PONDEROMOTIVE EFFECTS OF ELECTROMAGNETIC RADIATION

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We investigate the singularities of ponderomotive effects in resonant systems of electromagnetic radiation, when the frequencies of the mechanical system are much lower than the frequencies of the electromagnetic system. We discuss the nature of retardation effects and consider the conditions for the occurrence of ponderomotive instabilities near resonance of degenerate and nondegenerate radiation systems. The approach developed in the paper is applicable for the analysis of electrostriction and magnetostriction effects accompanying both dimensional and frequency resonances of an electromagnetic system.

THE stability of mechanical systems in which the electromagnetic-energy density is high was the subject of a number of investigations[1-3]. Braginskiĭ and Mikunin[1] have shown that the mirrors of an optical Fabry-Perot resonator can be set in vibration, while Karliner, Shekhtman, and the author[2] observed instabilities of the walls of a cavity resonator of an accelerator[1)], and Braginskiĭ and Minakova[3] investigated experimentally the self-oscillation of the plates of a capacitor of a lumped circuit. In all these cases, monochromatic radiation of frequency ω is applied to a linear (without account of the mechanical motion) electromagnetic system, and mechanical oscillations of frequency $\Omega \ll \omega$ are excited ($\Omega/\omega \sim 10^{-13}$ in[1] and $\Omega/\omega \sim 10^{-6}$ in[2,3]). The slow motion of the resonator walls in[1-3] changes, as a result of the change of the natural frequencies of the radiation, the state of the electromagnetic system, and the resultant change of the ponderomotive forces influences in turn the mechanical motion. The sharp dependence of the state of a high-Q radiation system on its configuration, the prolonged transient time, and the possibility of high concentration of radiation, cause an effective interaction between the mechanical and electromagnetic vibrational systems, which differ greatly in frequency.

The considered ponderomotive effects are of practical interest and can find an application. On the other hand, they are of independent interest, since they can be used as an example for tracing the mechanisms of interaction between slow and fast systems.

In[1-3] they considered manifestations of forces acting on the surfaces of the resonators, i.e., due to the Maxwellian stress tensor of the electromagnetic field. Besides the surface forces, in a number of cases volume forces are also important, for example, forces of electrostriction or magnetostriction type. The theoretical analysis calls for a generalization in the sense that in[1-3] the analysis was confined to the case when the behavior of the fast system can be characterized by means of one oscillator whose line width is much larger than the characteristic frequencies of the slow system.

1. Disregarding other nonlinear effects, we shall assume that in the absence of interaction the mechanical and electromagnetic systems are linear, the former being characterized by low frequencies and the latter by high frequencies, so that the effect of the high frequency mechanical vibrations are small and the ponderomotive forces are averaged over the fast motion.

The mechanical motion will be characterized by an aggregate of normal (in the absence of ponderomotive forces) Lagrangian coordinates $X = \{X_\nu\}$, the oscillation ν being determined by its form of mechanical motion and by the coefficients of stiffness, inertia, and friction (k_ν, m_ν, h_ν); the coordinates X correspond to generalized ponderomotive forces $F = \{F_\nu\}$. In cases that are not fundamental, the index ν will be omitted.

We consider first the case when the radiation system can be represented by a single mode of natural frequency ω_0 and $Q \gg 1$, which is excited by an external monochromatic force of frequency $\omega \sim \omega_0$. The radiation density that is established in the system is a function of the electric parameters (ω_0, Q ...) which in turn are functions of the equilibrium mechanical configuration X_0. The change of the radiation state and of the ponderomotive forces F, caused by the change of X, proceeds with a certain delay. Therefore $F = F[X(t-\tau)]$, and in the case of small and medium deviations $x = X - X_0$ we have

$$F \approx F_0 + \frac{\partial F}{\partial X}(x - \tau\dot{x}). \tag{1}$$

The term $x\partial F/\partial X$ introduces stiffness in the mechanical system, and the term $\tau\dot{x}\partial F/\partial X$ introduces friction.

The delay τ gives rise to two effects. One of them is connected with the finite velocity of light. For the case of forces that act on a moving body exposed to light—this is the so-called "light friction"; when $v \ll c$, the "light friction" is smaller by a factor of v/c than the light pressure (v—velocity of the irradiated body, c—velocity of light). In our case the "light friction" does not exceed $F_m\dot{x}/c$, where the speed of light c is expressed in units of $\dot{x}$ and F_m is the value of F_0 at resonance $\omega = \omega_0$.

