STRONG FLUCTUATIONS OF THE AMPLITUDE OF A LIGHT WAVE AND PROBABILITY OF FORMATION OF RANDOM CAUSTICS

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Attention is called to the fact that the start of the region of strong fluctuations of the intensity of light coincides with the region in which, with a probability on the order of unity, random caustics appear.

THE question of strong fluctuations of the amplitude of a plane light wave in a medium containing large (compared with the wavelength) random inhomogeneities is dealt with in[1-5]. It is noted in[4] that strong fluctuations are connected with random focusing of the light in such a medium, i.e., with the lens action of the large inhomogeneities. However, the geometrical picture of random focusings was not considered in detail in[4], although the ray representations give useful information concerning the physical nature of the strong fluctuations. The purpose of the present paper is to consider in greater detail than in[4] the geometry of the random foci (caustics) and to present an estimate showing that the start of the region of strong fluctuations coincides approximately with the region in which the probability of formation of the random caustics becomes noticeably different from zero.

In the case of propagation in a randomly inhomogeneous medium, each ray experiences random deviations and moves on the average farther away from the initial direction (in accordance with a diffusion law[6]). The individual rays can in this case intersect and form caustics (i.e., envelopes of the rays), as shown schematically in Fig. 1. Obviously, with appearance of the first caustic, the fluctuations of the amplitude of the light wave increase both as a result of the "swelling" of the field on the caustics and formation of the caustic shadow, and as a result of interference of the waves ahead of the caustic. We can therefore expect that in region b on Fig. 1 the fluctuations of the amplitudes will be larger than in region a which is free of caustics.

With further penetration into the layer, the number of caustics to which the ray is tangent will increase (see Fig. 2, which shows an approximate form of the caustic picture as a whole). However, as a result of the interference of a large number of waves, we cannot expect a noticeable increase of fluctuations in the depth of the turbulent layer (i.e., in the region c of Fig. 2). If this is indeed the case, then the region b, in which the probability of appearance of at least one caustic is already

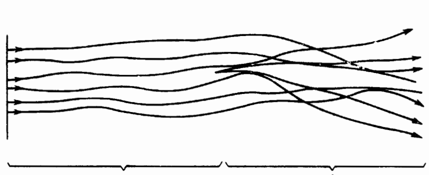

FIG. 1

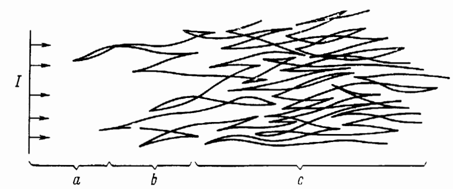

FIG. 2

noticeably different from zero, can be regarded as the start of the region of strong fluctuations. Of course, a study of the real course of the fluctuations of the amplitude inside the layer (region c) calls for an examination of the averaging action of the diffraction effects, which are not included in the geometrical scheme developed here. As to the region b, the qualitative considerations advanced above can be confirmed by quantitative estimates, which indicate that the first caustic on the ray appears approximately where the variance $\sigma^2 = \overline{\chi^2}$ of the level of the amplitude $\chi = \ln(A/A_0)$ becomes of the order of unity, i.e., where the region of strong fluctuations begins.

We start from the law of conservation of the energy flux in the ray tube, from which it follows that in the geometrical optics approximation the wave amplitude $u = Ae^{ik}$ equals

$$A = A_0/\sqrt{I}, \quad I = D(\tau)/D(0), \tag{1}$$

where $D(\tau) = \partial(x, y, z)/\partial(\xi, \eta, \tau)$, and $\mathbf{r}(\tau) = [x(\tau), y(\tau), z(\tau)]$ is the ray equation, determined from the characteristic equations

$$\frac{d\mathbf{r}}{d\tau} = \mathbf{p}, \qquad \frac{d\mathbf{p}}{d\tau} = \frac{1}{2}\nabla\varepsilon$$

with initial conditions

$$x(0) = \xi, \ y(0) = \eta, \ z(0) = 0,$$
$$p_x(0) = p_y(0) = 0, \ p_z(0) = [\varepsilon(\xi, \eta, 0)]^{1/2}$$

