INTERACTION OF ELECTRONS WITH PARAMAGNETIC IMPURITIES IN SUPERCONDUCTORS

S. L. GINZBURG

A. F. Ioffe Physicotechnical Institute, USSR Academy of Sciences

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We obtain the scattering amplitude of electrons in superconductors with paramagnetic impurities. We consider the influence of paramagnetic impurities on the temperature of the transition of the metal into the superconducting state and on the gap at zero temperature. We show that the transition temperature and the gap at zero temperature increase when the impurities are introduced if the exchange part of the interaction is negative and the temperature of the transition into the superconducting state is low compared with the characteristic Kondo temperature.

We consider in this paper the influence of paramagnetic impurities on the change of the temperature of transition of a metal into the superconducting state and on the change of the gap at zero temperature. This question was first considered by Abrikosov and Gor'kov[1]. In[1], the cross section for the scattering of an electron by an impurity, which must be known to solve the problem, was calculated in first order of perturbation theory. Recently, however, it became clear that perturbation theory is not applicable to the case of the scattering of a conduction electron[2]. In the present paper the problem is solved with the aid of a dispersion method. It is shown that the transition temperature and the gap at zero temperature increase when the impurities are introduced if the exchange part of the interaction is negative, and the temperature of the transition into the superconducting state is small compared with a certain characteristic temperature first introduced by Kondo[2]. On the other hand, when the exchange part of the interaction is positive or the exchange part of the interaction is negative but the transition temperature is large compared with the characteristic Kondo temperature, the results of[1] are obtained, namely that the transition temperature and the gap at zero temperature decreases.

Maleev[3] has shown that in the case of negative exchange part of an interaction, the non-exchange amplitude for the scattering of electrons in the normal metal on the Fermi surface is close to its maximum possible value ip_0^{-1} (p_0—Fermi momentum). It is essentially this circumstance which leads to an increase of the transition temperature and of the gap in the superconductor.

The amplitude of scattering of a conduction electron by a paramagnetic impurity in a superconductor was calculated by Maki[4]. However, the formulas by him are complicated and difficult to visualize. Furthermore, the equations obtained by Maki have incorrect analytic properties (energy pole on the physical sheet). In the present paper we use a different solution method, similar to that used by the author to calculate the scattering amplitude in a normal metal[5]. The problem will be solved for the case of zero non-exchange part of the interaction, and for integer spins. The generalization to the case of half-integer spins can be readily performed in the manner used by the author[5] for the normal metal, but the formulas obtained thereby are very cumbersome and difficult to use for concrete calculations, and will therefore not be written out here.

Maki has shown[4] that when a = 0 (a and b—non-exchange and exchange Born scattering amplitudes) there are four independent scattering amplitudes $t_\pm$ and $\tau_\pm$, for which there exist unitarity conditions

$$\operatorname{Im} t_\pm = \{|t_\pm|^2 + s(s+1)|\tau_\pm|^2\}\operatorname{Re} g_\pm,$$

$$\operatorname{Im} \tau_\pm = \left\{ t_\pm \tau_\pm^* + t_\pm^* \tau_\pm - |\tau_\pm|^2 \operatorname{th}\frac{\omega}{2T} \right\} \operatorname{Re} g_\pm; \tag{1}$$

Here

$$g_\pm = p_0(\omega \pm \Delta) / \sqrt{\omega^2 - \Delta^2},$$

ω—energy reckoned from the Fermi surface, Δ—gap, and T—temperature. We shall henceforth consider only the case when $|p_0 b| \ll 1$.

We introduce the functions

$$u_\pm = (1 + 2ig_\pm t_\pm) / \tau_\pm. \tag{2}$$

From (1) we readily get

$$u_\pm(\omega + i\delta) - u_\pm(\omega - i\delta) = 2i \operatorname{Re} g_\pm \cdot \operatorname{th}\frac{\omega}{2T}. \tag{3}$$

Formula (3) determines $u_\pm$ accurate to a rational function $P(\omega)$:

$$u_\pm = P(\omega) + ig_\pm - \frac{2}{\pi}\int_{-E_F}^{\infty} \frac{d\omega'}{\omega' - \omega} n(\omega') \operatorname{Re} g_\pm(\omega'); \tag{4}$$

$n(\omega) = (1/2)[1 - \tanh(\omega/2T)]$—Fermi distribution function. From (4) and from the definition of $g_\pm$ in (1) we see that $u_\pm$ are analytic functions with a cut on the real axis at $|\omega| > \Delta$.

