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TURBULENT HEATING OF IONS BY MAGNETOHYDRODYNAMIC WAVES

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We consider the problem of turbulent heating of a plasma by quasi-periodic mhd waves of finite amplitude. It is shown that oscillations in the frequency region $\omega_{Hi} \ll \omega \ll \omega_{He}$ are excited in the plasma as a result of the current instability. The excitation threshold with respect to the current velocity lies lower than the average-thermal velocity of the ions in the case of hot ions $T_i \gg T_e$. The interaction of the ions with the oscillations is analyzed on the basis of the equations for a weakly-turbulent plasma. The ions are scattered by the oscillations quasi-elastically. An equation is derived for the rate of their heating. The anomalous damping of mhd waves, observed in a number of experiments, is discussed.

In experiments on plasma heating by mhd waves[1,2] (magnetohydrodynamic waves), a strong heating of ions was observed and could not be attributed to dissipation in pair collisions. Thus, in experiments on plasma heating by a direct magnetosonic wave the proton temperature was $T_i = 100$ eV, and the electron temperature was $T_e \lesssim 10$ eV. Kovan and Spektor[1] indicate a high absorption rate of the energy of the magnetosonic wave. The damping decrement δ was of the order of 2×10^7 sec^{-1}. The magnetosonic resonance is realized at a frequency $f = 2 \times 10^7$ sec^{-1} in a plasma with density $n \sim 10^{13}$ cm^{-3}. The constant magnetic H_0 in the amplitude of the alternating field of the wave $H_\sim$ were respectively 2×10^3 and 60 Oe.

In the present paper we consider the question of current instability in the case when $T_i \gg T_e$, we derive a quasi-linear equation for the law governing the ion heating, and we consider the mechanism of the damping of the mhd wave. On the basis of nonlinear estimates, we establish the limiting temperatures of the electrons and the damping decrement. The experiment of[1] is discussed in the conclusion.

1. CURRENT INSTABILITY OF AN INHOMOGENEOUS PLASMA

A hydromagnetic wave produces in a plasma a current given by the relation

$$\mathbf{j} = -en\mathbf{u} = \frac{c}{4\pi}\operatorname{rot}\mathbf{H}_\sim \qquad (1)$$

where $\mathbf{u}$ is the current velocity of the electrons. In accordance with[3,4], we consider electrostatic oscillations ($\mathbf{E} = -\nabla\varphi$) in the frequency range

$$\omega_{Hi} \ll \omega \ll \omega_{He}.$$

We assume here that the hydromagnetic wave is excited in the plasma column whose density n and temperature T are radially inhomogeneous, and the electron temperature T_e is such that the effective frequency of the Coulomb collisions $\nu_{e,i}$ is sufficiently large and must be taken into account. In the case when $T_i \gg T_e$, we obtain the following dispersion equation:

$$\varepsilon \equiv 1 + \frac{k_\perp^2\omega_{pe}^2}{k^2\omega_{He}^2} - \frac{k_z^2\omega_{pe}^2}{k^2(\omega - \mathbf{ku})^2}\left(1 - \frac{i\nu_{e,i}}{\omega - \mathbf{ku}}\right)$$
$$+ \frac{\omega_{pi}^2}{k^2}\int\frac{(\mathbf{k}\partial f_0/\partial\mathbf{v} - k_\perp\omega_{Hi}^{-1}\partial f_0/\partial r)\,d\mathbf{v}}{(\omega - \mathbf{kv})} - \frac{\omega_{pe}^2 k_\perp d\ln n/dr}{\omega_{He}(\omega - \mathbf{ku})k^2}$$
$$+ \frac{i\sqrt{\pi}\omega_{pe}^2}{k^2 v_{Te}^2}\frac{(\omega - \mathbf{ku})}{k_z v_{Te}}\exp\left\{-\left(\frac{\omega - \mathbf{ku}}{k_z v_{Te}}\right)^2\right\} = 0, \qquad (2)$$

in the derivation of which it is assumed that

$$\frac{v_{Te}}{\omega_{He}} \ll |k|^{-1} \ll \frac{c}{\omega_{pe}}, \quad \frac{\omega}{k} \gg v_{Te},$$
$$\nu_{e,i} \ll |\omega - \mathbf{ku}|.$$

Here

$$\omega_{p\alpha}^2 = \frac{4\pi n e_\alpha^2}{m_\alpha}, \quad \omega_{H\alpha} = \frac{e_\alpha H_0}{m_\alpha c}, \quad v_{T\alpha} = \left(\frac{2p_\alpha}{m_\alpha n}\right)^{1/2},$$

$p_\alpha = n_\alpha T_\alpha$—partial pressures, α = e, i; $k_\perp$, k_z—projections of the wave vector on the directions perpendicular and parallel to the magnetic field respectively; f_0—ion distribution function.

