HIGH-FREQUENCY TURBULENCE SPECTRA OF A PLASMA AND ACCELERATION OF SUBCOSMIC RAYS

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Nonlinear equations describing the interaction of high-frequency turbulent pulsations of a plasma of the Langmuir type are used to determine the equilibrium spectrum of plasma turbulence in the region in which energy is transferred along the spectrum. It is assumed that a stationary source generates turbulent energy in the region of very large values of k; this energy is transferred to a region with $k = k_0$, which is the basic turbulence dimension, and decays in this region. An approximate power spectrum $W_h \sim k^{-\nu}$ is obtained in the transfer region $k \gg k_0$. For strong turbulence, when absorption and scattering processes are insignificant, $\nu \approx 2.84$ and in the general case it can rise to $\nu = 4$. Application of the results to astrophysical problems, and in particular to the problem of acceleration of subcosmic rays, is discussed.

1. INTRODUCTION

As is well known, besides the low-frequency turbulence of the hydrodynamic type, high frequency (hf) turbulence can also exist in a plasma. This turbulence was intensely investigated recently by many experimenters (see, for example,[1,2]). The study of the hf turbulence of a plasma is both of great practical interest (for example, in the problem of turbulent heating[2]) and of general interest (for example, in problems involving the acceleration of fast particles and the origin of cosmic and subcosmic rays[3]). The latest investigations of the mechanism of heating and ionization of interstellar gas[4] and isotropization of cosmic rays[5,6] make it possible to use even now astrophysical data for the determination of the spectrum and intensity of plasma turbulence. The spectrum of plasma turbulence can be obtained also with the aid of correlation measurements of fluctuation fields in a laboratory experiment[7]. The purpose of this paper is to calculate theoretically the spectrum of plasma turbulence and to consider the ensuing consequences for the mechanisms of isotropization and acceleration of cosmic and cosmic rays in the interstellar medium.

The general problem should include the self-consistent problem of the distribution of energy of plasma turbulence, of the ions and electrons accelerated by it, and the electromagnetic radiation produced by them. The electrons acquire in this case energy from the plasma turbulence much more effectively than the ions, and it can be assumed that the turbulence will be suppressed as a result of the energy consumed in electron acceleration (when $\epsilon \gg m_i c^2$, the acceleration of the electrons and of ions is the same). However, there exists processes, for example the inverse Compton effect on the plasma turbulence, in which the relativistic electron loses energy, producing low-frequency electromagnetic radiation, which can be effectively reconverted into plasma turbulence. A detailed discussion of these problems as applied to astrophysical conditions will be discussed in a separate article (in Astronomicheskiĭ Zhurnal [Sov. Astr. AJ]). In the present article, attention will be focused on the physical problems of establishment of the stationary turbulent spectrum and its response to electrons and ions of different energies. It will be assumed that the spectra of cosmic electrons and ions, determined by the balance of many processes, are specified, and the number of electrons is much smaller than that of protons of the same velocity. It should be stated that under astrophysical conditions the regions of intense plasma turbulence apparently are concentrated in individual small volumes behind the fronts of shock waves. It is possible in this case to carry out the averaging for the cosmic ions, but the electrons may be completely decelerated in the space between two such regions. A discussion of these questions is beyond the scope of the present article.

The formulation of the problem of hf plasma turbulence has much in common with the formulation of the problem of the turbulence of liquids. Stationary turbulence is the result of the balance of generation of turbulence in one spectral interval and its transfer to another interval, where the energy is absorbed. Thus, it is possible to single out three spectral regions in which the predominant processes are generation, transfer, and absorption of the turbulence. Unlike liquids, the theory of energy transfer over the spectrum of the plasma turbulence can be constructed on the basis of nonlinear equations for the interaction of turbulent pulsations[8]. Another essential difference from liquids lies in the effect of the interaction between the pulsations and fast charged particles (Landau absorption[9] on fast particles), since the high-frequency plasma pulsations carry sufficiently strong electric fields and therefore interact intensely with the charged particles.

In cosmic and laboratory conditions, fast particles are always present, and their influence on the turbulence spectrum can be appreciable. Since the spectra of particles accelerated in a turbulent plasma usually fall off towards larger energies, it follows that at

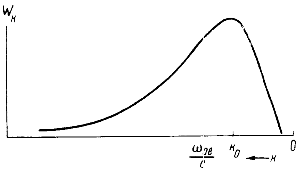

FIG. 1. Schematic form of the stationary spectrum of plasma turbulence.

large phase velocities v_{ph} of the turbulent pulsations, and all the more when $v_{ph} > c$, the main mechanism of their damping may be paired collisions of particles. We emphasize here also the difference between hf turbulence and the turbulence of liquids, due to the character of the absorption of the turbulent pulsations. In liquids, the absorption coefficient increases with increasing k, whereas for Langmuir pulsations it is independent of k.

The stationary spectrum of plasma turbulence is shown schematically in Fig. 1. The spectral turbulence function W_k equals the turbulence energy per cm^3 in the interval dk. The total turbulence energy will be

$$W = \int_0^\infty W_k dk. \tag{1.1}$$

The quantity W_k is connected with the number of plasmons N_k^l by the relation

$$W_k = \frac{\hbar k^2}{2\pi^2} \omega^l N_k^l, \tag{1.2}$$

where ω^l is the plasmon frequency. The region of direct generation of the plasma waves usually corresponds to small phase velocities, i.e., to large k. For example, under astrophysical conditions the main sources of hf turbulence are shock waves and various types of instability, which excite Langmuir oscillations with v_{ph} on the order of several times v_{Te} (v_{Te}—thermal velocity of the electrons). This corresponds to $v_{ph} \approx (0.01-0.03)\,c$, although in strong fields v_{ph} may reach 0.1c. On the other hand, for interaction with fast particles, particular importance attaches to the region with large phase velocities, $v_{ph} \approx (0.1-1)\,c$. It can be frequently assumed that there is no direct generation in this region, which can be called the energy-transfer region. The maximum energy in the right side of the spectrum corresponds to the main turbulence scale and the wave number k_0. Just as in liquids, the energy-containing region corresponds to small k, but the direction of the energy transfer is opposite here, i.e., from large k to small ones.