Of greater importance for resonant radiation systems is another nonrelativistic effect, which is due to the dissipative properties of the radiation systems. In the nonrelativistic approximation, the ponderomotive forces

[1)]For radio-frequency resonators with high radiation density, the ponderomotive effects turn out to be [2] quite appreciable, and loss of mechanical stability frequently sets in much before other limitations (breakdown, overheating) come into play. This imposes additional requirements on the construction of the resonators.

correspond to tension of the electromagnetic field for an immobile configuration X, i.e., without allowance for mechanical motion, and the influence of the latter on the radiation system is assumed to be equivalent to the change of the electric parameters (for example, the natural frequency ω_0) simultaneously with the mechanical motion. The nonrelativistic retardation τ is the relaxation time of the radiation under perturbations of the electric parameters and under changes of X. It is clear from physical considerations that this time is of the order of the time constant $\tau_0 = 2Q/\omega_0$. A rigorous account of the retarded forces in the nonrelativistic approximation, as well as the conditions of applicability (1), will be discussed later.

We now estimate the order of the nonrelativistic retarded forces and compare them with the "light friction."

The change of F under the influence of X is connected mainly with the change of the detuning $\omega - \omega_0$, since $(\partial F/\partial X)_{max} \approx F_m/\Delta X$, where ΔX corresponds to retuning of the radiation system by an amount equal to the width of the resonance curve ω_0/Q. In order of magnitude, $\Delta X \sim 2l/Q$, where l is the characteristic dimension (in units of X) of the volume in which the radiation is concentrated. Hence, putting $\tau = 2Q/\omega_0$, we find that the ratio of the maximum values of the nonrelativistic friction to the "light friction" amounts to

$$\frac{c\tau}{\Delta X} \sim Q^2 \frac{\lambda}{2\pi l}, \quad \lambda = \frac{2\pi c}{\omega_0}.$$

Thus, the possibility of neglecting relativistic retardation is determined by the smallness of the parameter $\alpha = 2\pi l Q^{-2}/\lambda$. In the cases of practical interest, α is small and we shall henceforth, just as in[1-3], neglect the relativistic retardation (in[1-3], $\alpha \sim 10^{-11}-10^{-9}$).

2. A rigorous account of the retardation of the ponderomotive forces can be obtained by solving the equation of motion of the radiation

$$\ddot{u} + \frac{\omega_0}{Q}\dot{u} + \omega_0^2 u = q e^{j\omega t}, \tag{2}$$

where u(t) is the generalized coordinates of the radiation mode, the right-hand term is the external source, the parameters ω_0, $\tau_0 = 2Q/\omega_0$ and q are functions of X(t), and j is the imaginary unit.

Putting $u = Ue^{j\omega t}$, we represent (2) in the form

$$\Delta U + \left(\frac{\omega_0}{Q} + 2j\omega\right)\dot{U} + \ddot{U} = q \tag{3}$$

where

$$\Delta(X) = \frac{\omega\omega_0}{Q}(j - y), \quad y(X) = Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right).$$

In the case of slow changes of $X(\mu = \Omega\tau_0 < 1$, Ω—characteristic frequency of the mechanical motion), the solution (2) can be sought in the form of a series $U = U_0 + U_1 + \ldots$, where U_k is of the order of smallness μ^k. In the zeroth approximation $U_0(X) = q\Delta^{-1}$, in the first approximation, $U_1(X, \dot{X}) \approx -2j\omega U_0 \Delta^{-1}$, etc.

The ponderomotive forces are quadratic in the field, so that when account is taken of the averaging over the frequency ω, the generalized forces can be represented in the form $F = (1/2)a|U|^2$, where $a = \{a_\nu\}$—weight coefficients that depend on the distribution of the electromagnetic field and the form of the mechanical oscillations. Confining ourselves to the first approximation we obtain for $F = (1/2)a(U_0U_0^* + U_0U_1^* + U_0^*U_1)$ an expression of the form (1), the time τ being dependent on the frequency difference $\omega - \omega_0$. Thus, the delay τ relative to the change of $\omega_0(X)$ equals $\tau = 2\tau_0/(1 + y^2)$. Near resonance, τ differs from τ_0 by not more than several times, and identification of τ with the time constant τ_0 is acceptable for estimates.

For a radiation system with $Q \gg 1$, the line width ω_0/Q may turn out to be commensurate with the frequency of the mechanical oscillations, so that the approximation $\mu \ll 1$ no longer holds. It is then advantageous to introduce in lieu of τ another dissipative characteristic of the radiation system, namely the active component R of the impedance at the frequencies $\omega \pm \Omega$ (see Sec. 4).

