sigma_j(q_1 t_1; q_2 t_2) = \sigma_j(\mathbf{r}_1 t_1; \mathbf{r}_2 t_2)\,\delta_{\sigma_1\sigma_2}. \tag{10}$$

The derivation of the generalized Boltzmann equation is made in exactly the same sequence of steps in[5]. Transforming to the new variables

$$\mathbf{R} = {}^1\!/\!_2(\mathbf{r}_1 + \mathbf{r}_2), \qquad T = {}^1\!/\!_2(t_1 + t_2),$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \qquad t = t_1 - t_2,$$

expanding all quantities in Eq. (9) in series in $\mathbf{r}$ and t about the values $\mathbf{R}$ and T, and carrying out the Fourier transformation with the canonical momentum $\mathbf{p}$, with account of the condition $\operatorname{div} \mathbf{A}_i = 0$, we obtain the following equations, after rather involved transformations,

$$\left[\omega-\frac{(\mathbf{p}-\mathbf{A}_j(\mathbf{R}T))^2}{2m_j}-U_j(\mathbf{R}T)-\operatorname{Re}\sigma_j(\mathbf{p}\omega;\mathbf{R}T),\,g_j^<(\mathbf{p}\omega;\mathbf{R}T)\right]_\Pi$$
$$+[\operatorname{Re}g_j(\mathbf{p}\omega;\mathbf{R}T),\,\sigma_j^<(\mathbf{p}\omega;\mathbf{R}T)]_\Pi=\sigma_j^<(\mathbf{p}\omega;\mathbf{R}T)g_j^>(\mathbf{p}\omega;\mathbf{R}T)$$
$$-\sigma_j^>(\mathbf{p}\omega;\mathbf{R}T)g_j^<(\mathbf{p}\omega;\mathbf{R}T). \quad (11)$$

The generalized Poisson brackets entering into these equations are defined in the following way:

$$[F_1,F_2]_\Pi=\frac{\partial F_1}{\partial\omega}\frac{\partial F_2}{\partial T}-\frac{\partial F_1}{\partial T}\frac{\partial F_2}{\partial\omega}-\nabla_{\mathbf{p}}F_1\cdot\nabla_{\mathbf{R}}F_2+\nabla_{\mathbf{R}}F_1\cdot\nabla_{\mathbf{p}}F_2. \quad (12)$$

Moreover, the notation

$$g_j(\mathbf{p}z;\mathbf{R}T)=\int\frac{d\omega}{2\pi}\frac{a_j(\mathbf{p}\omega;\mathbf{R}T)}{z-\omega},$$
$$\sigma_j(\mathbf{p}z;\mathbf{R}T)=\int\frac{d\omega}{2\pi}\frac{\Gamma_j(\mathbf{p}\omega;\mathbf{R}T)}{z-\omega}+\sigma_j^{HF}(\mathbf{p};\mathbf{R}T), \quad (13)$$

is introduced and the spectral functions a_j and Γ_j are defined as

$$a_j(\mathbf{p}\omega;\mathbf{R}T)=g_j^>(\mathbf{p}\omega;\mathbf{R}T)+g_j^<(\mathbf{p}\omega;\mathbf{R}T),$$
$$\Gamma_j(\mathbf{p}\omega;\mathbf{R}T)=\sigma_j^>(\mathbf{p}\omega;\mathbf{R}T)+\sigma_j^<(\mathbf{p}\omega;\mathbf{R}T). \quad (14)$$

The functions $g_j(\mathbf{p}_z;\mathbf{R}T)$ can be represented in the form

$$g_j(\mathbf{p}z;\mathbf{R}T)=\left\{z-\frac{(\mathbf{p}-\mathbf{A}_j(\mathbf{R}T))^2}{2m_j}-U_j(\mathbf{R}T)-\sigma_j(\mathbf{p}z;\mathbf{R}T)\right\}^{-1}. \quad (15)$$

The validity of this relation can be established by direct calculation of the values of a_j and Re g_j with account of (13). Expressions for σ_j are obtained from Eq. (6) with the help of approximations for the Green's function.