Inasmuch as the absorption does not depend on k, it is maximal where the energy is maximal, i.e., at small k. However, the rate of energy transfer also depends on the energy density, so that the absorption and transfer regions cannot be distinctly separated. Nevertheless, the intensity of transfer for Langmuir pulsations decreases as a rule with decreasing k, so that there should be such a $k_{min} \lesssim k_0$ for which the transfer and the absorption become comparable. The spectrum should subsequently fall off. The position of the maximum (k_0) is an essential parameter of the spectrum. Unfortunately, it is difficult to determine it by directly solving the equations, since we do not have the small parameter here.

A qualitative analysis of the question of determining k_0 and a number of other estimates for cosmic and subcosmic rays will be presented in another article, which will be published in Astronomicheskiĭ Zhurnal. Here we shall solve the problem of the stationary spectrum of an isotropic plasma turbulence in the region $k \gg k_0$. The assumption that the turbulence is isotropic is natural, since isotropization of the turbulence pulsations is the most rapid nonlinear process[8].

The general equation for the interaction of turbulent pulsations, which is valid in the region of the spectral transfer and absorption, can be written in the form

$$\frac{dW_k}{dt} = -\nu_e W_k - \gamma_k W_k + W_k \left[\int_k^\infty Q(k, k_1) W_{k_1} dk_1 - \int_0^k Q(k_1, k) W_{k_1} dk_1 \right] + \int dk_1\, dk_2\, dk_3 R(k, k_1, k_2, k_3) (k^2 W_{k_1} W_{k_2} W_{k_3} + k_1^2 W_{k_2} W_k W_{k_3} - k_2^2 W_k W_{k_1} W_{k_3} - k_3^2 W_k W_{k_1} W_{k_2}). \tag{1.3}$$

In the right side, the first term describes absorption due to pair collisions, the second Landau absorption on fast particles, the third induced scattering of pulsations by the plasma particles, and the last the scattering of pulsations by one another. For a stationary spectrum, the left side vanishes. Then (1.3) is an integral equation for W_k.

The scattering of pulsations by pulsations, with only plasma electrons taken into account, was considered in[10,11], while scattering with allowance of ion interactions was considered in[12,13].

2. THEORY OF THE ASYMPTOTIC TURBULENCE SPECTRUM FOR SMALL WAVE NUMBERS

In the region $v_{ph} > v_{ph}^* = 3v_{Te}^2/v_{Ti}$, which in interstellar gas corresponds to the most important interval $v_{ph} > 0.1c$, the expressions for the operators Q and R, which describe the energy transfer, simplify greatly[8]

$$Q(k, k_1) = \frac{3(k_1^2 - k^2)}{16\sqrt{2\pi} n_0 m_e v_{Ti}} \int d\Omega_1 \frac{(\mathbf{k}\mathbf{k}_1)^2}{k^2 k_1^2} \frac{1}{|\mathbf{k}_1 - \mathbf{k}|} \frac{T_e/T_i}{(1 + T_e/T_i)^2}, \tag{2.1}$$

$$R(k, k_1, k_2, k_3) = \frac{\omega_{0e}^3}{6\pi n_0^2 T_i^2 v_{Te}^2} \frac{\delta(k^2 + k_1^2 - k_2^2 - k_3^2)}{4^3(1 + T_e/T_i)^2} \int d\Omega_1\, d\Omega_2\, d\Omega_3 \times \delta(k_1^2 + k^2 - k_2^2 - k_3^2) \left(\frac{(\mathbf{k}\mathbf{k}_3)}{kk_3} \frac{(\mathbf{k}_2\mathbf{k}_1)}{k_2 k_1} + \frac{(\mathbf{k}_2\mathbf{k})}{k_2 k} \frac{(\mathbf{k}_3\mathbf{k}_1)}{k_3 k_1} \right)^2. \tag{2.2}$$

Here $d\Omega_1$, $d\Omega_2$, $d\Omega_3$—elements of solid angles of the vectors $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$; T_e and T_i—temperatures of the electrons and ions in ergs; $\omega_{0e}^2 = 4\pi n_0 e^2/m_e$; n_0—plasma concentration. The integration over the angles in (2.1) is elementary, but in (2.2) it is quite cumbersome. At first it is convenient to integrate, with the aid of a δ-function, over the angles of $\mathbf{k}_2$ and $\mathbf{k}_3$, choosing as the z axis the direction $\mathbf{k}_1 + \mathbf{k}_2$, after which one integrates over the angles of $\mathbf{k}_1$. Omitting the intermediate steps, we present the final result:

$$Q(k, k_1) = \frac{\sqrt{2\pi}}{8n_0} \frac{T_e}{m_e v_{Ti} T_i} \frac{(k_1^2 - k^2)}{(1 + T_e/T_i)^2 k_1^3} (k_1^2 + {}^2/_5 k^2). \tag{2.3}$$