For an unambiguous determination of a, we shall henceforth choose the coordinate u in such a way that the radiation energy stored in the system is $\mathscr{E} = (1/2)|U|^2$. Let us confine ourselves here to the adiabatic terms of the interaction of the mechanical system with the radiation, then

$$a_\nu = -\frac{1}{\omega_0}\frac{\partial\omega_0}{\partial X_\nu}, \tag{4}$$

which follows from energy considerations: in the adiabatic approximation (influx of radiation from the outside and dissipation of the radiation, both slow relative to the change in δX), the processes in the system occur with conservation of the adiabatic invariant $\mathscr{E}/\omega_0$, and consequently

$$\frac{\delta\mathscr{E}}{\mathscr{E}} = \frac{\delta\omega_0}{\omega_0} = \frac{d\omega_0}{dX}\frac{\delta X}{\omega_0}.$$

At the same time, this change $\delta\mathscr{E}$ under the influence of δX is equal to the work done by the ponderomotive forces, with the sign reversed: $\delta\mathscr{E} = -F\delta X$, where the forces F correspond to the equilibrium energy at the given X. Since $F = a\mathscr{E}$, this leads to (4).

3. We now consider the manifestations of the ponderomotive effects.

The behavior of the mechanical system under the influence of ponderomotive forces in the form (1) is described by an autonomous system of equations:

$$m_\nu\ddot{x}_\nu + 2h_\nu\dot{x}_\nu + k_\nu x_\nu = \sum_\alpha (x_\alpha - \tau\dot{x}_\alpha)\frac{\partial F_\nu}{\partial X_\alpha}, \quad \nu = 1, 2, \ldots, \tag{5}$$

where

$$F_\nu = -\frac{1}{\omega_0}\frac{\partial\omega_0}{\partial X_\nu}\frac{\mathscr{E}_m}{1 + y^2}, \quad y = y(X) = Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right),$$

$\mathscr{E}_m$—radiation energy at resonance (y = 0), and the values of $\partial F_\nu/\partial X_\alpha$, F_ν, and τ correspond to the unperturbed value $y = y(X_0)$. The equilibrium state of the mechanical system X_0 at a given radiation density is determined from the relation

$$k_\nu X_{0\nu} = F_\nu(X_0), \quad \nu = 1, 2, \ldots. \tag{6}$$

It is convenient to characterize the radiation system not by the energy $\mathscr{E}$, but by the power $W = \omega_0\mathscr{E}_m/Q(1 + y^2)$ proceeding from the source. Then

$$\frac{\partial F_\nu}{\partial X_\alpha} = -\frac{4y}{1 + y^2}\frac{\partial\omega_0}{\partial X_\nu}\frac{\partial\omega_0}{\partial X_\alpha}\frac{Q^2}{\omega_0^3}W. \tag{7}$$

The coupling between the oscillations of x_ν results from the fact that ω_0 is a function of the entire aggregate X.

Generally speaking, the appearance of the ponderomotive forces does not leave the mechanical system holonomic, so that gyroscopic forces $f_{\nu\alpha}\dot{x}_\alpha$ (where $f_{\nu\alpha} = -f_{\alpha\nu}$), can appear besides the terms written out on the right side of (5). Without considering here the effects due to the coupling of the oscillations, we shall take into account in the sums of (5) only the terms with $\alpha = \nu$.

It follows from (5) and (7) that on the right slope of the resonance curve ($\omega > \omega_0$) the ponderomotive forces, just as the elastic ones, are directed such as to return the system to the equilibrium position ($k_\nu k_{p\nu} \geq 0$, $k_{p\nu} = -\partial F_\nu/\partial X_\nu$), and when $\omega < \omega_0$ they move the system from the equilibrium position ($k_\nu k_{p\nu} < 0$). Accordingly, the increment introduced into the mechanical system as a result of the delay τ is negative when $\omega < \omega_0$, and the mechanical oscillations are additionally damped, and is positive when $\omega > \omega_0$. At large powers W and at $\omega > \omega_0$, when $2h_\nu + \tau\partial F_\nu/\partial X_\nu \leq 0$, self-oscillations arise in the system (mechanical vibration and modulation of the radiation).