Let us assume that near the Fermi surface, $P(\omega)$ is constant, and we choose it from the condition that when $\Delta \to 0$ the functions $u_\pm$ go over into the known expressions for the normal metal. When $|p_0 b| \ll 1$ we get[3,5] $P(\omega) = b^{-1}$, and then

$$u_\pm = \frac{1}{b} + ig_\pm - \frac{2}{\pi}\int_{-E_F}^{\infty} \frac{d\omega'}{\omega' - \omega} n(\omega') \operatorname{Re} g_\pm(\omega'). \tag{5}$$

When T = 0, the integral can be readily calculated and we have

$$u_\pm = \frac{1}{b} - \frac{2p_0}{\pi}\ln\frac{\Delta}{2E_F} + g_\pm\Big\{-\frac{2}{\pi}\ln\frac{\omega+\sqrt{\omega^2-\Delta^2}}{\Delta} + i\Big\}. \quad (6)$$

In the derivation of (6) we took into account the fact that $E_F \gg \Delta$. It is convenient to introduce in lieu of $u_\pm$ the dimensionless functions $\Phi_\pm$:

$$\Phi_\pm = u_\pm / 2ig_\pm. \quad (7)$$

From (6) and (7) at T = 0 we get

$$\Phi_\pm = -\frac{i}{\pi h}\frac{p_0}{g_\pm} + \frac{i}{\pi}\ln\frac{\omega+\sqrt{\omega^2-\Delta^2}}{\Delta} + \frac{1}{2},$$
$$h = \frac{2p_0 b}{\pi}\Big(1 - \frac{2p_0 b}{\pi}\ln\frac{\Delta}{2E_F}\Big)^{-1}. \quad (8)$$

We now proceed to calculate the amplitudes. We introduce new functions

$$Q_\pm = 2ig_\pm\tau_\pm. \quad (9)$$

From (2), (7), and (9) we readily obtain

$$t_\pm = \frac{1}{2ig_\pm}(Q_\pm\Phi_\pm - 1), \quad \tau_\pm = \frac{Q_\pm}{2ig_\pm}. \quad (10)$$

The unitarity conditions for $Q_\pm$ are obtained from formula (1):

$$|Q_\pm|^2 = \frac{1}{|\Phi_\pm|^2 + s(s+1)}. \quad (11)$$

Formula (11) defines $Q_\pm$ accurate to a factor with unity modulus on the cut.

We first find the amplitudes at zero temperature. We note first of all that the unitarity condition (1) is very similar to the corresponding unitarity condition for the normal metal in[5]. Therefore the solution is also very similar to the solution obtained for the normal metal. We shall not duplicate here the derivation given in[5], and write out the answer immediately. We consider only the case of integer spin.

Then for T = 0 we have

$$Q_\pm(s) = \frac{\varphi_\pm|s|}{\Phi_\pm + s}D_\pm(s), \quad |D_\pm(s)|^2 = 1,$$
$$\varphi_\pm(s) = \prod_{n=1}^{s}\frac{\Phi_\pm - 1/2 + (-1)^{n-1}[s+1/2-n]}{\Phi_\pm - 1/2 - (-1)^{n-1}[s+1/2-n]}; \quad (12)$$

$D_\pm(s)$ is a unimodular function.

It is easy to verify by direct substitution that the $Q_\pm$ determined in this manner satisfy the unitarity condition (11). We write out $\varphi_\pm$ for the cases s = 1 and 2:

$$\varphi_\pm(1) = \frac{\Phi_\pm}{\Phi_\pm - 1}, \quad \varphi_\pm(2) = \frac{\Phi_\pm^2 - 1}{\Phi_\pm(\Phi_\pm - 2)}. \quad (12a)$$

We now proceed to determine $D_\pm(s)$. In[5] we determined D from the conditions of analyticity with respect to energy and spin. It was shown there that in order for the amplitudes to satisfy the necessary spectral representations and to be analytic in the spin, $D_\pm$ must be constructed in such a way that $Q_\pm$ have no zeroes or poles at complex values of the energy. The poles of the amplitudes for real ω, when $|\omega| < \Delta$, do not contradict the spectral representations and the unitarity conditions, and therefore in general it is not necessary to exclude them.

Let the function φ_+ have a pole at the point ω_0. In the cases which we shall consider, ω_0 turns out to be pure imaginary: $\omega_0 = i\epsilon_0$. It is easy to show that the point $\omega = \omega_0^*$ is also a pole of the amplitude. In this case, obviously, the expression

$$\varphi_+\frac{\sqrt{\omega^2-\Delta^2} - i\sqrt{\varepsilon_0^2+\Delta^2}}{\sqrt{\omega^2-\Delta^2} + i\sqrt{\varepsilon_0^2+\Delta^2}}$$

has no poles and satisfies the unitarity condition and all the analyticity properties. It is possible to exclude in the same manner also the zeroes of $\varphi_\pm$ lying on the imaginary axis.