Here $k_1 > k_2$; when $k_1 < k_2$ it is necessary to make the substitution $k_1 \rightleftarrows k_2$. Further,

$$R(k, k_1, k_2, k_3) = \frac{\pi\omega_{0e}^3 \delta(k^2 + k_1^2 - k_2^2 - k_3^2)}{48 v_{Te}^2 n_0^2 T_i^2 (1 + T_e/T_i)^2 k^3 k_1^3 k_2^3 k_3^3} \left\{ \frac{(k_2^2 - k_3^2)^4 k_1^5}{5(k^2 - k_1^2)^3} \right.$$

$$\times(k^2-\frac{3}{7}k_1^2)+\frac{(k_2^2-k_3^2)^2k_1^3}{15(k^2-k_1^2)}(-5k^4+24k^2k_1^2-43k_1^4)$$
$$+\frac{k_1^3}{315}(622k_1^6+825k_1^4k^2+672k_1^2k^4+105k^6)\Big\}. \qquad (2.4)$$

Here $kk_1 < k_2k_3$; $k > k_1$. If $k < k_1$ but $kk_1 < k_2k_3$, then it is necessary to make the substitution $k \rightleftarrows k_1$. If $kk_2 > k_2k_3$ and $k_2 > k_3$, then $k \rightleftarrows k_2$ and $k \rightleftarrows k_1$, and if $kk_1 > k_2k_3$ and $k_2 < k_3$, then $k \rightleftarrows k_3$ and $k_2 \rightleftarrows k_1$.

The relative efficiency of the processes of induced scattering and scattering by plasmons (2.3) and (2.4) depends on the plasma parameters, on the magnitude of the energy flux, on the wave-number interval, and on the position of the maximum of the spectrum. As will be shown subsequently, the solution of the general equation reduces to a simple modification of an equation corresponding to the collision of plasmons with one another. We shall therefore take into account only plasmon collisions (2.4), i.e., we retain in the right side of (1.3) only the last term. We introduce the dimensionless variables $\xi = (k_1/k)^2$, $\eta = (k_2/k)^2$ for $kk_1 < k_2k_3$ and $\xi = (k_3/k)^2$, $\eta = (k_2/k)^2$ for $kk_1 > k_2k_3$. Taking into account the energy conservation law $k^2 + k_1^2 - k_2^2 - k_3^2 = 0$ and the conditions that follow (2.4), we obtain the sought integral equation, which determines the turbulence spectrum W_k, in the form

$$-\frac{1}{W_k}\frac{dW_k}{dt}=\frac{\pi}{6}\frac{\omega_{0e}}{(8n_0)^2}\frac{\omega_{0e}^2}{(T_i+T_e)^2v_{Te}^2}\Big\{\int_{\xi_0}^{1}d\eta\int_{\xi_0}^{\eta}d\xi L(\xi,\eta)V(\xi,\eta)$$
$$+\int_1^\infty d\eta\int_\eta^\infty d\xi M(\xi,\eta)V(\xi,\eta)+2\int_1^\infty d\eta\int_{\xi_0}^{1}d\xi N(\xi,\eta)U(\xi,\eta)\Big\}. \qquad (2.5)$$

Here $\xi_0 = (k_0/k)^2$, and the quantities $V(\xi,\eta)$ and $U(\xi,\eta)$ are expressed in terms of the spectral functions of the turbulence in the form

$$V(\xi,\eta)=\Big[\frac{w_\xi w_\eta w_{1+\xi-\eta}}{W_k}+\xi w_\eta w_{1+\xi-\eta}-\eta w_\xi w_{1+\xi-\eta}$$
$$-w_\xi w_\eta(1+\xi-\eta)\Big]\frac{1}{\eta^2\xi^2(1+\xi-\eta)^2},$$
$$U(\xi,\eta)=\Big[\frac{w_\xi w_\eta w_{\xi+\eta-1}}{W_k}+(\xi+\eta-1)w_\xi w_\eta-\xi w_{\xi+\eta-1}w_\eta$$
$$-\eta w_{\xi+\eta-1}w_\xi\Big]\frac{1}{\eta^2\xi^2(\xi+\eta-1)^2},$$
$$w_\xi = W_{k\sqrt{\xi}}.$$

Finally

$$L(\xi,\eta)=\frac{(2\eta-\xi-1)^4}{5(1-\xi)^3}\xi^{5/2}\Big(1-\frac{3}{7}\xi\Big)+\frac{\xi^{5/2}(2\eta-\xi-1)^2}{15(1-\xi)}\cdot$$
$$\times(-5+24\xi-43\xi^2)+\frac{\xi^{5/2}}{315}(622\xi^3+825\xi^2+672\xi+105),$$
$$M(\xi,\eta)=\frac{(2\eta-\xi-1)^4}{5(\xi-1)^3}\Big(\xi-\frac{3}{7}\Big)+\frac{(2\eta-\xi-1)^2}{15(\xi-1)}\cdot$$
$$\times(-5\xi^2+24\xi-43)+\frac{1}{315}(622+825\xi+672\xi^2+105\xi^3),$$
$$N(\xi,\eta)=\frac{(2-\xi-\eta)^4\xi^{5/2}}{5(\eta-\xi)^3}\Big(\eta-\frac{3}{7}\xi\Big)+\frac{(2-\eta-\xi)^2\xi^{5/2}}{15(\eta-\xi)}\cdot$$
$$\times(-5\eta^2+24\xi\eta-43\xi^2)+\frac{\xi^{5/2}}{315}(622\xi^3+825\xi^2\eta+672\xi\eta^2+105\eta^3). \qquad (2.6)$$