Assuming that the imaginary part of the frequency, $\gamma = \operatorname{Im}\omega$, is much smaller than $|\omega - \mathbf{k}\cdot\mathbf{u}|$, and expanding in the dispersion equation (2) in terms of the small quantity $\gamma/|\omega - \mathbf{k}\cdot\mathbf{u}|$, we obtain two relations, the first for the spectrum of the unstable oscillations and the second for the increment:

$$(1 + k_\perp^2\rho_e^2)(\omega - \mathbf{ku})^2 - \omega^*(\omega - \mathbf{ku}) - k_z^2 T_i/m = 0, \qquad (3a)$$

$$\gamma = -\left\{\frac{\nu_{e,i}}{2} + \frac{\sqrt{\pi}}{2}\frac{(\omega - \mathbf{ku})^3 m}{k_z^2 T_i}\left[\frac{\omega - \omega^*(1+\eta)}{kv_{Ti}}\right.\right.$$
$$\left.\left. + \frac{T_i}{T_e}\frac{(\omega - \mathbf{ku})}{k_z v_{Te}}\exp\left\{-\left(\frac{\omega - \mathbf{ku}}{k_z v_{Te}}\right)^2\right\}\right]\right\}\left[1 + \frac{\omega^*(\omega - \mathbf{ku})m}{2k_z^2 T_i}\right]^{-1}, \qquad (3b)$$

where

$$\omega^* = \frac{cT_i}{eH}k_\perp\varkappa, \quad \varkappa = \frac{d\ln n}{dr},$$
$$\eta = \frac{d\ln T_i}{d\ln n}, \quad \rho_e = \left(\frac{T_i}{m}\right)^{1/2}\frac{1}{\omega_{He}}, \quad \rho_{Hi} = \frac{v_{Ti}}{\omega_{Hi}}.$$

In the derivation of (3a) and (3b) it was assumed that the electrons and the ions have Maxwellian velocity distributions, and that $H_0^2 < 4\pi nmc^2$.

We note that in the absence of current ($u \equiv 0$), the oscillations in question with $\omega \gg \omega_{Hi}$ and $T_i \gg T_e$ have already been analyzed by Mikhaĭlovskiĭ[5], and the instability of the inhomogeneous plasma with current in the case $T_i \gg T_e$ were investigated in papers by the author[3] and by Sizonenko and Stepanov[4]. The influence of collisions on the instability of a current flowing across a magnetic field was investigated by Suzuki[6].

We confine ourselves below to two limiting cases, depending on the parameter

$$\frac{k_z}{\omega^*}\left(\frac{T_i}{m}\right)^{1/2} = \frac{k_z}{k}\left(\frac{M}{m}\right)^{1/2}\frac{1}{\varkappa\rho_{Hi}}.$$

In the first case, when $k_z \gg k(m/M)^{1/2}\kappa\rho_{Hi}$, the frequency ω and the increment γ of the unstable oscillations are determined by the expressions:

$$\omega = \mathbf{ku} + k_z(T_i/m)^{1/2}\alpha, \quad \alpha = (1 + k_\perp^2\rho_e^2)^{-1/2}, \tag{4}$$

$$\gamma = -\frac{\nu_{e,i}}{2} - \frac{\sqrt{\pi}}{2}k_z\left(\frac{T_i}{m}\right)^{1/2}\alpha^3\left[\frac{\omega}{kv_{Ti}} + \left(\frac{T_i}{T_e}\right)^{3/2}\alpha\exp\left\{-\left(\frac{T_i\alpha^2}{2T_e}\right)\right\}\right]. \tag{5}$$

We note that in[7] the ion heating was explained by considering oscillations with precisely the same values of ω and γ, but with $\nu_{e,i} \equiv 0$. The condition for the instability of such oscillations is

$$-\frac{\mathbf{ku} + k_z(T_i/m)^{1/2}\alpha}{kv_{Ti}} > \frac{\nu_{e,i}}{\sqrt{\pi}\,k_z(T_i/m)^{1/2}\alpha^3} + \left(\frac{T_i}{T_e}\right)^{3/2}\alpha\exp\left\{-\left(\frac{T_i\alpha}{2T_e}\right)\right\} \tag{6}$$

The second term in the right side describes Cerenkov damping of the oscillations on electrons, and the first damping in Coulomb collisions.

In the case when the plasma is highly non-isothermal, $T_i \gg T_e$, and the Coulomb collisions are sufficiently rare, $\nu_{e,i} \ll k_z(T_i/m)^{1/2}$, a low threshold for the build-up of the oscillations is possible when $\mathbf{k}\cdot\mathbf{u} < 0$ and

$$k_z < -\frac{\mathbf{ku}}{v_{Ti}}\left(\frac{m}{M}\right)^{1/2}.$$

If the electrons are sufficiently cold, then the damping of the oscillations will be determined primarily by the Coulomb collisions. This assumption is valid when

$$\nu_{e,i} \gg \sqrt{\pi}\,k_z\left(\frac{T_i}{m}\right)^{1/2}\alpha^4\left(\frac{T_i}{T_e}\right)^{3/2}\exp\left\{-\left(\frac{T_i\alpha^2}{2T_e}\right)\right\}. \tag{7}$$

In this case the threshold value of the current velocity

$$u^* = v_{Ti}\left[\frac{k_z(T_i/m)^{1/2}\alpha}{kv_{Ti}} + \frac{\nu_{e,i}}{\sqrt{\pi}\,k_z(T_i/m)^{1/2}\alpha^3}\right]$$

is minimal when

$$k_z = k\left(\frac{2m}{M}\right)^{1/4}\left(\frac{\nu_{e,i}}{\sqrt{\pi}\,kv_{Ti}\alpha^4}\right)^{1/2} \tag{8}$$

and equals

$$u^*_{min} = 2v_{Ti}\left(\frac{\nu_{e,i}}{\sqrt{\pi}\,kv_{Ti}}\right)^{1/2}\alpha \approx v_{Ti}\left(\frac{\nu_{e,i}}{\omega_{e,i}}\right)^{1/2}. \tag{9}$$

Here the maximization is with respect to $k \sim 1/\rho_e$, and $\omega_{e,i} = (\omega_{He}\omega_{Hi})^{1/2}$. (If $T_e \sim 1-2$ eV and $H_0 \sim 2\times10^3$ Oe, then $\nu_{e,i} \sim 10^8\ \text{sec}^{-1}$, $\omega_{e,i} \sim 5\times10^8\ \text{sec}^{-1}$, and $u^*_{min} \sim 0.5\, v_{Ti}$.)