From (12) we see that in order to find the zeroes and the poles of $\varphi_\pm(s)$ we must find the roots of the equation

$$\Phi_\pm \pm n = 0, \quad (13)$$

where n is an integer. The solution of (13) is in general difficult, so that we shall consider several particular cases.

Let $|h| \ll 1$, and we get from (8) and (13)

$$-\frac{i}{\pi h}\frac{p_0}{g_\pm} + \frac{i}{\pi}\ln\frac{\omega+\sqrt{\omega^2-\Delta^2}}{\Delta} + \frac{1}{2} \pm n = 0. \quad (14)$$

when $|h| \ll 1$, the first term is much smaller than unity everywhere except in a small region near the Fermi surface $|\omega - \Delta| \ll \Delta$, where $g_\pm^{-1}$ can vanish. But in this region the logarithm is much smaller than unity, and (14) yields

$$-\frac{i}{\pi h}\frac{p_0}{g_\pm} + \frac{1}{2} \pm n = 0. \quad (15)$$

Taking into account the definition of $g_\pm(1)$, we can readily show that (15) has a solution only at real values of ω in the interval $-\Delta < \omega < \Delta$. On the other hand, the zeroes of (13) at such values of ω are of no interest to us. Thus, in the region where the logarithm is on the order of unity, Eq. (14) has no solution in the region of interest to us.

Let us consider now the region $|\omega| \gg \Delta$. In this region we get from (14), taking (1) and (8) into account,

$$-\frac{i}{2p_0 b} + \frac{i}{\pi}\ln\frac{\omega}{E_F} + \frac{1}{2} \pm n = 0. \quad (16)$$

It is easy to see that this equation has when $b < 0$ a root only if n = 0. This root equals

$$\omega = \pm i\varepsilon_0, \quad \varepsilon_0 = E_F\exp\Big(-\frac{\pi}{2p_0|b|}\Big). \quad (17)$$

When $b > 0$ we obtain a root at $|\omega| \gg E_F$, and in this energy region all our formulas are not valid in general. In the energy region of interest to us ($|\omega| \sim \epsilon_0$) at $b > 0$ Eq. (16) has no solutions. All the arguments presented above are true, naturally, only if $2\omega_0 \gg \Delta$. If $2\omega_0 \ll \Delta$ then, as seen above, Eq. (13) has no solution in the region $\omega \sim \epsilon_0 \ll \Delta$.

Let us see now in what cases do we get $|h| \ll 1$. It is seen from (8) that when $b > 0$ we always have $|h| \ll 1$, since $|p_0 b| \ll 1$. When $b < 0$, the formula for h from (8) can be rewritten in the form

$$h = -\Big(\ln\frac{\Delta}{2\varepsilon_0}\Big)^{-1}. \quad (18)$$

It is clear that when $|\Delta| \gg 2\epsilon_0$ and $|\Delta| \ll 2\epsilon_0$ we have $|h| \ll 1$.

Thus, we have obtained the following result: Eq. (13) has no solution when $b > 0$, and also when $b < 0$ but $\Delta \gg 2\epsilon_0$; Eq. (13) has a solution at n = 0 and $\omega = \pm i\epsilon_0$ when $b < 0$ and $\Delta \ll 2\epsilon_0$.

We now consider the case $b < 0$, $\Delta = 2\epsilon_0$. It is seen

from (18) that in this case h = ∞ and Eq. (13) assumes the form

$$\frac{i}{\pi}\ln\frac{\omega+\sqrt{\omega^2-\Delta^2}}{\Delta}+\frac{1}{2}\pm n=0. \tag{19}$$

It has a solution only at n = 0 and ω = 0, and such a solution is of no interest to us.