Equating the left side of (2.5) to zero, we seek the solution of the equation in the form of a power function $W_k \sim k^{-\nu}$ in a certain interval of wave numbers. By virtue of the fact that the energy transfer over the spectrum increases the phase velocities of the turbulence pulsations, the main energy of the turbulence will be concentrated at values of k that are small but larger than k_0. Then

$$W_{k_1}=(\nu-1)\frac{W}{k_0}\Big(\frac{k_0}{k_1}\Big)^\nu,\quad w_\xi=\frac{W}{k_0}\xi_0^{\nu/2}\frac{1}{\xi^{\nu/2}},$$
$$W_k=(\nu-1)\frac{W}{k_0}\xi_0^{\nu/2},\quad \xi_0=\Big(\frac{k_0}{k}\Big)^2. \qquad (2.7)$$

Equation (2.5) takes the form

$$0=\frac{1}{W_k}\frac{dW_k}{dt}=\frac{\pi}{6}\omega_{0e}\Big[\frac{W}{8n_0(T_i+T_e)}\Big]^2\frac{\omega_{0e}^2(\nu-1)^2}{k_0^2v_{Te}^2}(G_1+G_2+G_3)\xi_0^\nu, \qquad (2.8)$$

$$G_1=\int_{\xi_0}^{1}d\eta\int_{\xi_0}^{\eta}d\xi L(\xi,\eta)v(\xi,\eta),\quad G_2=\int_1^\infty d\eta\int_\eta^\infty d\xi M(\xi,\eta)v(\xi,\eta), \qquad (2.9)$$
$$G_3=2\int_{\xi_0}^{1}d\xi\int_1^\infty d\eta N(\xi,\eta)u(\xi,\eta),$$

where

$$v(\xi,\eta)=\frac{1+\xi^{\nu/2+1}-\eta^{\nu/2+1}-(1+\xi-\eta)^{\nu/2+1}}{\eta^{\nu/2+2}\xi^{\nu/2+2}(1+\xi-\eta)^{\nu/2+2}},$$
$$u(\xi,\eta)=\frac{1+(\xi+\eta-1)^{\nu/2+1}-\xi^{\nu/2+1}-\eta^{\nu/2+1}}{\eta^{\nu/2+2}\xi^{\nu/2+2}(\xi+\eta-1)^{\nu/2+2}}. \qquad (2.10)$$

Let us consider the spectrum far from the energy-containing region ($k \gg k_0$), i.e., let us assume that the following inequality is satisfied:

$$\xi_0 \ll 1. \qquad (2.11)$$

From (2.9) and (2.10) it follows that the main contribution to G_1 is made by the region of small ξ and small η, when $\xi \approx \eta \approx \xi_0$, and the region of small ξ and η, close to unity, when $\xi \approx \xi_0$ and $1-\eta \approx \xi_0$. A contribution to G_3 is made only by the second region. The quantity G_2 does not contain large factors of the type $\xi_0^{-\alpha}$, and is negligibly small. In the region $\xi, \eta \ll 1$ we get from (2.6)

$$L(\xi,\eta)\approx {}^4/_3\xi^{5/2}(\eta+{}^{11}/_5\xi),\quad v(\xi,\eta)\approx\frac{(\nu/2+1)(\eta-\xi)}{\xi^{\nu/2+2}\eta^{\nu/2+2}}. \qquad (2.12)$$

Consequently, the contribution from this region of ξ and η to the integral G_1 amounts to, in the case when $\nu > {}^3/_2$,

$$\delta G_1=\frac{16}{15}\frac{4\nu-3}{\nu(\nu/2-1)(\nu-{}^3/_2)\xi_0^{\nu-3/2}}. \qquad (2.13)$$

It is possible to verify in the same manner that the contribution of the region $\xi \ll 1$, $1-\eta \ll 1$ to G_1 coincides with (2.13)[1], i.e., $G_1 \approx 2\delta G_1$. For G_3, it is convenient to go over to integration with respect to $\eta' = \xi + \eta - 1$. It is easy to see that for the region $\eta - 1 \approx \xi_0$, $\eta \ll 1$, we have

$$u(\xi,\eta') \approx -v(\xi,\eta'),\ N(\xi,\eta') \approx L(\xi,\eta'),$$

where $v(\xi,\eta)$ and $L(\xi,\eta)$ are specified by Eqs. (2.13). Calculating G_3, we find that in this approximation G_3 and G_1 cancel each other exactly: $G_3 = -G_1$.

From this we can obtain a number of consequences. We can take into account the following terms of the expansion in ξ and η in (2.12). It is obvious that they should give a result that differs from (2.13) by a factor ξ_0^{-1}, i.e., $G_1 \sim \xi_0^{-(\nu-5/2)}$. This does not take place when

[1] This is easiest to verify by reversing the order of integration and making the substitution $1+\xi-\eta=\eta'$.

$\nu > {}^5/_2$. For G_3 we also have $G_3 \sim \xi_0^{-(\nu-5/2)}$, and, as will be seen subsequently, in this approximation G_1 no longer cancels out G_3. Besides the region of small η, it is necessary to consider also the region of finite and arbitrary η, but small ξ. Since L and N are proportional to $\xi^{3/2}$, and ν and u are proportional to $\xi^{-(\nu/2+2)}$ the result will be $G_1 + G_2 \sim \xi_0^{-(\nu/2-1/2)}$. Comparing this with $\xi_0^{-(\nu-5/2)}$ and equating them to each other, we verify that the terms with $\xi_0^{-(\nu-5/2)}$ prevail, when $\nu > 4$ and $\xi_0^{-(\nu/2-1/2)}$ prevail when $\nu < 4$.