The limitation on the threshold value of the current velocity follows also from another condition. In the derivation of the dispersion equation (2) it was assumed that the ions are not magnetized in the oscillations. To this end, it is not sufficient to stipulate $\omega \gg \omega_{Hi}$. A criterion for this is one of the following inequalities: either $\gamma > \omega_{Hi}$, or $k_z v_{Ti} > \omega_{Hi}$. This means that the cyclotron resonances should be smeared out.

The maximum value γ_{max} is reached in (5) when

for $$k_z = \frac{|\mathbf{ku}|}{2\alpha}\left(\frac{T_i}{m}\right)^{-1/2}, \quad k \sim \frac{1}{\rho_e}$$

and equals

$$|\gamma_{max}| = \frac{\sqrt{\pi}}{2}\frac{(\mathbf{ku})^2}{kv_{Ti}}. \tag{10}$$

From the condition $\gamma > \omega_{Hi}$ at the limiting values $k \sim 1/\rho_e$ we get

$$u/v_{Ti} > (m/M)^{1/4}. \tag{11}$$

Let us consider the second limiting case

$$k_z \ll k(m/M)^{1/2}\kappa\rho_{Hi}.$$

The frequency ω and the increment γ for such k_z are respectively

$$\omega = \mathbf{ku} + \omega^*\alpha, \tag{12}$$

$$\gamma = -\frac{\nu_{e,i}(k_z/k)^2 M/m}{(\kappa\rho_{Hi})^2\alpha} - \sqrt{\pi}\,\omega^*\alpha^2\left[\frac{\omega - \omega^*(1+\eta)}{kv_{Ti}}\right]. \tag{13}$$

We have left out here a term describing the Cerenkov damping of the oscillations by the electrons, which is always small when $T_i \gg T_e$ and

$$k_z \ll k(m/M)^{1/2}\kappa\rho_{Hi}.$$

In this case the oscillations are unstable when

$$-\frac{(\mathbf{ku} - \eta\omega^*)}{kv_{Ti}} > \frac{\nu_{e,i}k_z^2 M}{\sqrt{\pi}\,\alpha^3k^2m(\kappa\rho_{Hi})^2}. \tag{14}$$

Let us explain the physical meaning of the foregoing inequality. Its right side contains a term causing the damping of the oscillations as a result of the Coulomb collisions. The left side contains terms responsible for the buildup of the oscillations. The first term results from the current motion of the electrons in the hydromagnetic wave, and the second from the Larmor current which exists in a spatially-inhomogeneous plasma. It is seen from (14) that the Larmor drift always leads to instability if $\nabla T_i \neq 0$. For purely transverse propagation $k_z \equiv 0$, we can neglect the damping of the oscillations in (14). From the applicability condition, formula (13) for $\gamma > \omega_{Hi}$ reduces to the inequality

$$(u/v_{Ti} + \kappa\rho_{Hi}\eta)\kappa\rho_{Hi} > (m/M)^{1/2}, \tag{15}$$

which is obtained when $\mathbf{k}\cdot\mathbf{u} = -\mathbf{ku}$, and $k \sim 1/\rho_e$. It is seen from this inequality that even in the absence of current motion of the electrons ($\mathbf{u} \equiv 0$) the instability of the oscillations in question remains in force if the inhomogeneity of the plasma is such that

$$\rho_{Hi}\kappa\eta > (m/M)^{1/4}. \tag{16}$$

In conclusion it should be noted that at temperatures $T_i \sim T_e$ the Cerenkov damping of the oscillations on the electrons may exceed the damping resulting from the Coulomb collisions. This case was investigated for a homogeneous plasma in[3], where it was shown that the threshold value of the current velocity is $(1-2)((T_e + T_i)/M)^{1/2}$. The maximum increment of the unstable oscillations is of the order of the hybrid frequency $\gamma \approx (\omega_{He}\omega_{Hi})^{1/2}$.

2. LAW GOVERNING ION HEATING

The entire subsequent analysis will be presented with a plasma column of cylindrical geometry as an example. Under these conditions the direct magnetosonic wave will have radial and azimuthal electric-field components E_r and E_φ, the electron current flows azimuthally, and the magnetic field of the wave $H_\sim$ is directed along the constant magnetic field H_0 (the z axis).