As a result we get for D

$$D_\pm(\omega,s)=\frac{\sqrt{\omega^2-\Delta^2}+i\sqrt{\varepsilon_0^2+\Delta^2}}{\sqrt{\omega^2-\Delta^2}-i\sqrt{\varepsilon_0^2+\Delta^2}} \tag{20a}$$

$$\text{or}\quad b<0,\quad \Delta\ll 2\varepsilon_0,\quad s=2k+1;$$

$$D_\pm(\omega,s)=\frac{\sqrt{\omega^2-\Delta^2}-i\sqrt{\varepsilon_0^2+\Delta^2}}{\sqrt{\omega^2-\Delta^2}+i\sqrt{\varepsilon_0^2+\Delta^2}} \tag{20b}$$

$$\text{or}\quad b<0,\quad \Delta\ll 2\varepsilon_0,\quad s=2k;$$

$$D_\pm(\omega,s)=1\quad \text{or}\quad b>0; \tag{20c}$$

$$D_\pm(\omega,s)=1$$

$$\text{or}\quad b<0,\quad \Delta\gg 2\varepsilon_0\quad \text{or}\quad b<0,\quad \Delta=2\varepsilon_0. \tag{20d}$$

It is easy to see that with such a choice of $D_\pm(\omega)$ the quantities $Q_\pm(\omega)$ have no zeroes and poles at complex values of the energy in the particular cases under consideration.

Let us write out also expressions for the amplitudes in the considered particular cases at s = 1. From (10), (12), (12a) and (20c), (20d), we get for b > 0, or for b < 0 and $\Delta \gg 2\epsilon_0$:

$$t_\pm=\frac{1}{2ig_\pm}\frac{1}{\Phi_\pm^2-1},\qquad \tau_\pm=\frac{1}{2ig_\pm}\frac{\Phi_\pm}{\Phi_\pm^2-1}. \tag{21a}$$

When b < 0 and $\Delta \ll 2\epsilon_0$ we have

$$t_\pm=\frac{1}{2ig_\pm}\left\{\frac{\Phi_\pm^2}{\Phi_\pm^2-1}\frac{\sqrt{\omega^2-\Delta^2}+i\sqrt{\varepsilon_0^2+\Delta^2}}{\sqrt{\omega^2-\Delta^2}-i\sqrt{\varepsilon_0^2+\Delta^2}}-1\right\},$$

$$\tau_\pm=\frac{1}{2ig_\pm}\frac{\Phi_\pm}{\Phi_\pm^2-1}\frac{\sqrt{\omega^2-\Delta^2}+i\sqrt{\varepsilon_1^2+\Delta^2}}{\sqrt{\omega^2-\Delta^2}-i\sqrt{\varepsilon_1^2+\Delta^2}}. \tag{21b}$$

Finally, when b < 0 and $\Delta = 2\epsilon_0$ we get

$$t_\pm=\frac{1}{2ig_\pm}\left\{\left(\frac{i}{\pi}\ln\frac{\omega+\sqrt{\omega^2-\Delta^2}}{\Delta}+\frac{1}{2}\right)^2-1\right\}^{-1},$$

$$\tau_\pm=\frac{1}{2ig_\pm}\left[\frac{i}{\pi}\ln\frac{\omega+\sqrt{\omega^2-\Delta^2}}{\Delta}+\frac{1}{2}\right]$$

$$\times\left\{\left(\frac{i}{\pi}\ln\frac{\omega+\sqrt{\omega^2-\Delta^2}}{\Delta}+\frac{1}{2}\right)^2-1\right\}^{-1}. \tag{21c}$$

Formulas (21) and (21b) are valid when $|h| \ll 1$. From the definition (8) of $\Phi_\pm$ we see that we can expand in terms of h everywhere in the region $|\omega| \sim \Delta$, with the exception of the region $|\omega - \Delta| \ll \Delta$ (where $g_\pm^{-1}$ can vanish). We then get from (21) and (21b) when $|\omega| \sim \Delta$ in the case b > 0, or in the case b < 0 and $\Delta \gg 2\epsilon_0$,

$$t_\pm=i\frac{\pi^2h^2}{2p_0^2}g_\pm,\qquad \tau_\pm=\frac{\pi h}{2p_0}; \tag{22a}$$

and in the case b < 0 and $\Delta \ll 2\epsilon_0$ we get

$$t_\pm=\frac{i}{g_\pm}-i\frac{\pi^2h^2}{2p_0^2}g_\pm,\qquad \tau_\pm=-\frac{\pi h}{2p_0}. \tag{22b}$$

In the derivation of (22b) we took into account the fact that (21b) is true only when $\Delta \ll 2\epsilon_0$.