We consider first $\nu > 4$. Using a more accurate expansion for L, M and v, u than (2.12), we get

$$G_1 + G_3 = \frac{16}{15}\frac{(\nu/2+2)(16\nu-34)}{(\nu-{}^5/_2)\nu(\nu/2-1)(\nu/2-2)\xi_0^{\nu-5/2}}. \tag{2.14}$$

This expression vanishes only when $\nu = {}^{34}/_{16}$, which contradicts the condition $\nu > 4$. Thus, Eq. (2.18) has no solutions in the region $\nu > 4$.

We now consider $\nu < 4$. We transform G_1, reversing the order of integration with respect to ξ and η, and introducing a new variable $\eta' = 2\eta/(1+\xi) - 1$. The integration with respect to the variable η' is carried out within symmetrical limits, and the integrand is even. Writing the integral in the form of double the value of the integral from zero to $\eta'_{\max}$ and returning to the old variables, we get

$$G_1 = 2\int_{\xi_0}^{1} d\xi \int_{\xi}^{(1+\xi)/2} d\eta L(\xi,\eta)\, v(\xi,\eta). \tag{2.15}$$

This integral has a singularity only at small ξ. The singularity at small η is compensated in (2.15) by a similar singularity in G_3, i.e., the total integral $G_1 + G_3$ has no diverging singularity at small η. This makes it possible to use the expansion of L and N at small ξ but arbitrary η:

$$L(\xi,\eta) \approx {}^4/_3\eta(1-\eta)\xi^{3/2}.$$

Hence

$$G_1 = \frac{8}{3}\int_{\xi_0}^{1}\frac{d\xi}{\xi^{(\nu+1)/2}}\int_{\xi}^{1/2}\frac{d\eta}{\eta^{\nu/2+1}(1-\eta)^{\nu/2+1}} \times (1-\eta^{\nu/2+1}-(1-\eta)^{\nu/2+1}).$$

Similarly

$$G_3 = \frac{8}{3}\int_{\xi_0}^{1}\frac{d\xi}{\xi^{(\nu+1)/2}}\int_{\xi}^{\infty}\frac{d\eta}{\eta^{\nu/2+1}(1+\eta)^{\nu/2+1}} \times (1+\eta^{\nu/2+1}-(1+\eta)^{\nu/2+1}).$$

Using the value of the integral

$$I_{\pm} = \int_{\xi}^{\infty}\frac{d\eta}{\eta^{\nu/2+1}(1\pm\eta)^{\nu/2+1}} = \frac{2}{\xi^{\nu/2}\nu}F\left(\frac{\nu}{2}+1,\ -\frac{\nu}{2},\ -\frac{\nu}{2}+1,\ \mp\xi\right)$$

where F is the hypergeometric function, and its asymptotic expansion at small ξ

$$I_{\pm} \approx \frac{2}{\xi^{\nu/2}\nu} \mp \frac{\nu/2+1}{(\nu/2-1)\xi^{\nu/2-1}}$$

we get

$$G_1 + G_3 = \frac{2^{\nu/2+5}}{3\nu(\nu-1)\xi_0^{(\nu-1)/2}}\left[2\left(1-\frac{1}{2^{\nu/2}}\right) - F\left(\frac{\nu}{2}+1,\ -\frac{\nu}{2},\ -\frac{\nu}{2}+1,\ \frac{1}{2}\right)\right]. \tag{2.16}$$

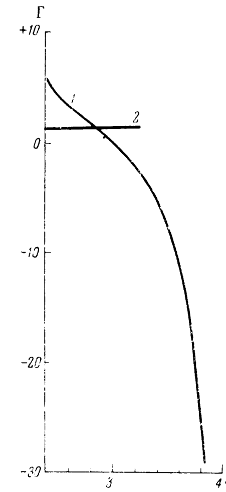

FIG. 2. Left (1) and right (2) parts of (2.17).

Thus, the sought equation, with allowance for collisions of the plasmons, is of the form

$$F\left(\frac{\nu}{2}+1,\ -\frac{\nu}{2},\ -\frac{\nu}{2}+1,\ \frac{1}{2}\right) = \Gamma_0, \tag{2.17}$$

where

$$\Gamma_0 = 2(1-2^{-\nu/2}).$$

Figure 2 shows the right (curve 1) and left (curve 2) parts of Eq. (2.17) as functions of ξ. The intersection of these curves yields $\nu = 2.84$.

We now take into account the induced scattering from ions, and also the influence exerted on the spectrum by absorption by fast particles and pair collisions entering in Eq. (1.3). This influence can be expressed as the difference between the right side of (2.16) and zero, i.e., as the change of the quantity Γ_0 in (2.17). Since this change depends on k, the quantity ν will also depend on k, i.e., the spectrum will be of the power-law type only approximately. On the other hand, curve 1 is very steep, and has a singularity at $\nu = 4$. Therefore, even an appreciable change in Γ_0 does not change ν very strongly.

The total Γ can be written in the form

$$\Gamma = \Gamma_0 - \delta\Gamma_Q - \delta\Gamma_\nu - \delta\Gamma_\gamma, \tag{2.18}$$

where $\delta\Gamma_Q$ describes the effect of scattering by ions, $\delta\Gamma_\nu$ the absorption due to the collisions, and $\delta\Gamma_\gamma$ the effect of fast particles (subcosmic rays). From (2.3) and (2.7) we get

$$\delta\Gamma_Q = \frac{n_0T_i}{W}\frac{9\nu}{\sqrt{\pi}2^{(\nu+1)/2}(\nu-1)}\left(\frac{k_0v_{Te}}{\omega_{0e}}\right)^3\frac{v_{Te}}{v_{Ti}}\frac{1}{\xi_0^{\nu/2+1}}. \tag{2.19}$$

It is only easy to obtain

$$\delta\Gamma_\nu = \frac{18}{\pi}\frac{\nu_e}{\omega_{0e}}\frac{n_0^2(T_e+T_i)^2}{W^2}\frac{\nu}{(\nu-1)2^{\nu/2-1}}\left(\frac{k_0v_{Te}}{\omega_{0e}}\right)^2\frac{1}{\xi_0^{(\nu+1)/2}}$$

and a similar expression for $\delta\Gamma_\gamma$, which differs in that ν_e is replaced by γ_k.