We shall consider the ion heating on the basis of the equations for a weakly turbulent plasma

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{e}{Mc}[\mathbf{v}\mathbf{H}_0]\frac{\partial f}{\partial \mathbf{v}}=\frac{\partial}{\partial v_\alpha}D_{\alpha\beta}\frac{\partial f}{\partial v_\beta}, \tag{17}$$

$$D_{\alpha\beta}=\frac{2\pi\omega_{pi}^2}{nM}\int\frac{k_\alpha k_\beta}{k^2}\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}N_k\delta(\omega-\mathbf{kv})\,d\mathbf{k};$$

$$\frac{1}{2}\frac{d}{dt}\ln N_k=\frac{\pi}{2}\tilde{\alpha}^2(\omega-\mathbf{ku})v_{Ti}^2\int\left(\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\right)\delta(\omega-\mathbf{kv})\,d\mathbf{v}$$

$$-\frac{\sqrt{\pi}\,\tilde{\alpha}^2(\omega-\mathbf{ku})}{2k_z v_{Te}}\left(\frac{T_i}{T_e}\right)\exp\left\{-\frac{T_i\alpha^2}{2T_e}\right\}\Big\};$$

$$\tilde{\alpha}=\left[1+k_\perp^2\rho_e^2+\frac{v_{Ti}^2k_\perp\varkappa}{2\omega_{Hi}(\omega-\mathbf{ku})}\right]^{-1/2}\quad\text{if } \nu_{e,i}\equiv 0. \tag{18}$$

Here N(t, **k**, **r**)—spectral density of the oscillations (number of waves in the **k**, **r** space), normalized as follows:

$$\int N_k\omega_k\,d\mathbf{k}=\int w_k\,d\mathbf{k}=w, \tag{19}$$

where w_k—oscillation (noise) energy density:

$$w_k=\frac{k^2|\varphi_k|^2}{8\pi}\omega_k\frac{\partial\varepsilon(\omega_k)}{\partial\omega_k}, \tag{20}$$

φ_k—amplitude of the Fourier component of the potential disturbance.

Equation (17) describes the change of the ion distribution function as a result of induced Cerenkov emission (absorption) of oscillations, whose frequency ω and increment γ are determined by expressions (4), (5), (12), and (13). Equation (18) describes the noise buildup process. In this equation, we have left out terms of higher order in the nonlinearity, which take into account particle-oscillation and wave-wave scattering. The nonlinearities will be taken into consideration later.

The oscillations with frequency ω and wave vector **k** are excited by resonant ions, the velocity of which is determined by the phase relation $\omega = \mathbf{k}\cdot\mathbf{v}$. In the language of quasi-particles, in each act of Cerenkov emission (absorption) of oscillations, the changes of the momentum and energy of the resonant ion are respectively equal to $M\Delta\mathbf{v} = \hbar\mathbf{k}$ and $M\Delta\mathbf{v}\cdot\mathbf{v} = \hbar(\mathbf{k}\cdot\mathbf{v}) = \hbar\omega$.

We shall consider below a particular case, but one of physical interest, when the phase velocity of the oscillations is much smaller than the velocity of the resonant particles $\omega/k \ll v$. In this case the relative change of the ion momentum exceeds by kv/ω times the relative change of its energy. This means that the ions will be scattered by the oscillations quasi-elastically (accurate to a small quantity ω/kv). In other words, with the same accuracy, the resonant ions can interact only with those oscillations, whose wave vectors lie in the plane perpendicular to the velocity vector **v**. This is seen also from the form of the $\delta(\mathbf{k}\cdot\mathbf{v})$ function.

This important circumstance greatly facilitates the investigation of Eq. (17). We rewrite it in a cylindrical coordinate system in velocity space ($v_\perp$, φ, v_z) and wave-vector space ($k_\perp$, φ', k_z). The angle φ' is reckoned from the direction of the current velocity: $\varphi' = (\mathbf{ku})$. We obtain:

$$\frac{\partial f}{\partial t}+v_\perp\frac{\partial f}{\partial r}\sin\varphi-\omega_{Hi}\frac{\partial f}{\partial\varphi}$$

$$=\frac{1}{v_\perp^3}\frac{\partial}{\partial\varphi}\left[D_{\varphi\varphi}\frac{\partial f}{\partial\varphi}+D_{\varphi v}\frac{\partial f}{\partial v_\perp}\right]+\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{1}{v_\perp^2}\left[D_{v\varphi}\frac{\partial f}{\partial\varphi}+D_{vv}\frac{\partial f}{\partial v_\perp}\right], \tag{21}$$

where

$$D_{\varphi\varphi}=\frac{2\pi\omega_{pi}^2}{nM}\int k_\varphi^2N_k\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}\delta(\omega-\mathbf{kv})\,v\,d\mathbf{k},$$

$$D_{\varphi v}=D_{v\varphi}=\frac{2\pi\omega_{pi}^2}{nM}\int\omega k_\varphi N_k\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}\delta(\omega-\mathbf{kv})\,v\,d\mathbf{k},$$

$$D_{vv}=\frac{2\pi\omega_{pi}^2}{nM}\int\omega^2N_k\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}\delta(\omega-\mathbf{kv})\,v\,d\mathbf{k},$$

$$k_\varphi=-k_\perp\sin(\varphi-\varphi'),\qquad\omega=k_vv_\perp,\qquad k_v=k_\perp\cos(\varphi-\varphi').$$