2. EXTENSION OF THE GENERALIZED BOLTZMANN EQUATIONS TO THE CASE OF A COULOMB POTENTIAL

It has already been noted above, in the case of a Coulomb interaction potential between particles of the system considered, the separate terms of the expansion (6) for the energy eigenfunctions σ_j are divergent. To remove this divergence, it is necessary to carry out a re-expansion of the expression for σ_j, i.e., a regrouping of the different terms of the expansion with subsequation summation of the separate sequences of the diverging terms. Practically, such a re-expansion can be made by replacing the variation of the G_h functions over the external field U_j by variation over the effective field U_j^{eff} defined as follows:

$$U_j^{\text{eff}}(\mathbf{r}t;t_0)=U_j(\mathbf{r}t)-i\sum_{k=1}^{n}\int_{t_0}^{t_0-i\beta}dt_1dq_1V_{jk}(\mathbf{r}t;\mathbf{r}_1t_1)G_k(q_1t_1;q_1t_1+0;t_0). \quad (16)$$

Such a substitution is due to the fact that the actual physical quantity determining the behavior of the Coulomb system is the effective field in which the particles move, but not the applied external field, which is connected with the long-range character of the Coulomb potential. The introduction of $U_j^{\text{eff}}(\mathbf{r}t;t_0)$ essentially means an explicit account of the dynamic screening; that part of the Coulomb interaction between the particles which associated with all polarization processes is separated, while the remaining interaction between the particles is described by the screened potential. Such an approach to the theory of Coulomb systems was proposed by Silin.[2] The detailed discussion of this question is contained in[5,7].

The replacement of the variation is performed according to the following rule:

$$\frac{\delta G_j(q_1t_1;q_2t_2;t_0)}{\delta U_k(\mathbf{r}_3t_3)}=\sum_{l=1}^{n}\int_{t_0}^{t_0-i\beta}dt_4d\mathbf{r}_4\frac{\delta G_j(q_1t_1;q_2t_2;t_0)}{\delta U_l^{\text{eff}}(\mathbf{r}_4t_4)}\frac{\delta U_l^{\text{eff}}(\mathbf{r}_4t_4)}{\delta U_k(\mathbf{r}_3t_3)}. \quad (17)$$

The equation for the function $\sigma_j(q_1t_1;q_2t_2;t_0)$, which is the difference between the exact energy eigenfunctions of particles of type j and the energy eigenfunctions in the Hartree approximation, has the form

$$\sigma_j'(q_1t_1;q_2t_2;t_0)=iV_{jj}^s(\mathbf{r}_1t_1;\mathbf{r}_2t_2;t_0)G_j(q_1t_1;q_2t_2;t_0)$$
$$+i\sum_{k=1}^{n}\int_{t_0}^{t_0-i\beta}dt_3dq_3dt_4d\mathbf{r}_4V_{jk}^s(\mathbf{r}_1t_1;\mathbf{r}_4t_4;t_0)G_j(q_1t_1;q_3t_3;t_0)\frac{\delta\sigma_j'(q_3t_3;q_2t_2;t_0)}{\delta U_k^{\text{eff}}(\mathbf{r}_4t_4)}. \quad (18)$$