In the general case, the Cerenkov absorption by fast particles is described by the formula

$$\gamma_k = \frac{\pi}{2}\sum_\alpha\frac{Z^4\omega_{0e}^4m_e}{n_0k^3}\int_{\varepsilon_{cr\alpha}}^{\infty}\left(m_\alpha+\frac{\varepsilon}{c^2}\right)^2\frac{\partial}{\partial\varepsilon}\frac{f_\alpha(\varepsilon)}{\sqrt{\varepsilon}\,(m_\alpha+\varepsilon/c^2)^{3/2}}d\varepsilon. \tag{2.20}$$

Here $f_\alpha(\epsilon)$ is the distribution function α of the fast particles, normalized to their kinetic energy ϵ:

$$\int_{\varepsilon_0}^{\infty} f_\alpha(\varepsilon)\,d\varepsilon = n_{1\alpha}, \quad \varepsilon = m_\alpha c^2\left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right), \quad \varepsilon_{cr}{}^\alpha = \varepsilon|_{v=\omega_{0e}/k}.$$

For nonrelativistic protons, if the number of fast electrons is sufficiently small[2)], we have

$$\gamma_k = \frac{\pi}{2}\frac{Z^2\omega_{0e}{}^4 m_e m^{1/2}}{n_0 k^3}\frac{f(\varepsilon_{cr})}{\sqrt{\varepsilon_{cr}}}.$$

If we specify a power-law spectrum of the fast particles at $\epsilon > \epsilon_0$, with an exponent γ,

$$f(\varepsilon) = \frac{\gamma - 1}{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)^\gamma n_1, \tag{2.21}$$

where n_1 is the number of fast particles with energy larger than $\epsilon_0 (\gamma > 1)$, then we get

$$\gamma_k = \sqrt{2}\,\pi(\gamma - 1)\frac{n_1}{n_0}\omega_{0e}\left(\frac{2\varepsilon_0 k^2}{m\omega_{0e}{}^2}\right)^{\gamma-1}\frac{m_e}{m}Z^2. \tag{2.22}$$

Hence

$$\delta\Gamma_\gamma = 18\sqrt{2}\frac{\nu(\gamma-1)}{(\nu-1)2^{\nu/2-1}}\left(\frac{2\varepsilon_0 k_0^2}{m\omega_{0e}{}^2}\right)^{\gamma-1}\cdot$$
$$\times\left(\frac{k_0 v_{Te}}{\omega_{0e}}\right)^2\frac{n_1}{n_0}\frac{n_0^2(T_e+T_i)^2}{W^2}\frac{1}{\xi_0^{\gamma+(\nu-1)/2}}\frac{m_e}{m}Z^2.$$

All the obtained quantities $\delta\Gamma$ depend on a number of parameters: ϵ_0, k_0, W, γ, ν_e, n_1, etc. Actually these parameters are interrelated. In principle they should be expressed in terms of the plasma parameters, in terms of the flux of the turbulent energy along the spectrum, in terms of the number of fast particles that depend on the injection conditions and on the turbulence itself. The main difficulty lies here in the existence of the influence of the position of the maximum, i.e., k_0, which so far can be estimated only from qualitative considerations. We note that with decreasing k the value of $\delta\Gamma$ decrease and they can become asymptotically negligible. Finally, from (2.18) we see that the contributions of all the $\delta\Gamma$ decrease Γ and consequently increase ν.

The obtained solutions (in particular Eq. (2.17) retain their form even if the turbulence spectrum is greatly modified in the energy-containing region $k \approx k_0$. In the derivation of (2.17), the only important factor is the appreciable increase of the turbulence energy when k approaches k_0. This gives grounds for assuming that, in any case for the indicated class of spectra, the obtained solution is stable. It should also be noted that the pair collisions of the particles stop the spectral transfer connected with scattering by ions when k is approximately equal to k_0. This can serve as a rough estimate for k_0. The larger ν_e, the larger k_0, i.e., the flatter the spectrum in the energy-containing region. The form of the spectrum k_0 has little influence on the asymptotic form of the spectrum, on which the collisions influence in the opposite direction, making it steeper. The increase of the spectrum index can be intuitively explained as follows. Scattering by the ions is integral in the region under consideration, i.e., the energy is transferred to the main turbulence scale, from which it is "spread" by plasmon-plasmon scattering over the entire spectrum, and the amount of energy decreases strongly with increasing k.