In writing out (21), we have left out terms that take into account the scattering of the particles in the magnetic-field direction. The rate of change of the longitudinal component of the energy of the resonant ions turns out to be smaller by a factor $(k_z/k)^2k^2v^2/\omega^2 \approx m/M$ than the rate of change of the transverse energy component. For this reason, in particular, we have chosen the cylindrical coordinate system. Equation (18) has in the cylindrical system the form

$$\frac{\partial N_k}{\partial t}=\frac{\partial\omega}{\partial k_\perp}\frac{1}{r}\frac{\partial N_k}{\partial\varphi'}+2[\gamma_i(t,\varphi')+\gamma_e(N)]N_k, \tag{22}$$

where

$$\gamma_i=\tilde{\alpha}^2\pi(\omega-\mathbf{ku})v_{Ti}^2\int\left(\frac{\omega}{v_\perp}\frac{\partial}{\partial v_\perp}+\frac{k_\varphi}{v_\perp}\frac{\partial}{\partial\varphi}\right)f\delta(\omega-\mathbf{kv})\,d\mathbf{v},$$

$$\frac{\partial\omega}{\partial k_\perp}=u\cos\varphi'+\frac{k_z}{k_\perp}\left(\frac{T_i}{m}\right)^{1/2}k_\perp^2\rho_e^2,\quad\text{if } k_z\gg k\left(\frac{m}{M}\right)^{1/2}\varkappa\rho_{Hi}.$$

Here $\gamma_e(N)$—damping decrement of the oscillations on electrons, taking into account both the linear Cerenkov damping and the nonlinear effects.

The first term in the right side of (21) is the principal one. It describes the elastic scattering of particles over the angles and exceeds by a factor $(kv/\omega)^2$ the term with the coefficient D_{vv}, which describes the inelastic scattering.

We note that the frequency of elastic scattering of ions by oscillations, $\nu_{\varphi\varphi} \approx v_\perp^{-3}D_{\varphi\varphi}$ depends on the velocity like v^{-3}, i.e., just as in Coulomb scattering of ions (protons) by heavy impurity ions. However, generally speaking $\nu_{\varphi\varphi}$ has a complicated angular dependence. In our problem the frequency $\nu_{\varphi\varphi}$ of ion scattering by oscillations, greatly exceeds the average ion-ion collision frequency.

We shall assume that ion heating occurs within a time much longer than the period of their cyclotron revolution. Then Eq. (21) can be solved by successive approximations in terms of the small parameters

$$\nu_{\varphi\varphi}/\omega_{Hi}\ll 1,\qquad\rho_{Hi}/a\ll 1\qquad(a=\varkappa^{-1}).$$

In the zeroth approximation, the distribution function $f_0(v)$ does not depend on the angle φ. The first correction, which is connected with the Larmor drift and with the collisions, is

$$f_1(v_\perp,\varphi)=-\frac{v_\perp}{\omega_{Hi}}\frac{\partial f_0}{\partial r}\cos\varphi-\frac{1}{v_\perp^3\omega_{Hi}}D_{v\varphi}\frac{\partial f_0}{\partial v_\perp}+C(v). \tag{23}$$

The integration constant C(v) must be set equal to zero, on the basis of the fact that the particle flux due to the deviation of the distribution function from symmetrical be equal to zero, i.e.,

$$\int_0^{2\pi}d\varphi f_1(v_\perp,\varphi)=0.$$

The second correction, as can be readily established, equals

$$f_2(v_\perp,\varphi)=-\frac{\sin\varphi}{\omega_{Hi}v_\perp}\frac{\partial f_1}{\partial\varphi}D_{\varphi\varphi}.$$

We have left out here other inessential terms, which drop out from the final results during the subsequent averaging. Substituting f_1 and f_2 in (21) and (22), and averaging over the angle φ in (21), we get

$$\frac{\partial f_0}{\partial t}=\int dk\,d\varphi'\left(\frac{\omega}{k_\perp v_\perp}\frac{\partial}{\partial v_\perp}-\frac{\cos\varphi'}{\omega_{Hi}}\frac{\partial}{\partial r}\right)$$

$$\times D_{k,\varphi'}\left(\frac{\omega}{k_\perp v_\perp}\frac{\partial f_0}{\partial v_\perp}-\frac{\cos\varphi'}{\omega_{Hi}}\frac{\partial f_0}{\partial r}\right),$$

$$D_{k,\varphi'}=\frac{2\pi\omega_{pi}^2}{nM}\frac{N_{k,\varphi'}}{v_\perp}\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}; \quad (24)$$

$$\gamma_i=\tilde{\alpha}^2\pi(\omega-\mathbf{ku})v_{Ti}^2\int\left(\frac{\omega}{k_\perp v_\perp}\frac{\partial}{\partial v_\perp}-\frac{\cos\varphi'}{\omega_{Hi}}\frac{\partial}{\partial r}\right)f_0\delta(\omega-\mathbf{kv})\,d\mathbf{v} \quad (25)$$

The effective collision frequencies $\nu_{\varphi\varphi}$ and ν_{vv} are determined here by the expressions

$$\nu_{\varphi\varphi}=\frac{2\pi\omega_{pi}^2}{nM}\frac{1}{v_\perp^3}\int N_{k,\varphi'}\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}\cos^2\varphi'\,dk\,d\varphi',$$

$$\nu_{vv}=\frac{2\pi\omega_{pi}^2}{nM}\frac{1}{v_\perp^3}\int N_{k,\varphi'}\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}\frac{\omega^2}{k^2v_{Ti}^2}\,dk\,d\varphi'. \quad (26)$$