The quantity

$$V_{jk}^s(\mathbf{r}_1t_1;\mathbf{r}_2t_2;t_0)=\sum_{l=1}^{n}\int_{t_0}^{t_0-i\beta}dt_3d\mathbf{r}_3V_{jl}(\mathbf{r}_1t_1;\mathbf{r}_3t_3)\frac{\delta U_k^{\text{eff}}(\mathbf{r}_2t_2;t_0)}{\delta U_l(\mathbf{r}_3t_3)} \quad (19)$$

entering into this equation describes the time dependent effective interaction between particles of type j and k. This interaction is effected both directly and by a change in the potential created by all the remaining particles of all sorts. The radius of this interaction turns out to be much less, as a consequence of which it is called the screened potential. The equation satisfied by the V_{jk}^s can be obtained by means of the definitions (16) and (19):

$$V_{jk}^s(\mathbf{r}_1t_1;\mathbf{r}_2t_2;t_0)=V_{jk}(\mathbf{r}_1t_1;\mathbf{r}_2t_2)-i\sum_{l=1}^{n}\int_{t_0}^{t_0-i\beta}dt_3dq_3dt_4dq_4$$
$$\times V_{jl}^s(\mathbf{r}_1t_1;\mathbf{r}_4t_4;t_0)G_l(q_4t_4;q_3t_3;t_0)G_l(q_3t_3;q_4t_4;t_0)V_{kl}(\mathbf{r}_2t_2;\mathbf{r}_3t_3)$$
$$-i\sum_{l,\,p=1}^{n}\int_{t_0}^{t_0-i\beta}dt_3d\mathbf{r}_3dt_4dq_4dt_5dq_5dt_6dq_6V_{jl}^s(\mathbf{r}_1t_1;\mathbf{r}_3t_3;t_0)G_p(q_4t_4;q_5t_5;t_0)$$
$$\times G_p(q_6t_6;q_4t_4;t_0)V_{kp}(\mathbf{r}_2t_2;\mathbf{r}_4t_4)\frac{\delta\sigma_p'(q_5t_5;q_6t_6;t_0)}{\delta U_l^{\text{eff}}(\mathbf{r}_3t_3)}. \quad (20)$$

The self-consistent solutions of Eqs. (18) and (20) can be found by the method of iteration.

As before, it is seen from Eqs. (18) and (20) that one can separate the energy eigenfunctions in the Fock approximation from the energy eigenfunction σ_j', while the remaining part of σ_j' describing the correlation interaction between particles is decomposed into two analytic functions of the time arguments. In the derivation of generalized kinetic equations, only the analytic properties of the functions σ_j^c were used; therefore, Eqs. (11) and (15) are valid also in the case of interaction between particles in the sense that the equations formally preserve their form, only the expressions for the functions $\sigma_j^{>,<}$ are obtained not from Eq. (6) but from Eqs. (18) and (20).

3. DERIVATION OF THE KINETIC EQUATIONS OF THE NORMAL ELECTRON FERMI LIQUID IN THE ABSENCE OF A MAGNETIC FIELD

The quantum-mechanical description of transport phenomena can be made formally completely analogous to the classical description by the introduction of "quantum distribution functions,"[8] which are also known in the literature under the name "mixed representation of the density matrix":

$$f_j(\mathbf{p},\mathbf{R},T)=\int\frac{d\omega}{2\pi}g_j^<(\mathbf{p}\omega;\mathbf{R}T). \quad (21)$$

We write the functions $g_j^<$ in the form

$$g_j^<(\mathbf{p}\omega;\,\mathbf{R}T)=a_j(\mathbf{p}\omega;\,\mathbf{R}T)f_j(\mathbf{p}\omega;\,\mathbf{R}T). \tag{22}$$

With the help of the relation (15), the spectral function a_j can be written in the following way:

$$a_j(\mathbf{p}\omega;\mathbf{R}T)=\frac{\Gamma_j(\mathbf{p}\omega;\mathbf{R}T)}{\{\omega-(\mathbf{p}-\mathbf{A}_j(\mathbf{R}T))^2/2m_j-U_j(\mathbf{R}T)-\mathrm{Re}\,\sigma_j(\mathbf{p}\omega;\mathbf{R}T)\}^2+{}^1/_4\Gamma_j^2(\mathbf{p}\omega;\mathbf{R}T)}. \tag{23}$$