We shall also write out the asymptotic form of the amplitudes for arbitrary integer s. From (10), (12), and (20a)–(20d) we can readily obtain, by expanding the amplitudes in powers of h for b > 0 or for b < 0 and $\Delta \gg 2\epsilon_0$, in the energy region $|\omega| \sim \Delta$:

$$t_\pm=i\frac{\pi^2h^2}{4p_0^2}g_\pm s(s+1),\qquad \tau_\pm=\frac{\pi h}{2p_0}; \tag{23a}$$

for b < 0 and $\Delta \ll 2\epsilon_0$ we have

$$t_\pm=\frac{i}{g_\pm}-i\frac{\pi^2h^2}{4p_0^2}g_\pm s(s+1),\qquad \tau_\pm=-\frac{\pi h}{2p_0}. \tag{23b}$$

It follows from (12) that the amplitudes have poles on the real axis inside the gap ($|\omega| < \Delta$). Obviously, these poles correspond to bound states of the electron on the impurity. Let us consider these poles in greater detail for s = 1. When ω is real and $|\omega| < \Delta$, it is convenient to rewrite $\Phi_\pm$ in the form

$$\Phi_\pm=\frac{1}{\pi h}\frac{\sqrt{\Delta^2-\omega^2}}{\omega\pm\Delta}+\frac{1}{\pi}\arcsin\frac{\omega}{\Delta}. \tag{24}$$

From (21a) and (21b) we see that the equations for the bound state have the form $\Phi_\pm \pm 1 = 0$. These equations have when $|h| \ll 1$ the following approximate solutions:

$$\omega_{1,2}=\pm\Delta(1-\pi^2h^2/2). \tag{25}$$

However, within the framework of the dispersion method, it is impossible to ascertain whether such bound states actually exist. Indeed, in our case $Q_\pm$ was determined accurate to the unimodular functions $D_\pm$. We have chosen $D_\pm$ such as to exclude the complex poles, but it is also possible to exclude the poles lying in the gap. For example, when we multiply the expression for $Q_\pm$ by the factor

$$D'=\frac{\sqrt{\omega^2-\Delta^2}-i\sqrt{\Delta^2-\omega_1^2}}{\sqrt{\omega^2-\Delta^2}+i\sqrt{\Delta^2-\omega_1^2}}, \tag{26}$$

we exclude the pole at the point ω_1, and then $Q_\pm$ satisfies as before the necessary unitarity conditions. Therefore the question of the existence of bound states cannot be solved within the framework of the dispersion method.

However, the main result (the dependence of the temperature of the transition and of the gap on the impurity concentration) is not affected by the presence or absence of bound states. Indeed, it is seen from (25) and (26) that D′ differs from unity in a very narrow region $|\omega - \Delta| \sim \Delta\pi^2h^2$. On the other hand, as will be shown later, the change of the transition temperature depends on the behavior of the amplitude in a much broader region $|\omega - \Delta| \sim \Delta$. In this region, by virtue of the statements made above, the influence of the pole of the amplitude is negligibly small.

Let us consider now the amplitudes at finite temperatures. In this case the $Q_\pm$ defined in (12) do not satisfy the unitarity condition. We proceed in exactly the same manner as in[5]. We write for $Q_\pm$:

$$Q_\pm(s)=\frac{\varphi_\pm(s)}{\Phi_\pm+s}M_\pm(s)D_\pm(s),\qquad |D_\pm|^2=1, \tag{27}$$

where $\varphi_\pm$ is defined in (12). The functions $\Phi_\pm$ in (27) were calculated already at finite temperatures. From (11) and (27) we get

$$|M_\pm|^2=\frac{|\Phi_\pm+s|^2}{|\Phi_\pm|^2+s(s+1)}|\varphi_\pm(s)|^{-2}. \tag{28}$$

We now choose, in analogy with the procedure used in[5], $M_\pm$ in the form

$$M_\pm=e^{2i\delta_\pm}, \tag{29}$$

where

$$\delta_\pm = -\frac{g_\pm}{4\pi}\int_{-E_F}^{-\Delta}\frac{d\omega'}{g_\pm(\omega')(\omega'-\omega)}\ln|M_\pm(\omega')|^2$$

$$-\frac{g_\pm}{4\pi}\int_{\Delta}^{\infty}\frac{d\omega'}{g_\pm(\omega')(\omega'-\omega)}\ln|M_\pm(\omega')|^2. \qquad (30)$$

At arbitrary values of the parameters, M_+ is a very complicated function. It is easy to show, however, as was done in[5], that in the case when the quantity

$$f=\frac{2p_0b}{\pi}\left(1-\frac{2p_0b}{\pi}\ln\frac{\theta}{E_F}\right)^{-1},\quad \theta=\max(T,\Delta) \qquad (31)$$

is small then $|M| \sim 1 + O(f^4)$. Therefore when $|f| \ll 1$ we get at $s = 1$ and $b > 0$, or $b < 0$ but $\Delta, T \gg \epsilon_1$ ($i\epsilon_1$—root of the equation $\Phi_+ = 0$ at finite T):

$$t_\pm=\frac{1}{2ig_\pm}\frac{1}{\Phi_\pm^2-1},\quad \tau_\pm=\frac{1}{2ig_\pm}\frac{\Phi_\pm}{\Phi_\pm^2-1}. \qquad (32)$$