Equation (24) describes the heating and diffusion of the plasma across the magnetic field. It was assumed in its derivation that the radial diffusion in the plasma column is the main form of ion energy loss. Equating the heating rate to the diffusion rate, we obtain an important estimate for the limiting ion temperatures:

$$\frac{\rho_{Hi}^2}{(r_0/2)^2}\cong\frac{\nu_{vv}}{\nu_{\varphi\varphi}}\approx\frac{\omega^2}{k^2v_{Ti}^2}\lesssim\frac{u^2}{v_{Ti}^2}; \quad (27)$$

or

$$T_i\lesssim M\omega_{Hi}r_0u/2,$$

where r_0—radius of the plasma column. The obtained estimate does not depend on the noise density, and it contains parameters that can be readily controlled in experiments.

If we neglect spatial diffusion in (24), then we obtain the heating equation in the form

$$\frac{\partial f_0}{\partial t}=\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp v_{Ti}^2\,\nu_{vv}\frac{\partial f_0}{\partial v_\perp}\right).$$

It admits of a simple self-similar solution

$$f_0=\text{const}\cdot\exp\{-(Av^5/t)\},$$

where A is a numerical coefficient. With the aid of this solution we can qualitatively trace the time evolution of the ion distribution function. It is seen from it, in particular, that in the course of time the ion distribution function will vary in velocity space in such a manner, that the distribution function becomes denser, in the region of velocities lower than the mean-thermal ion velocity, and in the region of larger velocities it drops off sharply, so that in the course of time the distribution function will acquire the form of a step.

3. MECHANISM OF ABSORPTION OF mhd-WAVE ENERGY. ELECTRON HEATING

As was established above, the heating of the ions is described fully by Eq. (24). The rate of heating is determined by the effective frequency of the inelastic scattering of the ions ν_{vv}, and the velocity of spatial diffusion is determined by the elastic-scattering frequency $\nu_{\varphi\varphi}$. Taking the corresponding moments of the distribution function in the heating equation, we obtain the momentum balance and the energy balance for the plasma plus wave system. From the energy balance it follows, in particular, that the thermal energy in the plasma can increase only as a result of a loss in the wave energy. In order for the wave energy to be maintained at a definite level, it is necessary to perform work on the waves. We shall show that this work is performed by the electric field of the hydromagnetic wave, and that such work is performed on the waves only during the process of their radiation (absorption).

To this end, in order to clarify the picture of the phenomenon, we use for the description of the plasma the equations of two-fluid hydrodynamics, in which we average over the scales of the time $\tau\gg 1/\omega$ and length $l\gg v_{Te}/\omega_{He}$. The balance of the forces acting on the ions and the electrons, with account of the cylindrical symmetry, takes the form

$$Mn\frac{dv_{ri}}{dt}=enE_r,$$

$$Mn\frac{dv_{\varphi i}}{dt}=enE_\varphi+\langle F_i\rangle,$$

$$0=-enE_r-\frac{env_{\varphi e}H}{c},$$

$$0=-enE_\varphi+\frac{env_{re}H}{c}+\langle F_e\rangle. \quad (28)$$

Here v_{ri}, $v_{\varphi i}$, v_{re}, and $v_{\varphi e}$ are the amplitude values of the electron and ion velocity components in the magnetosonic wave. The radial field E_r due to charge separation in the wave causes the electrons to drift in the azimuthal direction with a velocity $v_{\varphi e}=cE_r/H$, which we called the current velocity. The quantities $\langle F_i\rangle$ and $\langle F_e\rangle$ are the effective forces of "friction" of the ions and electrons against the oscillations. Thus $\langle F_i\rangle$ denotes physically the summary effect of a large number of acts of exchange of momentum between the resonant ions and the oscillations, and is obtained as a result of averaging the expression

$$\langle\mathbf{F}_i\rangle=\langle en_{i\sim}\mathbf{E}_\sim\rangle=\int d\mathbf{k}\frac{1}{4}\int e(f_k^*\mathbf{E}_k+\text{к. с.})\,d\mathbf{v}$$

$$=\int d\mathbf{k}\frac{\pi e^2}{M}|E_k|^2\int\left(\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\right)\delta(\omega-\mathbf{kv})\,d\mathbf{v}=2\int\gamma_i\mathbf{k}N_k\,d\mathbf{k}, \quad (29)$$

where

$$n_i=\int f_\sim d\mathbf{v},\quad \mathbf{E}_k=\int dt\,d\mathbf{r}\mathbf{E}_\sim(\mathbf{r},t)\exp\{-i(\omega t-\mathbf{kr})\},$$

$$f_k=\int dt\,d\mathbf{r}\,f(\mathbf{r},t)\exp\{-i(\omega t-\mathbf{kr})\}.$$

Similarly, we determine $\langle\mathbf{F}_e\rangle$ as a result of averaging of the expression $-\langle en_{e\sim}\mathbf{E}_\sim\rangle$. However, in the derivation of $\langle\mathbf{F}_e\rangle$ we cannot confine ourselves only to allowance for the linear absorption of the oscillations by the resonant electrons. As will be shown below, the nonlinear effects of absorption of oscillations by the electrons become much more important, when the amplitude of the noise is sufficiently large. In the noise saturation regime ($\dot{N}_k\equiv 0$), the linear increment $\gamma_i+\gamma_e$ of the buildup of the oscillations by the ions and electrons becomes comparable with the nonlinear decrement $\gamma_e(E)$ of oscillation damping by the electrons. Therefore the friction force $\langle\mathbf{F}_e\rangle$ is best introduced phenomenologically:

$$\langle\mathbf{F}_e\rangle=-2\int\gamma_e(N)\mathbf{k}N_k\,d\mathbf{k}.$$

It is easy to verify that $\langle\mathbf{F}_e\rangle=-\langle\mathbf{F}_i\rangle$. It is clear that the effective friction force $\langle\mathbf{F}_e\rangle$ can arise in the case when $\gamma_i+\gamma_e\neq 0$.