If $\Gamma_j(\mathbf{p}\omega;\,\mathbf{RT})\to 0$ (conditions under which this can be valid are discussed below), then

$$a_j(\mathbf{p}\omega;\mathbf{R}T)=2\pi\delta\left(\omega-\frac{(\mathbf{p}-\mathbf{A}_j(\mathbf{R}T))^2}{2m_j}-U_j(\mathbf{R}T)-\mathrm{Re}\,\sigma_j(\mathbf{p}\omega;\mathbf{R}T)\right)$$

In this case, using (21) and (22), we have

$$f_j(\mathbf{p},\mathbf{R},T)=\sum_k f_j(\mathbf{p}\omega_k^{(j)};\mathbf{R}T), \tag{24}$$

where $\omega_k^{(j)}$ represent the solutions of the following equation:

$$\omega_k^{(j)}=\frac{(\mathbf{p}-\mathbf{A}_j(\mathbf{R}T))^2}{2m_j}+U_j(\mathbf{R}T)+\mathrm{Re}\,\sigma_j(\mathbf{p}\omega_k^{(j)};\mathbf{R}T). \tag{25}$$

We now proceed to the derivation of the Landau-Silin equation in the absence of a magnetic field. For simplicity, we consider a one-component Fermi system, consisting of charged particles; electrical neutrality of the system as a whole is guaranteed by the introduction of a homogeneous compensating background. The behavior of the considered system is greatly simplified at low temperatures. For an equilibrium system at zero temperature, we have

$$f(\mathbf{p}\omega;\mathbf{R}T)=f(\omega)=\begin{cases}1, & \omega<\mu,\\ 0, & \omega>\mu,\end{cases} \tag{26}$$

where μ is the chemical potential of the system concerned. In this case,

$$\begin{aligned}\sigma^>(\mathbf{p}\omega;\,\mathbf{R}T)&=\sigma^>(\mathbf{p}\omega)=0, \quad \omega<\mu,\\ \sigma^<(\mathbf{p}\omega;\,\mathbf{R}T)&=\sigma^<(\mathbf{p}\omega)=0, \quad \omega>\mu.\end{aligned} \tag{27}$$

For temperatures different from absolute zero, but sufficiently low, the function $f(\omega)$ differs from zero and unity only for values of ω very close to μ.

We shall assume that at sufficiently low temperatures, the function f has the same properties also in the presence of an external perturbation. In other words, the existence of such a function $\mu(\mathbf{RT})$ is assumed, also that $f(\mathbf{p}\omega;\,\mathbf{RT})$ differs from zero and unity only in the immediate vicinity of $\mu(\mathbf{RT})$; it is equal to zero for ω much greater than $\mu(\mathbf{RT})$ and unity for a much less than $\mu(\mathbf{RT})$.

Inasmuch as the functions σ_j satisfy the same boundary conditions as the G_j (which is evident from their definition), then, in the same way as in the equilibrium case, we have

$$\begin{aligned}\sigma^>(\mathbf{p}\omega;\,\mathbf{R}T)&=0, \quad \omega<\mu(\mathbf{R}T),\\ \sigma^<(\mathbf{p}\omega;\,\mathbf{R}T)&=0, \quad \omega>\mu(\mathbf{R}T).\end{aligned} \tag{28}$$

We shall consider normal Fermi systems, i.e., such for which the $\sigma^{<,>}$ are continuous functions for $\omega\approx\mu$. In this case, in the vicinity of $\omega=\mu$, these functions are small, i.e., $\Gamma(\mathbf{p}\omega;\,\mathbf{RT})\approx 0$. In other words, the spectral function $a(\mathbf{p}a;\,\mathbf{RT})$ possesses a δ-like singularity for $\omega=\mu$. The argument of the δ function in the vicinity of $\omega=\mu$ is a monotonically increasing function of ω, since the derivative of $\mathrm{Re}\,\sigma(\mathbf{p}\omega;\,\mathbf{RT})$ with respect to ω is negative. Therefore, in the given region, Eq. (25) has only the single root