When $b < 0$ and $\Delta, T \gg \epsilon_1$ we have

$$t_\pm=\frac{1}{2ig_\pm}\left\{\frac{\Phi_\pm^2}{\Phi_\pm^2-1}\frac{\sqrt{\omega^2-\Delta^2}+i\sqrt{\varepsilon_1^2+\Delta^2}}{\sqrt{\omega^2-\Delta^2}-i\sqrt{\varepsilon_1^2+\Delta^2}}-1\right\}$$

$$\tau_\pm=\frac{1}{2ig_\pm}\frac{\Phi_\pm}{\Phi_\pm^2-1}\frac{\sqrt{\omega^2-\Delta^2}+i\sqrt{\varepsilon_1^2+\Delta^2}}{\sqrt{\omega^2-\Delta^2}-i\sqrt{\varepsilon_1^2+\Delta^2}} \qquad (33)$$

We shall need subsequently expressions for the amplitudes near the transition point. Let us calculate them. At the transition $\Delta \to 0$, and then we get from (1) (T_c—transition point)

$$g_\pm \approx p_0(1\pm\Delta/\omega),\qquad T\to T_c. \qquad (34)$$

From (5) we get

$$u_\pm = \mathrm{Re}\,u_0 + ip_0(1\pm\Delta/\omega)\,\mathrm{th}\,(\omega/2T). \qquad (35)$$

In the derivation of (35) we have neglected the real part of the correction, since it is small compared with the real part of $u_0 = u(\Delta = 0)$ (it contains a small quantity of the type $f \ll 1$).

At $\omega \lesssim T$, taking into account the form of Re u_0 in the normal metal[5],

$$\mathrm{Re}\,u_0=\frac{1}{b}-\frac{2p_0}{\pi}\ln\frac{\pi T}{2\gamma E_F},\qquad |\omega|<T,$$

$$\ln\gamma = C \approx 0{,}577, \qquad (36)$$

we get

$$\Phi_\pm=-\frac{i}{\pi h_1}\frac{p_0}{g_\pm}+\frac{1}{2}\,\mathrm{th}\frac{\omega}{2T},$$

$$g_\pm = p_0\left(1\pm\frac{\Delta}{\omega}\right),$$

$$h_1=\frac{2p_0b}{\pi}\left(1-\frac{2p_0b}{\pi}\ln\frac{\pi T}{2\gamma E_F}\right)^{-1}. \qquad (37)$$

On the basis of all the foregoing we have near the transition point, from (32), (33), and (37),

$$t_\pm = i\frac{\pi^2h_1^2}{2p_0^2}g_\pm,\qquad \tau_\pm=\frac{\pi h_1}{2p_0}$$

$$\text{when } b>0 \text{ or } b<0,\ \pi T/2\gamma \gg \varepsilon_0; \qquad (38a)$$

$$t_\pm=\frac{i}{g_\pm}-i\frac{\pi^2h_1^2}{2p_0^2}g_\pm,\qquad \tau_\pm=-\frac{\pi h_1}{2p_0}$$

$$\text{when } b<0,\ \pi T/2\chi \ll \varepsilon_0. \qquad (38b)$$

At arbitrary integer s we obtain from (27) in the case when $b > 0$ or $b < 0$ and $\pi T/2\gamma \gg \epsilon_0$:

$$t_\pm = i\frac{\pi^2h_1^2}{4p_0^2}g_\pm s(s+1),\qquad \tau_\pm=\frac{\pi h_1}{2p_0}; \qquad (39a)$$

and in the case $b < 0$ and $\pi T/2\gamma \ll \epsilon_0$:

$$t_\pm=\frac{i}{g_\pm}-i\frac{\pi^2h_1^2}{4p_0^2}g_\pm s(s+1),\qquad \tau_\pm=-\frac{\pi h_1}{2p_0}. \qquad (39b)$$

We now proceed to calculate the Green's functions. As is well known, the Green's functions equal[1,4]

$$G=\frac{\tilde\omega+\xi}{\tilde\omega^2-\tilde\Delta^2-\xi^2},\qquad F=-\frac{\tilde\Delta}{\tilde\omega^2-\tilde\Delta^2-\xi^2}. \qquad (40)$$