When the amplitude value of the current velocity cE_r/H rises to a value exceeding the threshold value u^*, electrostatic oscillations begin to be excited. As a result of interaction with the oscillations, a friction force $\langle F_e \rangle$ is produced, causing radial drift of the electrons. In the magnetosonic wave, the ions are not magnetized and their drift current can be neglected. The radial drift current of the electrons equals

$$\delta j_r = -en\delta v_{er} = \frac{c}{H}\int k_\perp \gamma_e(N) N_k \, d\mathbf{k}. \quad (30)$$

The work performed by the current δj_r in the radial electric field E_r of the hydromagnetic wave, is obviously

$$\delta W = \delta j_r E_r = \frac{c}{H} E_r \int k_\perp \gamma_e N_k \, d\mathbf{k} = \int \mathbf{ku}\gamma_e N_k \, d\mathbf{k}. \quad (31)$$

This work is always positive $\delta \dot{W} > 0$, i.e., the mhd wave gives up energy. On the other hand, if $\gamma_i + \gamma_e \equiv 0$, then $\delta \dot{W} \equiv 0$. There is no adiabatic work on the oscillations under conditions when $N_k \equiv 0$.

In the foregoing analysis, it was tacitly assumed throughout that in a quasi-stationary magnetosonic wave, the amplitude of which is maintained by the work of an external circuit, a quasi-stationary noise level is established. This statement must be justified.

Indeed, two physically different possibilities arise in the solution of the equation (18) for the noise spectrum. Either the noise growth rate is small, so that no limitation of the amplitude growth is produced by the nonlinear effect during a half-cycle of the current oscillation π/Ω ($\Omega = 2\pi f$), or else the rate of noise increase is large and saturation sets in within a time τ which is much shorter than the half-cycle π/Ω of the current oscillations. Let us derive the condition under which the second assumption is valid. From (18) we obtain the following estimate:

$$\ln\left(\frac{w_{k,\,max}}{w_0}\right) = 2\int_0^\tau \gamma_{max}(t)\,dt \leqslant 2\int_0^{\pi/\Omega} \gamma_{max}(t)\,dt, \quad (32)$$

where w_0 is the initial level of the fluctuations, and $w_{k,max}$ is the maximum energy density of the noise.

In accordance with formula (10) for $\gamma_{max}(t)$, we obtain for the quasi-periodic process $u(t) = u_0 \sin \Omega t$

$$|\gamma_{i\,max}| = \frac{\sqrt{\pi}}{2}\frac{(\mathbf{ku})^2}{kv_{Ti}} = \frac{\sqrt{\pi}}{2}\frac{ku_0^2}{v_{Ti}}\sin^2\Omega t \lesssim \omega_{e,i}\frac{u_0}{v_{Ti}}\sin^2\Omega t. \quad (33)$$

Averaging in (32) $\gamma_{i\,max}$ over the half-cycle π/Ω, we get

$$\int_0^{\pi/\Omega} \gamma_{i\,max}(t)\,dt = \frac{\pi}{2\Omega}\omega_{e,\,i}\frac{u_0}{v_{Ti}} = \frac{u_0}{v_{Ti}}\frac{\omega_{e,\,i}}{4f}. \quad (34)$$

Substituting (34) in the estimate (32), we obtain the following criterion:

$$\frac{u_0}{v_{Ti}} \gg \frac{2f}{\omega_{e,\,i}}\ln\left(\frac{w_{k,\,max}}{w_0}\right). \quad (35)$$

When this criterion is satisfied, the noise has time to reach saturation at the instants of time when the current velocity is greatly in excess of the threshold value u^*. This criterion establishes also a certain threshold value of the current velocity u^*, in addition to (9) and (11).

The energy given up by the hydromagnetic wave to the plasma per unit time will go in the quasi-stationary mode to increase the density of the thermal energy of the electrons and the ions. The energy balance is given by

$$n\dot{T}_e + n\dot{T}_i = \int \mathbf{ku}\gamma_e N_k \, d\mathbf{k}. \quad (36)$$

The heating of the electrons will be the result of nonlinear processes. To estimate this heating it is necessary to know the rate of heating of the ions $n\dot{T}_i$. It obviously equals the work of the resonant ions in the field of the excited oscillations. We have

$$n\dot{T}_i = \int d\mathbf{k}\frac{1}{4}\int e\mathbf{v}(f_k\mathbf{E}_k^* + \text{c.c.})\,d\mathbf{v}$$
$$= \int d\mathbf{k}\frac{\pi e^2|E_k|^2\omega}{Mk^2}\int\left(\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\right)\delta(\omega - \mathbf{kv})\,d\mathbf{v} = \int \gamma_i\omega N_k d\mathbf{k}. \quad (37)$$

with the aid of (36) and (37) we get

$$\frac{n\dot{T}_i}{n\dot{T}_e + n\dot{T}_i} = \frac{\int \gamma_i\,\omega_k\,N_k\,d\mathbf{k}}{\int \gamma_e\,\mathbf{ku}N_k\,d\mathbf{k}} \quad (38)$$