3. SPECTRUM OF ACCELERATED FAST PARTICLES

Inasmuch as the present theory is limited by the assumption $v_{ph} > v_{ph}^* = 3v_{Te}^2/v_{Ti}$, we can choose for ϵ_0

$$\varepsilon_0 = \frac{1}{2}m(v_{ph}^*)^2 = \frac{9}{2}m\frac{v_{Te}{}^2}{v_{Ti}{}^2}v_{Te}{}^2. \tag{3.1}$$

Then

$$\delta\Gamma_\gamma = 18\sqrt{2}\frac{\nu(\gamma-1)}{(\nu-1)2^{\nu/2-1}}\left(\frac{3v_{Te}}{v_{Ti}}\right)^{2(\gamma-1)}$$
$$\times\left(\frac{k_0 v_{Te}}{\omega_{0e}}\right)^{2\gamma}\frac{n_1}{n_0}\frac{n_0^2(T_i+T_e)^2}{W^2}\frac{Z^2 m_e/m}{\xi_0^{\gamma+(\nu-1)/2}}.$$

The value of γ for protons can in principle be taken from observations, but for low-energy cosmic rays they are quite inaccurate. Cosmic rays are hardly accelerated directly in the interstellar medium, but even if they arrive there from some sources, the interstellar turbulence does influence them. Although we are not considering here the self-consistent problem of the distribution of cosmic electrons and ions and of plasma turbulence, we shall estimate, by way of an example, the character of the spectrum of the protons accelerated by the plasma turbulence, without allowance for their ionization laws. The acceleration of nonrelativistic particles by turbulent pulsations is described by the expression[8]:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial\varepsilon}\left(D\frac{\partial}{\partial\varepsilon}\frac{f}{\sqrt{\varepsilon}}\right), \tag{3.2}$$

where the diffusion coefficient D for Langmuir pulsations is given by

$$D = \frac{\pi}{2\sqrt{2}}\frac{Z^2\omega_{0e}{}^4 m_e\sqrt{m}}{n_0}\int_{\omega_{0e}\sqrt{m/2\varepsilon}}^{\infty}\frac{W_{k_1}dk_1}{k_1{}^3}.$$

Substituting here the turbulence spectrum in the form (2.7), we obtain

$$D = \frac{2^{\nu/2-1/2}\pi(\nu-1)Z^2}{(\nu+2)}\left(\frac{m_e}{m}\right)^2\frac{W}{n_0 T_e}\left(\frac{v_{Te}k_0}{\omega_{0e}}\right)^2\left(\frac{\omega_{0e}\sqrt{m}}{k_0}\right)^{3-\nu}\varepsilon^{\nu/2+1}\omega_{0e}. \tag{3.3}$$

As seen from (3.3), when $3 \le \nu < 4$ the diffusion coefficient is a rapidly increasing function of ϵ. This is due to the fact that the phase velocity of the waves that are at resonance with the particle increases with increasing ϵ, and the spectral density of the energy of the turbulence increases rapidly towards small, k, i.e., large v_{ph}.

Equating the right side of (3.2) to zero, we obtain for the stationary spectrum of the particles the condition for diffusion along the energy axis with a constant flux

$$D\frac{\partial}{\partial\varepsilon}\frac{f(\varepsilon)}{\sqrt{\varepsilon}} = \text{const.} \tag{3.4}$$

Substituting in lieu of D its value from (3.3), integrating, and comparing f with (2.21), we get

$$\gamma = (\nu - 1)/2 \approx 1 - 1.5,$$

i.e., the spectrum is softer than for ordinary cosmic

[2)]It is seen from (2.20) that if the number of relativistic particles, is specified then $\nu_k \sim n_1\epsilon^{-2}$. Therefore, if the fast electrons are concentra centrated in the region of large energies, their contribution to the absorption is small.

rays. This is natural when the maximum is approached. In the calculation of the stationary spectrum of subcosmic rays it is necessary to take into account the ionization losses. Allowance for these losses modifies the spectrum, which is no longer purely powerlaw.

4. DISCUSSION OF RESULTS

The asymptotic solution obtained above for $k \gg k_0$ has an approximate powerlaw character with $\nu \approx 3-4$. The latter value is more probable, and for soft cosmic rays it yields $\gamma \approx 1.5$. The steep character of the turbulent spectrum is of significance for the problem of acceleration of fast particles. When the particle becomes accelerated, it interacts with faster and faster waves, whose energy consequently increases. Therefore the efficiency of the acceleration increases with increasing particle energy. For example, the nonrelativistic-particle acceleration, which can be obtained from (3.3), is equal to

$$\overline{d\varepsilon / dt} = \overline{a\varepsilon^{(\nu-1)/2}}, \tag{4.1}$$

where

$$a = \frac{\pi}{2}(\nu - 1)2^{(\nu-1)/2}\left(\frac{k_0}{\omega_{0e}\sqrt{m}}\right)^{\nu-3} Z^2 \left(\frac{k_0 v_{Te}}{\omega_{0e}}\right)^2 \left(\frac{m_e}{m_i}\right)^2 \frac{W}{n_0 T}\omega_{0e}. \tag{4.2}$$

When $\nu > 3$, the rate of acceleration increases with increasing ϵ more rapidly than in the acceleration of nonrelativistic particles by the Fermi mechanism, for which

$$d\varepsilon / dt = a_F\sqrt{\varepsilon}.$$

The previously made estimate[3], which leads to the conclusion that the acceleration decreases with increasing ϵ, corresponded to the assumption that the energy W_k in the spectral interval, which corresponds to the particle energy, is constant, i.e., no account was taken of the concrete spectrum of the turbulence.

The conclusion that the acceleration increases with energy corresponds, naturally, to nonrelativistic particles. When $\epsilon \gg mc^2$ and $v \approx c$, the acceleration effect is determined by the value of W_k in the entire spectral interval $v_{ph} < c$. The region effective in practice is $v_{ph} \approx c$, which does not change with further increase of particle energy. Consequently, in the region of relativistic energy this mechanism results in a decrease of the acceleration efficiency with increasing particle energy, although at high plasma-turbulence intensity, for example in supernova shells, the acceleration can occur up to relatively high energies[3].