$$\omega_1=E(\mathbf{p},\mathbf{R},T)+U(\mathbf{R}T), \tag{29}$$

where

$$E(\mathbf{p},\,\mathbf{R},\,T)=\frac{p^2}{2m}+\mathrm{Re}\,\sigma(\mathbf{p}\omega_1;\,\mathbf{R}T)\big|_{\omega_1=E(\mathbf{p},\,\mathbf{R},\,T)+U(\mathbf{R}T)},$$

while $f(\mathbf{p}\omega;\,\mathbf{RT})$, as a consequence of (24), is identical with $f(\mathbf{p},\,\mathbf{R},\,T)$. The equilibrium function $f(\mathbf{p})$ is spherically symmetric. At equilibrium, $E(\mathbf{p},\,\mathbf{R},\,T)$ goes over into $E(\mathbf{p})$. The reaction of the system to the external perturbation in the approximation used in the present approximation is manifest in the change of occupation numbers of the one-particle levels for $\omega\approx\mu$, i.e., for those momenta for which $E(\mathbf{p})\approx\mu$. We assume that there exists a unique momentum p_0, called the Fermi momentum, such that $E(p_0)=\mu$. As has been shown, in the vicinity of $\omega=\mu$, the one-particle states of the system are weakly damped. This makes it possible to speak of weakly excited states of the considered system in the language of weakly interacting quasi-particles, described by the distribution function $n(\mathbf{p},\,\mathbf{R},\,T)$. This is determined in the vicinity of the Fermi level as follows:

$$n(\mathbf{p},\,\mathbf{R},\,T)=f(\mathbf{p}\omega_1;\,\mathbf{R}T)\big|_{\omega_1=E(\mathbf{p},\,\mathbf{R},\,T)+U(\mathbf{R}T)}. \tag{30}$$

The equation for the function $n(\mathbf{p},\,\mathbf{R},\,T)$ can be obtained from the generalized kinetic equations (11) if in the latter we take into account all the approximations made in that section. Neglecting the functions $\sigma^{>,<}$ in (11), using the δ-like character of the spectral function $a(\mathbf{p}\omega;\,\mathbf{RT})$, the property ((12) of the generalized Poisson brackets, and the definition (30), we get the following equation in the vicinity of $\omega=\mu$:

$$\delta\left(\omega-\frac{p^2}{2m}-U(\mathbf{R}T)-\mathrm{Re}\,\sigma(\mathbf{p}\omega;\mathbf{R}T)\right)\left[\omega-\frac{p^2}{2m}-U(\mathbf{R}T)-\mathrm{Re}\,\sigma(\mathbf{p}\omega;\mathbf{R}T),\,n(\mathbf{p},\mathbf{R},T)\right]_{\mathrm{P}}=0. \tag{31}$$

Opening the generalized Poisson brackets according to the definition (12) and expanding $\mathrm{Re}\,\sigma$ in a series in ω about the value $\omega=\mu$, we obtain the following equation for the quasiparticle distribution function, after several transformations,

$$\frac{\partial n(\mathbf{p},\,\mathbf{R},\,T)}{\partial T}+\nabla_{\mathbf{p}}E(\mathbf{p},\,\mathbf{R},\,T)\cdot\nabla_{\mathbf{R}}n(\mathbf{p},\,\mathbf{R},\,T)-\nabla_{\mathbf{R}}E(\mathbf{p},\,\mathbf{R},\,T)\cdot\nabla_{\mathbf{p}}n(\mathbf{p},\mathbf{R},T)-\nabla_{\mathbf{R}}U(\mathbf{R}T)\cdot\nabla_{\mathbf{p}}n(\mathbf{p},\,\mathbf{R},\,T)=0. \tag{32}$$