It still does not follow from the obtained estimate that the electrons are heated, but if they are heated, then the estimate (38) makes it possible to estimate roughly the rate of their heating. It can be said that the rate of electron heating, at least, is not much larger than the rate of ion heating. However, this is already sufficient to establish the limiting temperature of the electrons under the conditions of the experiments of[1]. We shall assume that the main source of losses for the electrons is cooling by the "cold" end surfaces of the system as a result of the thermal conductivity along the magnetic field.

The temperature can be estimated on the basis of the following equation:

$$n\dot{T}_e \cong n\dot{T}_i + \frac{\partial}{\partial z}\left(3\frac{nT_e\tau_e}{m}\right)\frac{\partial T_e}{\partial z}, \quad (39)$$

which includes heating of the electrons as a result of nonlinear processes, the order of magnitude of which is $n\dot{T}_i$ (it will be established below that $n\dot{T}_i \lesssim \omega_{e,i}nmu^2/2$). Here, $\tau_e = \nu_{e,i}^{-1} = 3.5 \times 10^4 T_e^{3/2}/n$ sec. In the case of low electron temperatures $T_e \sim (1-2)$ eV we have $\nu_{e,i} \sim \omega_{e,i}$. Comparing the heating and cooling rates in (39), we obtain a simple estimate for the limiting temperature of the electrons

$$1.6\cdot 10^{-12}\tau_e T_e \lesssim Lmu, \quad (40)$$

where L is the characteristic longitudinal dimension ($L \sim 1/k_z$). Substituting in (40) the numerical values $L \sim 50$ cm and $u \sim 10^7$ cm/sec from[1], we get $T_e \lesssim 3$ eV.

4. CONCLUSION

As shown above, a hydrodynamic wave gives up energy to the plasma, and its quasi-stationary level is maintained by the generator. By wave damping it is natural to mean the reciprocal of the time over which the wave energy $H_\sim^2/8\pi$ is completely absorbed by the plasma. The Q-factor connected with this damping is defined as $\Omega/2\delta$. To determine these quantities it is necessary to know the energy density of the oscillations at any instant of time. To this end, strictly speaking, it is necessary to solve Eq. (22) for the spectral density $N_k(t)$. The noise amplitude is limited by nonlinear effects, a fact accounted for in this equation by introduction of the nonlinear decrement $\gamma_e(N)$. We note also one important feature of the oscillations considered here. If we use the real part Re ϵ in (2) and calculate with the aid of (20) the energy of the oscillations, then it turns out that the unstable waves in this case will be those with negative energy $w_k = N_k\omega_k < 0$.

The nonlinearities connected with the ion oscillations cannot lead to limitation of the amplitude. To the contrary, when account is taken of the nonlinear interaction, the noise amplitude can only increase. Therefore the main nonlinearities that lead to limitation of the amplitude must be sought in the allowance for the electron oscillations.

The noise energy density w can be roughly estimated as follows. It is obvious that the amplitude of the electron velocity oscillations v cannot exceed the phase velocity of the unstable oscillations $\omega/k \approx u$. The energy density of the oscillatory motion of the electrons $nmv_{\sim}^2/2$ can be assumed to be the limiting energy density of the oscillations. If we substitute this estimate in (37), we obtain the following expression for the ion heating rate:

$$n\dot{T}_i \approx \int \gamma_i w_k dk \lesssim \omega_{e,i}\frac{nmu^2}{2}. \qquad (41)$$

It is now easy to determine the damping δ and the factor Q. They are equal to

$$\delta = \frac{8\pi n\dot{T}_i}{H_{\sim}^2} \lesssim \frac{4\pi\omega_{e,\,i}\, nmu^2}{H_{\sim}^2}, \quad Q = \frac{\Omega}{2\delta}. \qquad (42)$$

For a direct magnetosonic wave we obtain with the aid of (1) the current velocity

$$u = \frac{\Omega}{\omega_{Hi}}\frac{H_{\sim}}{\sqrt{4\pi nM}}.$$

If we substitute this value in (42), we get

$$\delta = \frac{\Omega^2}{\omega_{e,\,i}}, \quad Q = \frac{\omega_{e,\,i}}{2\Omega}. \qquad (43)$$

To compare the present theory with experiment[1], we use the numerical values given at the beginning of the article, and also the formulas (27) and (40) for the limiting temperatures T_i and T_e, and (43) for the Q and the damping. Such a comparison of the present theory with experiment is performed in detail in[7]. We therefore only summarize here the already performed comparison. As a result we get that for protons $T_i \lesssim 200$ eV, for the electrons $T_e \sim 3$ eV, so that the condition $T_i \gg T_e$ is certainly satisfied; $u/vT_i \sim \sim \rho_{Hi}/(r_0/2) \sim 1/3$, $\delta \sim 2 \times 10^7\ \mathrm{sec}^{-1}$, and $Q = 3.5$.

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