The excited mechanical-vibration mode is the one having maximum $h_\nu(\partial\omega_0/\partial X_\nu)^{-2}$. Self-oscillations set in when the radiation power fed to the system is $W \geq W_{vib}$

$$W_{\text{vib}} = \frac{(1+y^2)^2}{8y}\frac{\omega_0^2}{Q\Omega^2}W_{\text{fr}}^{\nu}, \tag{8}$$

where $W_{fr}^{\nu} = \Omega^2\omega_0^2 h_\nu(Q\partial\omega_0/\partial X_\nu)^{-2}$ is the power lost to mechanical friction at an oscillation swing ΔX_ν, corresponding to a frequency change $\Delta\omega_0 \approx \omega_0/Q$. The threshold W_{vib} is maximal when $y = 1/\sqrt{3}$, where $(1 + y^2)^2/86 \approx 0.4$. The frequency of the arising oscillations is $\Omega \approx (k_\nu + k_{p\nu})^{1/2}/m_\nu$; for a high-Q mode ν ($2h_\nu \ll k_\nu\tau$) the frequency Ω is close to $\Omega_\nu = (k_\nu/m_\nu)^{1/2}$.

The steady state of the self-oscillations is determined by the nonlinearity of the resonance curve, so that near the optimal frequency deviation ($y \sim 1/\sqrt{3}$) the modulation of the radiation becomes appreciable even when W exceeds W_{vib} slightly. In[2], the depth of modulation reached 80–90%.

As already indicated, when $\omega < \omega_0$ the oscillations become deformed, but the fact that the stiffness introduced thereby by the ponderomotive forces is negative ($k_\nu k_{p\nu} < 0$) can lead in the case of large W to a reversal of the sign of $k_\nu + k_{p\nu}$. When $k_\nu + k_{p\nu} \leq 0$, the initial state X_0 becomes unstable. The left slope of the resonance curve then takes on a hysteresis form, and there is no stationary regime on the steep section of the slope. When the radiation system is tuned, jumps were observed in the change of the steady-state radiation energy at $\omega < \omega_0$, due to the transition of the mechanical system to new equilibrium states.

Inasmuch as the motion is not periodic, the threshold of the static instability is determined by the contribution of all the mechanical-oscillation modes (for details see[2]):

$$W_{\text{stat}} = -\frac{1+y^2}{2y}\omega_0 K, \quad K = \frac{\omega_0^2}{Q^2}\left[\sum \frac{2}{k_\nu}\left(\frac{\partial\omega_0}{\partial X_\nu}\right)^2\right]^{-1}.$$

If the contribution of the mode ν predominates in the sum in K, then $K \approx (1/2)k_\nu \Delta X^2$, i.e., it equals the energy of elastic deformation following a change ΔX_ν in which the frequency of the radiation system is changed by an amount equal to the line width.

Thus, at sufficiently large radiation densities, the ponderomotive effects cause both steep sections of the slopes of the resonance curve to be unstable[2)]. The threshold W_{thr} depends on the mechanical friction, while W_{stat} depends on the stiffness. In[2], the minimum values of W_{vib} and W_{stat} were of the same order. The situation is approximately the same also for the systems considered in[1,3], so that static instability can be expected there, too.

4. In the preceding analysis, the radiation system was assumed to be single-mode and the vibrational stability was calculated in the approximation in which $\Omega\tau \ll 1$. Let us stop to discuss the more general case.

Representing the radiation system by means of a set of independent (at $X = X_0$) oscillators with natural frequencies $\omega_i(X_0)$, and confining ourselves as before to the adiabatic terms of the interaction with the mechanical system, we can represent the radiation energy in the form

$$\mathscr{E} = \frac{1}{2}\sum[\dot{u}_i^2 + \omega_i^2 u_i^2 - b_{ik}u_i u_k - c_{ik}\dot{u}_i\dot{u}_k],$$

where $u_i(t)$—coordinate of i-th oscillator, b_{ik}, c_{ik}—functions of $x = X - X_0$, which vanish when $x = 0$. The equations of motion can be represented in the form

$$\ddot{u}_i + \frac{\omega_i}{Q_i}\dot{u}_i + \omega_i^2 u_i \approx q_i(t) - \sum(b_{ik}u_k + c_{ik}\ddot{u}_k), \tag{9}$$

$$m\ddot{x} + 2h\dot{x} + kx = -\frac{\partial\mathscr{E}}{\partial X} = \frac{1}{2}\sum\left(\frac{db_{ik}}{dX}u_i u_k + \frac{dc_{ik}}{dX}\dot{u}_i\dot{u}_k\right). \tag{10}$$

For simplicity, the mechanical motion is assumed single-mode and index ν is omitted. Generally speaking, in Eqs. (9), just as in (5), additional terms due to the appearance of the gyroscopic potential and not taken into account by the Hamiltonian are possible. We shall disregard effects connected with violation of the holonomy of the couplings. As before, we confine ourselves to the case when the external source is monochromatic: $q_i(t) = q_i\exp(j\omega t)$, and the quality factors are $Q_i \gg 1$.