$\tilde\omega$ and $\tilde\Delta$ are determined from the equations (n—impurity concentration)

$$\tilde\omega=\omega+\varkappa t(\tilde\omega,\tilde\Delta),\qquad \tilde\Delta=\Delta-\varkappa u(\tilde\omega,\tilde\Delta), \qquad (41)$$

where

$$t=\tfrac{1}{2}(t_+ + t_-),\quad u=\tfrac{1}{2}(t_+ - t_-),\quad \varkappa = 2\pi n/m.$$

From (23) and (39) we readily get

$$\tilde\omega=\omega+i\gamma_1\tilde\omega/(\tilde\omega^2-\tilde\Delta^2)^{1/2},$$
$$\tilde\Delta=\Delta+i\gamma_2\tilde\Delta/(\tilde\omega^2-\tilde\Delta^2)^{1/2}. \qquad (42)$$

Let us write out γ_1 and γ_2, as well as $\Gamma = \gamma_1 - \gamma_2$, for the different particular cases. For zero temperature at $b > 0$ or $b < 0$ and $\Delta \gg 2\epsilon_0$, we have

$$\gamma_1=\frac{\pi^2h^2}{4p_0}\varkappa s(s+1),\quad \gamma_2=-\gamma_1,\quad \Gamma=2\gamma_1; \qquad (43)$$

at $b < 0$ and $\Delta \ll 2\epsilon_0$ we have

$$\gamma_1=\left\{\frac{1}{p_0}-\frac{\pi^2h^2}{4p_0}s(s+1)\right\}\varkappa,$$
$$\gamma_2=\left\{\frac{1}{p_0}+\frac{\pi^2h^2}{4p_0}s(s+1)\right\}\varkappa,$$
$$\Gamma=-\frac{\pi^2h^2}{2p_0}s(s+1)\varkappa. \qquad (44)$$

Near the transition point we obtain for $b > 0$ or $b < 0$, $T_c \gg \epsilon_0$:

$$\gamma_1=\frac{\pi^2h_1^2}{4p_0}s(s+1)\varkappa,\quad \gamma_2=-\gamma_1,\quad \Gamma=2\gamma_1; \qquad (45)$$

when $b < 0$, $T_c \ll \epsilon_0$

$$\gamma_1=\left\{\frac{1}{p_0}-\frac{\pi^2h_1^2}{4p_0}s(s+1)\right\}\varkappa,$$
$$\gamma_2=\left\{\frac{1}{p_0}+\frac{\pi^2h_1^2}{4p_0}s(s+1)\right\}\varkappa,$$
$$\Gamma=-\frac{\pi^2h_1^2}{2p_0}s(s+1)\varkappa. \qquad (46)$$

We now proceed to calculate the transition temperature. In[1] the transition temperature was calculated by summing a definite aggregate of diagrams. Instead of summing diagrams, we shall use a simpler method, which is perfectly equivalent to that used in[1]. We expand the function F in terms of Δ. From (42), recognizing that

$$\sqrt{\omega^2-\Delta^2}\to\omega\,\mathrm{sign}\,\mathrm{Im}\,\omega,\quad \Delta\to 0,$$

we get

$$\tilde\omega=\omega+i\gamma_1\,\mathrm{sign}\,\mathrm{Im}\,\omega,$$
$$\tilde\Delta=\Delta+i\gamma_2\frac{\tilde\Delta}{\tilde\omega\,\mathrm{sign}\,\mathrm{Im}\,\omega}$$

Solving these equations, we get

$$\tilde{\omega} = \omega + i\gamma_1 \operatorname{sign} \operatorname{Im} \omega,$$

$$\tilde{\Delta} = \Delta \frac{\omega + i\gamma_1 \operatorname{sign} \operatorname{Im} \omega}{\omega + i\Gamma \operatorname{sign} \operatorname{Im} \omega}. \quad (47)$$

We substitute (47) in (40) and expand G and F in terms of Δ:

$$G = \frac{1}{\tilde{\omega} - \xi}, \quad F = -\frac{\Delta}{\tilde{\omega}^2 - \xi^2}. \quad (48)$$

The equation for the gap is

$$\Delta = |\lambda| F(+0).$$

Representing F in the form $F_0 + F - F_0$, going over to imaginary frequencies, integrating with respect to ξ, and summing over $i\omega_n$, we get after straightforward calculations the well known formula[1]

$$\ln \frac{T_{c0}}{T_c} = \psi\left(\frac{\rho}{2} + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right), \quad \rho = \frac{\Gamma}{\pi T_c} \quad (49)$$

(ψ—logarithmic derivative of the Γ-function).