Equation (32) is the Landau-Silin kinetic equation for a charged Fermi liquid for neglect of the spin part and the collision integral of the quasiparticles. Equation (32) refers to the complete distribution function of all quasiparticles. But the distribution function of the quasiparticles is determined only near the Fermi surface. Therefore, writing the distribution function in the form

$$n(\mathbf{p},\mathbf{R},T)=n_0(\mathbf{p})+\delta n(\mathbf{p},\mathbf{R},T), \tag{33}$$

we identify $\delta n(\mathbf{p},\,\mathbf{R},\,T)$ with the distribution function of the quasiparticles close to the Fermi surface. Substituting (33) in Eq. (32) and keeping terms only of first order in δn and $U(\mathbf{R}, T)$, we obtain a linearized kinetic equation, which describes quasiparticles with momenta p close to the Fermi surface:

$$\frac{\partial \delta n}{\partial T} + \nabla_{\mathbf{p}} E \cdot \nabla_{\mathbf{R}} \delta n - \nabla_{\mathbf{R}} E \cdot \nabla_{\mathbf{p}} n_0 - \nabla_{\mathbf{R}} U \cdot \nabla_{\mathbf{p}} n_0 = 0. \qquad (34)$$

In obtaining (34), we neglected the term $\nabla_R E \cdot \nabla_p \delta n$, inasmuch as it is quadratic in δn, as will be seen later. Using the definition of the velocity of quasiparticles near the Fermi surface,

$$\mathbf{v} = \nabla_{\mathbf{p}} E(\mathbf{p}, \mathbf{R}, T),$$

the relation (29), and the equation

$$\nabla_{\mathbf{p}} n_0(\mathbf{p}) = -\mathbf{v}\delta(E(p_0) - \mu),$$

we can rewrite Eq. (34) in the following form:

$$\frac{\partial \delta n}{\partial T} + \mathbf{v}\nabla_{\mathbf{R}} \delta n + \mathbf{v}(\nabla_{\mathbf{R}} \operatorname{Re} \sigma' - e\mathbf{E})\,\delta(E(\mathbf{p}_0) - \mu) = 0, \qquad (35)$$

where

$$\sigma' = \sigma + U - U^{\text{eff}}, \quad e\mathbf{E} = -\nabla_{\mathbf{R}} U^{\text{eff}},$$

while the remaining terms are functions of **p**, **R**, and T. Thus E is the intensity of the self-consistent electric field acting on the particles of the system. By equating Eq. (35) with the Landau-Silin kinetic equation for δn[7] we obtain the following expression for the function $f(\mathbf{p}, \mathbf{p}')$, which represents the interaction energy of quasiparticles with momenta **p** and **p**′:

$$\operatorname{Re} \sigma'(\mathbf{p}, \mathbf{R}, T) = \int f(\mathbf{p}, \mathbf{p}')\,\delta n(\mathbf{p}', \mathbf{R}, T)\,d\mathbf{p}'. \qquad (36)$$

4. DERIVATION OF THE KINETIC EQUATIONS OF A NORMAL ELECTRON FERMI LIQUID IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD

The derivation of the kinetic equation of a normal electron Fermi liquid in the presence of an external magnetic field is based on the same assumptions and carried out in the same fashion as in the absence of a magnetic field. The magnetic field was assumed to be so weak that the equilibrium function $n_0(\mathbf{p})$ could be regarded as spherically symmetric. The difference lies in the fact that now the function $E(\mathbf{p}, \mathbf{R}, T)$, determined by the relation (29), has the form

$$E(\mathbf{p}, \mathbf{R}, T) = \frac{(\mathbf{p} - \mathbf{A}(\mathbf{R}\,T))^2}{2m} + \operatorname{Re}\sigma(\mathbf{p}\omega_1; \mathbf{R}T)\,|_{\omega_1 = E(\mathbf{p}, \mathbf{R}, T) + U(\mathbf{R}, T)}. \qquad (37)$$