Let us determine the power transferred to the mechanical system in the case of periodic motion $x = x_0\exp(j\Omega t) + x_0^*\exp(-j\Omega t)$. In the approximation linear in x, the solution of (9) takes the form

$$u_i = u_i^{(0)} + u_i^{(1)}.$$

$$u_i^{(1)} = \sum_k a_{ik}u_k^{(0)}[Z_i(\omega+\Omega)x_0 e^{j\Omega t} + Z_i(\omega-\Omega)x_0^* e^{-j\Omega t}],$$

$$Z_i(\nu) = \frac{\omega^2}{\omega_i^2 - \nu^2 + j\nu\omega_i/Q_i}, \quad 2a_{ik} = \frac{dc_{ik}}{dx} - \frac{1}{\omega^2}\frac{db_{ik}}{dx},$$

where $u_i^{(0)} \sim e^{j\omega t}$ is the zeroth approximation ($x = 0$), and $Z_i(\nu) = Y_i - jR_i$ is the normalized impedance of the radiation mode i at the frequency ν. The power transferred to the mechanical system during the oscillation period $2\pi/\Omega$ is

$$\overline{F\dot{x}} = 2x_0x_0^*\Omega\omega^2\sum_{i,k,l} a_{ik}a_{kl}[R_k(\omega-\Omega) - R_k(\omega+\Omega)]\,u_i^{(0)}u_l^{(0)*}.$$

When this quantity exceeds the power dissipated in friction $W_{fr} = 4h\Omega^2 x_0 x_0^*$, vibrational instability occurs in this system. If, for example, one radiation mode with index i is excited in the zeroth approximation, then the increment transferred to the mechanical system is

2)The mechanism of vibrational instability was first proposed in [3]. The analysis presented here is based on the result of [2], in which, besides the vibrational instability, the static instability was observed and explained for the first time.

$$\delta = \frac{Q_i W_i}{\omega_i \Omega m} \sum_k a_{ik}^2 [R_k(\omega - \Omega) - R_k(\omega + \Omega)], \quad (11)$$

where W_i is the radiation power coming from the outside.

For those forms of mechanical motion, at which the energy transferred to the radiation is stored in the energy of coupling between the oscillators i and k, the off-diagonal coefficients a_{ik} differ from zero. If the radiation system is close to degeneracy then, as seen from (11), the conditions for vibrational stability (8) are radically changed. If it is possible to retain in the sum of (11) one term $a_{ii} = a$, then the one-mode approximation is valid, and in particular, when $\Omega/\omega \to 0$, we get

$$R_{(\omega-\Omega)} - R_{(\omega+\Omega)} = -2\Omega \frac{\partial R}{\partial \omega} = \frac{2y}{1+y^2} \Omega \tau Q$$

and we arrive at the case considered in Sec. 3.

5. For ponderomotive forces of electrostriction or magnetostriction nature, the effects considered above can take place not only when the radiation system is excited (size resonances), but when the natural frequencies of the polarization or magnetization are excited (frequency resonances). These are, for example, the magnetoelastic instabilities of ferromagnets in ferromagnetic resonance, which were investigated in[4,5].

We have assumed that the electromagnetic system is linear in the absence of mechanical motion, but the discussed effects are typical also of the more general case, when the state of the electromagnetic system changes strongly upon deformation and a retardation of the ponderomotive forces relative to the deformation takes place. In particular, a similar situation can arise when the electromagnetic system is excited at multiple or fractional resonances. Such instabilities were observed in[6] in parametric resonance ($\omega \sim 2\omega_0$, $\Omega \ll \omega_0$).

In conclusion we note that the considered instability mechanisms can be used to interpret the interaction between fast and slow vibrational systems of a different nature, such as the experimentally investigated interactions between transverse and longitudinal elastic oscillations[7], spin-spin instability[8], etc. It can be shown from general considerations that in the zeroth approximation ($\Omega/\omega \to 0$) the fast system serves as a reservoir of potential energy for the slow system (and sometimes also kinetic energy), thus causing the static instabilities; retardation effects appear in the next higher approximation, and one of their characteristic manifestations is the vibrational instability.

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