The number of accelerated fast particles is regulated by the turbulence level, and vice versa. The main part of the energy flux along the turbulent spectrum dissipates in the region of the minimum energy, i.e., near $k \approx k_0$. The absorption is the result of collisions also by fast particles. Let Q be the energy generated in 1 cm^3 per second at low phase velocities. Then, from the stationarity condition we get

$$Q = \nu_e W + \int \gamma_k W_k \, dk = (\nu_e + \overline{\gamma}_k) W. \tag{4.3}$$

This makes it possible to express W in the energy-containing region $k \approx k_0$ in terms of the dissipation power Q:

$$W \approx Q / (\nu_e + \overline{\gamma}_k),$$

where $\overline{\gamma}_k$ can be obtained from (2.22), taking into consideration the fact that the absorption of the turbulence by fast particles vanishes when $ck < \omega_{0e}$, i.e., when $v_{ph} > c$. This yields ($\nu > 3$ for nonrelativistic particles)

$$\overline{\gamma}_k = \sum_\alpha \frac{\sqrt{2}\pi(\nu - 1)(\nu - 3)}{4}\omega_{0e}\frac{m_e}{m_\alpha}Z_\alpha^2\frac{n_{1\alpha}}{n_0}\frac{c^2}{v_{Te}^2}\left(\frac{3v_{Te}}{v_{Ti}}\right)^{\nu-3}\left(\frac{k_0 v_{Te}}{\omega_{0e}}\right)^{\nu-1}.$$

The number of fast nonrelativistic electrons is apparently small. It should be noted that is $ck_0 \ll \omega_{0e}$, then the main dissipation, which occurs in the region $k \approx k_0$, can be due only to collisions. When the number n_1 of the cosmic rays increases, $\overline{\gamma}_k$ increases and W decreases in the region of the minimum. As a result, the diffusion coefficient D (3.3) decreases, i.e., the acceleration rate decreases.

The steep character of the turbulent spectrum can be qualitatively confirmed by astrophysical data. The isotropy of the cosmic rays was explained as being due to the action of plasma waves[5,6]. In order for the isotropization to be effective, it should act within a time $t \lesssim 10^5$ years. To this end, the isotropization increment for the main mass of the cosmic rays should be of the order of 3×10^{-13} sec^{-1}. The isotropization of the cosmic rays is described by the diffusion coefficient D^t [3]:

$$D^t = \pi^2 Z^2 \frac{e^2}{v}\int_{\omega_{0e}/v}^{\infty}\frac{W_k}{k}\left(1 - \frac{\omega_{0e}^2}{k^2 v^2}\right)dk.$$

When $v \approx c$ and $W_k \sim k^{-\nu}$, we have

$$\gamma_i \approx \frac{D^t}{m^2c^2} = \frac{\pi}{2}\left(\frac{m_e}{m}\right)^2\left(\frac{v_{Te}}{c}\right)^2 Z^2 \frac{\nu - 1}{\nu(\nu + 2)}\left(\frac{k_0 c}{\omega_{0e}}\right)^{\nu-1}\frac{W}{n_0 T}\omega_{0e}. \tag{4.4}$$

This quantity determines the energy of pulsations with $v_{ph} < c$. The increment of acceleration of the subcosmic rays with energy $\epsilon \ll mc^2$ is determined by waves with $v_{ph} < c$, and the steeper the spectrum the smaller the fraction of the turbulent energy which is responsible for the acceleration of the low-velocity particles. From (3.2) we find the increment, i.e., the reciprocal of the acceleration time

$$\gamma_a \approx \frac{D}{\varepsilon^{3/2}} \approx \pi\left(\frac{m_e}{m}\right)^2\left(\frac{v_{Te}}{c}\right)^2\frac{\nu - 1}{\nu + 2}2^{(\nu-1)/2}Z^2\left(\frac{k_0 c}{\omega_{0e}}\right)^{\nu-1}\frac{W}{n_0 T}\omega_{0e}\left(\frac{\varepsilon_0}{mc^2}\right)^{(\nu-3)/2}.$$

Comparing with (4.4), we determine γ_a and obtain the increment of the energy of the subcosmic rays per cm^3 per second:

$$n_1\varepsilon_0\gamma_a = \langle n_1\varepsilon\rangle \nu\frac{5 - \nu}{\nu - 3}\frac{\varepsilon_0}{mc^2}2^{(\nu+1)/2}\gamma_i. \tag{4.5}$$

Such an increment should be obtained if the number of fast electrons is sufficiently small and the isotropization of the cosmic rays is carried out by plasma waves.

This energy can be consumed by the ionization loss and by the acceleration, i.e., by the transformation of the subcosmic particles into ordinary cosmic particles. From an analysis of the properties of the interstellar gas we find that the ionization losses are on the average equal to 2×10^{-26} erg/cm^3 sec[4]. The transition into the region of cosmic rays should be smaller. Consequently, the quantity (4.5) has the indicated upper limit. From the condition of stability in the magnetic field of the galaxy we have $\langle n_1\epsilon\rangle \lesssim 1$ eV/cm^3. For ϵ_0 we assume 5 MeV, i.e., $v \approx 0.1c$. It is easy to see that when $\nu = 4$ and $\gamma_i \approx 3 \times 10^{-13}$ sec^{-1} the energy increment is on the order of 5×10^{-26} erg/cm^3 sec. This agrees qualitatively with the ionization loss, particularly if account is taken of the fact that the real increment of the energy should be decreased somewhat, for when $v_{ph} < 0.1c$, where the character of the interaction changes, the turbulent spectrum drops more steeply than when $\nu = 4$, and this is of particular importance for subcosmic rays. The particle spectrum should also decrease steeply in the region $v < 0.1c$. This agrees with the results of measurements of the upper limit of the L_α radiation that can be produced upon charge exchange of particles having such energies[14].

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