Carrying out the same operations as previously, we obtain a kinetic equation for the quasiparticle distribution function in the form (32), but written in the space of the canonical and not the kinetic momentum. In order to put it in its usual form, it is necessary to transform to the kinetic momentum **P**, which is connected with the canonical by the usual relation

$$\mathbf{p} = \mathbf{P} + \mathbf{A}(\mathbf{R}T).$$

In the transformation to the kinetic equation, derivatives with respect to **p**, **R**, and T are transformed as follows:

$$\left(\frac{\partial F(\mathbf{p}, \mathbf{R}, T)}{\partial p_\alpha}\right)_{\mathbf{R}} = \left(\frac{\partial F(\mathbf{P}, \mathbf{R}, T)}{\partial P_\alpha}\right)_{\mathbf{R}},$$

$$\left(\frac{\partial F(\mathbf{p}, \mathbf{R}, T)}{\partial R_\alpha}\right)_{\mathbf{p}} = \left(\frac{\partial F(\mathbf{P}, \mathbf{R}, T)}{\partial R_\alpha}\right)_{\mathbf{P}} - \frac{\partial F}{\partial P_\beta}\,\frac{\partial A_\beta}{\partial R_\alpha},$$

$$\left(\frac{\partial F(\mathbf{p}, \mathbf{R}, T)}{\partial T}\right)_{\mathbf{p}, \mathbf{R}} = \left(\frac{\partial F(\mathbf{P}, \mathbf{R}, T)}{\partial T}\right)_{\mathbf{P}, \mathbf{R}} - \frac{\partial F}{\partial P_\alpha}\frac{\partial A_\alpha}{\partial T}.$$

Taking Eq. (37) and the definition of the velocity of the quasi-particles close to the Fermi surface into account

$$\mathbf{v} = \nabla_{\mathbf{p}} E(\mathbf{p}, \mathbf{R}, T) = \frac{\mathbf{p} - \mathbf{A}(\mathbf{R}\,T)}{m} + \nabla_{\mathbf{p}} \operatorname{Re}\sigma(\mathbf{p}, \mathbf{R}, T),$$

we obtain the following equation for the function $n(\mathbf{p}, \mathbf{R}, T)$ after rather cumbersome algebraic transformations:

$$\frac{\partial n}{\partial T} + e\mathbf{E}\nabla_{\mathbf{P}} n + e[\mathbf{v}\mathbf{H}]\,\nabla_{\mathbf{P}} n + \mathbf{v}\nabla_{\mathbf{R}} n - \nabla_{\mathbf{R}} \operatorname{Re}\sigma \cdot \nabla_{\mathbf{P}} n = 0. \qquad (38)$$

In (38), the following notation has been used:

$$e\mathbf{E} = -\frac{\partial \mathbf{A}(\mathbf{R}\,T)}{\partial T} - \nabla_{\mathbf{R}} U(\mathbf{R}\,T),$$

$$e\mathbf{H} = \operatorname{rot} \mathbf{A}(\mathbf{R}T),$$

while all the remaining quantities are functions of **P**, **R**, and T. Comparing (38) with the Landau-Silin kinetic equation,[7] we obtain an expression for the function $f(\mathbf{P}, \mathbf{P}')$, which represents the interaction energy of quasiparticles with momenta **P** and **P**′:

$$\operatorname{Re}\sigma'(\mathbf{P}, \mathbf{R}, T) = \int f(\mathbf{P}, \mathbf{P}')\,n(\mathbf{P}', \mathbf{R}, T)\,d\mathbf{P}'. \qquad (39)$$

It is taken into account in Eq. (39), that the Landau-Silin equation contains self-consistent field that differs from the external field in (38) by the value of the polarization field as determined by Eq. (16). Therefore σ^H—the energy eigenfunction in the Hartree approximation—is excluded from $\operatorname{Re}\sigma$.

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