We note that if $\Gamma < 0$, our formulas become inapplicable when $\rho \sim -1$, since it turns out that $\tilde{\omega}$ and $\tilde{\Delta}$ cannot be expanded in terms of Δ in this case. We shall therefore consider only the case $|\rho| \ll 1$ and $|\Gamma| \ll \pi T_c$. Then (49) yields

$$T_c = T_{c0} - \pi\Gamma/4. \quad (50)$$

When $b > 0$, or $b < 0$ and $T_c \gg \epsilon_0$, we get $\Gamma > 0$ and consequently $T_c < T_{c0}$. When $T_c \gg \epsilon_0$, as seen from (37), $h_1 \approx 2p_0 b/\pi$ and we get from (44)

$$\Gamma = 2p_0 b^2 s(s+1)\varkappa. \quad (51)$$

Substituting (51) in (50), we obtain the same result as in[1] (we recall that we are considering the case when the non-exchange coupling constant vanishes).

When $b > 0$, or $b < 0$ and $T_c \gg \epsilon_0$, we have from (37) and (44)

$$\Gamma = 2p_0 b^2 s(s+1)\left[1 - \frac{2p_0 b}{\pi} \ln \frac{\pi T_c}{2\gamma E_F}\right]^{-2} \varkappa. \quad (52)$$

In the case $b < 0$ and $T_c \ll \epsilon_0$, as seen from (45), we get $\Gamma < 0$, and consequently $T_c > T_{c0}$. Then Γ equals

$$\Gamma = -2p_0 b^2 s(s+1)\left[1 - \frac{2p_0 b}{\pi} \ln \frac{\pi T_c}{2\gamma E_F}\right]^{-2} \varkappa. \quad (53)$$

We now consider the change of Δ at T = 0. Again we confine ourselves to small concentrations ($|\Gamma| \ll \Delta$). Duplicating verbatim the derivation given in[1], we get

$$\Delta = \Delta_0 - \pi\Gamma/4. \quad (54)$$

We see from (44) that $\Gamma < 0$ when $b < 0$ and $\Delta \ll 2\epsilon_0$, and the gap increases in the presence of impurities. We shall not write out the formulas for Γ in different limiting cases, since they are perfectly analogous to formulas (51)—(53), if we replace in them $\pi T_c/\gamma$ by Δ.

Thus we see that at a negative coupling constant and a low transition temperature (compared with the Kondo energy), the transition temperature and the value of Δ at T = 0 increase when paramagnetic impurities are introduced.

From the derivation of the formula for the change of the transition temperature we see that the increase of the transition temperature is connected with the asymptotic form of $t_\pm$ given in (38) and (39), namely with the fact that the term proportional to $g_\pm$ has at $b < 0$ and $T_c \ll \epsilon_0$ a sign opposite to that in the case $b < 0$ and $T_c \gg \epsilon_0$, or else when $b > 0$. Let $\Delta = 0$, and then (38) and (39) take on the form

$$t_+ = t_- = i\frac{\pi^2 h_1^2}{4p_0} s(s+1)$$
$$\text{when } b > 0 \text{ or } b < 0,\ T \gg \varepsilon_0,$$
$$t_+ = t_- = \frac{i}{p_0} - i\frac{\pi^2 h_1^2}{4p_0} s(s+1)$$
$$\text{when } b < 0,\ T \ll \varepsilon_0. \quad (55)$$

The second term in the second formula of (55) must be negative, since p_0^{-1} is the maximum possible value of the negative part of $t_\pm$, at $\Delta = 0$, compatible with the unitarity condition (as is customarily said, $t = ip_0^{-1}$ saturates the unitarity condition). On the other hand, it is precisely the negativity of this term which leads in final analysis to the change of T_c obtained above.

It is shown in[3] that the solution (54) for a normal metal leads to the maximum resistance that is experimentally observed in many cases. We see therefore that the dependence of T_c and Δ on the concentration of the paramagnetic impurities derived in the present paper is very closely related with the maximum of the resistance of the normal metal.

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[1] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **39**, 1781 (1961) [Sov. Phys.-JETP **12**, 1243 (1962)].
[2] J. Kondo, Progr. Theor. Phys. **32**, 37 (1964).
[3] S. V. Maleev, Zh. Eksp. Teor. Fiz. **53**, 1038 (1967) [Sov. Phys.-JETP **26**, 620 (1968)].
[4] K. Maki, Phys. Rev. **153**, 428 (1967).
[5] S. L. Ginzburg, Zh. Eksp. Teor. Fiz. **54**, 1455 (1968) [Sov. Phys.-JETP **27**, 780 (1968)].

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