(here $\mathbf{p} \equiv \nabla\varphi$, where φ—eikonal). Putting $\epsilon = 1 + \nu$ (ν—random component of the dielectric constant of the medium) and assuming the fluctuations to be weak, $(\overline{\nu^2})^{1/2} \ll 1$, we obtain in first order of perturbation theory with respect to:

$$x(\tau) = \xi + \frac{1}{2}\int_0^\tau (\tau - \tau_1)\frac{\partial\nu}{\partial x}(\xi, \eta, \tau_1)\,d\tau_1,$$

$$y(\tau) = \eta + \frac{1}{2}\int_0^\tau (\tau - \tau_1)\frac{\partial\nu}{\partial y}(\xi, \eta, \tau_1)\,d\tau_1,$$

$$z(\tau)=\tau+\frac{1}{2}\int_0^{\tau}\nu(\xi,\eta,\tau_1)\,d\tau_1. \tag{2}$$

In the same approximation, as simple calculations show, we have

$$D(\tau)=1-2\chi,\quad D(0)=1,$$
$$I=1-2\chi, \tag{3}$$

where

$$\chi=\frac{1}{4}\int_0^{\tau}(\tau-\tau_1)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\nu(\xi,\eta,\tau_1)\,d\tau_1 \tag{4}$$

is the level of the amplitude determined in the first approximation of the geometric-optics method[4]. With the obtained value of I, formula (1) with $|\chi|\ll 1$ is equivalent to the customarily employed expression $A = A_0e^{\chi}$, but has the advantage that it gives a clear indication of the appearance of caustics at $\chi \sim 1/2$, when the Jacobian $D(\tau)$ vanishes. Of course, when $\chi \sim 1/2$ it is no longer possible to confine oneself to the first approximation of perturbation theory, but formulas (1), (3), and (4) are suitable for estimates of the probability of appearance of caustics in order of magnitude.

Let us find the probability P of the fact that $I \le 0$, or, what is the same, $\chi \ge 1/2$. Assuming that χ has a normal distribution with a mean value $\overline{\chi} = 0$ (this follows from (4)) and a variance $\sigma^2 = \overline{\chi^2}$ (by virtue of the large-numbers law, χ has a normal distribution also in the case of negative fluctuations of ν), i.e., that $w(\chi) = (1/\sqrt{2\pi}\sigma)\exp\{-\chi^2/2\sigma^2\}$, we obtain

$$P=\int_{1/2}^{\infty}w(\chi)\,d\chi=\frac{1}{2}\left[1-\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)\right]. \tag{5}$$

According to (5), when $\sigma \ll 1$ (weak fluctuations of the level amplitude), the probability P is exponentially small: $P \cong \sqrt{2/\pi}\,\sigma\exp(-1/8\sigma^2)$, but when $\sigma \sim 1$ it reaches already the appreciable magnitude $P \cong 0.3$. The probability of appearance of at least one caustic will obviously be larger than P, since χ can exceed the level $\chi = 1/2$ more than once along the ray. It therefore follows that when $\sigma \sim 1$ the probability of appearance of the caustic becomes comparable with unity.

This conclusion is applicable to a spherical wave, for which formulas (1) and (3) are valid, but the trajectories of the rays and the variance σ^2 are calculated by means of different formulas than for a plane wave. The difference lies in the fact that in the case of a spherical wave the variance σ^2 reaches a value on the order of unity at much larger distances than in the case of the plane wave. The increased distance of the region of strong fluctuations and the region of appearance of caustics, which practically coincides with it, is attributed to the divergence of the rays in the spherical waves. In this connection, it should be pointed out that an interpretation of experiments similar to those described in[7], on the basis of the theory developed in[1-5] for plane waves, calls for great caution.

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[1] V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. **49**, 1581 (1965) [Sov. Phys.-JETP **22**, 1083 (1966)].

[2] V. I. Tatarskiĭ, Izv. vuzov, Radiofizika **10**, 48 (1967).

[3] V. I. Tatarskiĭ, ibid. **10**, 231 (1967).

[4] V. I. Tatarskiĭ, Rasprostranenie voln v turbulentnoĭ atmosfere (Wave Propagation in the Turbulent Atmosphere), Nauka, 1967.

[5] V. I. Klyatskin and V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. **55**, 662 (1968) [Sov. Phys.-JETP **28**, 346 (1969)].

[6] L. A. Chernov, Rasprostranenie voln v srede so sluchaĭnymi neodnorodnostyami (Propagation of Waves in a Medium with Random Inhomogeneities), AN SSSR, 1958.

[7] M. E. Gracheva and A. S. Gurvich, Izv. vuzov, Radiofizika **8**, 717 (1965).

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